# Lie identities on symmetric elements of restricted enveloping algebras

Salvatore Siciliano[1], Hamid Usefi[2]

[1]Università del Salento

[2]Memorial University of Newfoundland

Polynomial Identities in Algebras II, St. John's, 2011

# Lifting Identities

- Let $A$ be an algebra with involution $*$ over a field $\mathbb{F}$. We denote by $A^+ := \{x \in A|\, x^* = x\}$ the set of symmetric elements of $A$ under $*$ and by $A^- := \{x \in A|\, x^* = -x\}$ the set of skew-symmetric elements.

# Lifting Identities

- Let $A$ be an algebra with involution $*$ over a field $\mathbb{F}$. We denote by $A^+ := \{x \in A |\, x^* = x\}$ the set of symmetric elements of $A$ under $*$ and by $A^- := \{x \in A |\, x^* = -x\}$ the set of skew-symmetric elements.
- A question of general interest is which properties of $A^+$ or $A^-$ can be lifted to the whole algebra $A$.

# Lifting Identities

- Let $A$ be an algebra with involution $*$ over a field $\mathbb{F}$. We denote by $A^+ := \{x \in A |\, x^* = x\}$ the set of symmetric elements of $A$ under $*$ and by $A^- := \{x \in A |\, x^* = -x\}$ the set of skew-symmetric elements.
- A question of general interest is which properties of $A^+$ or $A^-$ can be lifted to the whole algebra $A$.
- The history of this problem goes back to Herstein where he had conjectured that if the symmetric or skew-symmetric elements of a ring $R$ satisfy a polynomial identity, then so does $R$.

# Lifting Identities

- Let $A$ be an algebra with involution $*$ over a field $\mathbb{F}$. We denote by $A^+ := \{x \in A|\, x^* = x\}$ the set of symmetric elements of $A$ under $*$ and by $A^- := \{x \in A|\, x^* = -x\}$ the set of skew-symmetric elements.
- A question of general interest is which properties of $A^+$ or $A^-$ can be lifted to the whole algebra $A$.
- The history of this problem goes back to Herstein where he had conjectured that if the symmetric or skew-symmetric elements of a ring $R$ satisfy a polynomial identity, then so does $R$.
- Simple case and semi-prime algebras due to Herstein and Martindale.

# Lifting Identities

- Let $A$ be an algebra with involution $*$ over a field $\mathbb{F}$. We denote by $A^+ := \{x \in A | \, x^* = x\}$ the set of symmetric elements of $A$ under $*$ and by $A^- := \{x \in A | \, x^* = -x\}$ the set of skew-symmetric elements.
- A question of general interest is which properties of $A^+$ or $A^-$ can be lifted to the whole algebra $A$.
- The history of this problem goes back to Herstein where he had conjectured that if the symmetric or skew-symmetric elements of a ring $R$ satisfy a polynomial identity, then so does $R$.
- Simple case and semi-prime algebras due to Herstein and Martindale.
- Notably this conjecture was proved by Amitsur in 1968 and subsequently generalized by himself in 1969.

# Definitions

- A subset $S$ of $A$ is said to be Lie nilpotent if there exists a positive integer $n$ such that

$$[x_1, \ldots, x_n] = 0$$

for every $x_1, \ldots, x_n \in S$, while $S$ is said to be bounded Lie Engel if there is an $n$ such that

$$[x,_n y] = 0$$

for every $x, y \in S$.

# Group Algebras

- Let $G$ be a group and consider the group algebra $\mathbb{F}G$ over a field $\mathbb{F}$. The canonical involution is induced by $g \mapsto g^{-1}$, for every $g \in G$.

# Group Algebras

- Let $G$ be a group and consider the group algebra $\mathbb{F}G$ over a field $\mathbb{F}$. The canonical involution is induced by $g \mapsto g^{-1}$, for every $g \in G$.
- There has been an intensive investigation devoted to demonstrate the extent to which the symmetric or skew-symmetric elements of $\mathbb{F}G$ under the canonical involution determine the structure of the group algebra.

# Group Algebras

- Let $G$ be a group and consider the group algebra $\mathbb{F}G$ over a field $\mathbb{F}$. The canonical involution is induced by $g \mapsto g^{-1}$, for every $g \in G$.
- There has been an intensive investigation devoted to demonstrate the extent to which the symmetric or skew-symmetric elements of $\mathbb{F}G$ under the canonical involution determine the structure of the group algebra.
- In particular, the chacterization of groups $G$ for which $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is Lie nilpotent was carried out by Giambruno, Sehgal, and Lee during 1993-2006.

# Group Algebras

- Let $G$ be a group and consider the group algebra $\mathbb{F}G$ over a field $\mathbb{F}$. The canonical involution is induced by $g \mapsto g^{-1}$, for every $g \in G$.
- There has been an intensive investigation devoted to demonstrate the extent to which the symmetric or skew-symmetric elements of $\mathbb{F}G$ under the canonical involution determine the structure of the group algebra.
- In particular, the chacterization of groups $G$ for which $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is Lie nilpotent was carried out by Giambruno, Sehgal, and Lee during 1993-2006.
- Furthermore, if either $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is bounded Lie Engel, and $G$ is devoid of 2-elements, Lee in 2000 showed that $\mathbb{F}G$ is bounded Lie Engel. He also classified remaining groups for which $\mathbb{F}G^+$ is bounded Lie Engel.

# Group Algebras

- Let $G$ be a group and consider the group algebra $\mathbb{F}G$ over a field $\mathbb{F}$. The canonical involution is induced by $g \mapsto g^{-1}$, for every $g \in G$.
- There has been an intensive investigation devoted to demonstrate the extent to which the symmetric or skew-symmetric elements of $\mathbb{F}G$ under the canonical involution determine the structure of the group algebra.
- In particular, the chacterization of groups $G$ for which $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is Lie nilpotent was carried out by Giambruno, Sehgal, and Lee during 1993-2006.
- Furthermore, if either $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is bounded Lie Engel, and $G$ is devoid of 2-elements, Lee in 2000 showed that $\mathbb{F}G$ is bounded Lie Engel. He also classified remaining groups for which $\mathbb{F}G^+$ is bounded Lie Engel.
- The Lie solvable case has been considered, however a complete answer to this case seems still under way.

# Enveloping Algebras

- Let $L$ be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$ and let $u(L)$ be the restricted enveloping algebra of $L$.

# Enveloping Algebras

- Let $L$ be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$ and let $u(L)$ be the restricted enveloping algebra of $L$.
- We denote by $\top$ the *principal involution* of $u(L)$, that is, the unique $\mathbb{F}$-antiautomorphism of $u(L)$ such that $x^{\top} = -x$ for every $x$ in $L$. We recall that $\top$ is just the antipode of the $\mathbb{F}$-Hopf algebra $u(L)$.

# Enveloping Algebras

- Let $L$ be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$ and let $u(L)$ be the restricted enveloping algebra of $L$.
- We denote by $\top$ the *principal involution* of $u(L)$, that is, the unique $\mathbb{F}$-antiautomorphism of $u(L)$ such that $x^\top = -x$ for every $x$ in $L$. We recall that $\top$ is just the antipode of the $\mathbb{F}$-Hopf algebra $u(L)$.
- The symmetric elements do not form a Lie subalgebra of $u(L)$ in general but they form a Jordan subalgebra under the Jordan bracket $x \circ y = \frac{1}{2}(xy + yx)$.

# Enveloping Algebras

- Let $L$ be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$ and let $u(L)$ be the restricted enveloping algebra of $L$.
- We denote by $\top$ the *principal involution* of $u(L)$, that is, the unique $\mathbb{F}$-antiautomorphism of $u(L)$ such that $x^\top = -x$ for every $x$ in $L$. We recall that $\top$ is just the antipode of the $\mathbb{F}$-Hopf algebra $u(L)$.
- The symmetric elements do not form a Lie subalgebra of $u(L)$ in general but they form a Jordan subalgebra under the Jordan bracket $x \circ y = \frac{1}{2}(xy + yx)$.
- Recently, Siciliano established the conditions under which $u(L)^-$ is Lie solvable, Lie nilpotent or bounded Lie Engel.

# Main Results

## Theorem

*Let $L$ be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following conditions are equivalent:*

1) *$u(L)^{+}$ is bounded Lie Engel;*
2) *$u(L)$ is bounded Lie Engel;*
3) *$L$ is nilpotent, $L'$ is $p$-nilpotent, and $L$ contains a restricted ideal $I$ such that $L/I$ and $I'$ are finite-dimensional.*

**Theorem**

*Let $L$ be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following conditions are equivalent:*

1) *$u(L)^+$ is Lie nilpotent;*
2) *$u(L)$ is Lie nilpotent;*
3) *$L$ is nilpotent and $L'$ is finite-dimensional and $p$-nilpotent.*

**Theorem**

*Let $L$ be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following conditions are equivalent:*

1) $u(L)^+$ *is Lie solvable;*
2) $u(L)$ *is Lie solvable;*
3) $L'$ *is finite-dimensional and p-nilpotent.*

# Comments

- The equivalence of 2) and 3) in our Theorems was established by Riley and Shalev in 1993.

# Comments

- The equivalence of 2) and 3) in our Theorems was established by Riley and Shalev in 1993.
- Our main contribution is to prove that 1) implies 3) in these theorems.

# Comments

- The equivalence of 2) and 3) in our Theorems was established by Riley and Shalev in 1993.
- Our main contribution is to prove that 1) implies 3) in these theorems.
- We can conclude that, in odd characteristic, if either $u(L)^+$ or $u(L)^-$ is Lie solvable (respectively, bounded Lie Engel or Lie nilpotent) then so is the whole algebra $u(L)$. Note that $[A^+, A^+] \subseteq A^-$.

# Comments

- The equivalence of 2) and 3) in our Theorems was established by Riley and Shalev in 1993.
- Our main contribution is to prove that 1) implies 3) in these theorems.
- We can conclude that, in odd characteristic, if either $u(L)^+$ or $u(L)^-$ is Lie solvable (respectively, bounded Lie Engel or Lie nilpotent) then so is the whole algebra $u(L)$. Note that $[A^+, A^+] \subseteq A^-$.
- Such conclusions are no longer true in characteristic 2.

# Some Known Results

**Theorem (Amitsur)**

*Let $A$ be an associative algebra with involution. If $A^+$ (or $A^-$) satisfies a PI then $A$ satisfies a polynomial identity.*

**Theorem (Passman and Petrogradsky)**

*The enveloping algebra $u(L)$ satisfies a PI if and only if $L$ has a restricted ideal $A$ of finite codimension in $L$ such that $A$ is nilpotent of class two and $A'$ is finite-dimensional and $p$-nilpotent.*

# Some Known Results

**Theorem (Stewart)**

*Let $L$ be a Lie algebra with a nilpotent ideal $M$ such that $L/M'$ is nilpotent. Then $L$ is nilpotent.*

**Corollary**

*Let $L$ be a restricted Lie algebra with a nilpotent ideal $M$ such that $L/(M')_p$ is nilpotent. Then $L$ is nilpotent.*

*Proof.* Suppose that $\gamma_{c+1}(L) \subseteq (M')_p$. Then $\gamma_{c+2}(L) \subseteq [(M')_p, L] \subseteq M'$. Thus, $L/M'$ is a nilpotent Lie algebra and it follows from the Theorem that $L$ is nilpotent.

**Lemma**

*Let $L = \langle H \rangle_p$ where $H$ is a finite-dimensional Lie algebra. If $u(L)^+$ is bounded Lie Engel then $L$ is nilpotent.*

**Lemma**

*Let $L$ be a metabelian restricted Lie algebra containing an abelian ideal $N$ of finite codimension. If $u(L)^+$ is bounded Lie Engel, then $L$ is nilpotent.*

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is bounded Lie Engel. Then $L$ is nilpotent.

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is bounded Lie Engel. Then $L$ is nilpotent.
- *Proof.* It follows from Amitsur's theorem that $u(L)$ is PI.

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is bounded Lie Engel. Then $L$ is nilpotent.
- *Proof.* It follows from Amitsur's theorem that $u(L)$ is PI.
- Thus, by Passman and Petrogradsky's theorem, there exists a nilpotent restricted ideal $N$ of $L$ of finite codimension. Hence, by Stewart's theorem, it is enough to prove that $L/(N')_p$ is nilpotent.

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is bounded Lie Engel. Then $L$ is nilpotent.
- *Proof.* It follows from Amitsur's theorem that $u(L)$ is PI.
- Thus, by Passman and Petrogradsky's theorem, there exists a nilpotent restricted ideal $N$ of $L$ of finite codimension. Hence, by Stewart's theorem, it is enough to prove that $L/(N')_p$ is nilpotent.
- So, we can assume that $N$ is abelian.

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is bounded Lie Engel. Then $L$ is nilpotent.
- *Proof.* It follows from Amitsur's theorem that $u(L)$ is PI.
- Thus, by Passman and Petrogradsky's theorem, there exists a nilpotent restricted ideal $N$ of $L$ of finite codimension. Hence, by Stewart's theorem, it is enough to prove that $L/(N')_p$ is nilpotent.
- So, we can assume that $N$ is abelian.
- Since $L/N$ is finite-dimensional, by Lemma 1 we see that $L/N$ is nilpotent. Thus $L$ is solvable.

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is bounded Lie Engel. Then $L$ is nilpotent.
- *Proof.* It follows from Amitsur's theorem that $u(L)$ is PI.
- Thus, by Passman and Petrogradsky's theorem, there exists a nilpotent restricted ideal $N$ of $L$ of finite codimension. Hence, by Stewart's theorem, it is enough to prove that $L/(N')_p$ is nilpotent.
- So, we can assume that $N$ is abelian.
- Since $L/N$ is finite-dimensional, by Lemma 1 we see that $L/N$ is nilpotent. Thus $L$ is solvable.
- Let $d$ be the derived length of $L$ and consider $M = \delta_{d-2}(L)_p$. By Lemma 2, we know that $M$ is nilpotent. Moreover, by induction on the derived length we have that $L/(M')_p = L/\delta_{d-1}(L)_p$ is nilpotent. Therefore, by Stewart's theorem, we conclude that $L$ is nilpotent. □

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L$ is solvable.

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L$ is solvable.
- *Proof.* We can assume that $\mathbb{F}$ is algebraically closed. By Amitsur's theorem, $u(L)$ is PI and then, by Passman and Petrogradsky's theorem, there exists a nilpotent restricted ideal $N$ of $L$ of finite codimension. Thus we can replace $L$ by $L/N$ and assume that $L$ is finite-dimensional.

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L$ is solvable.
- *Proof.* We can assume that $\mathbb{F}$ is algebraically closed. By Amitsur's theorem, $u(L)$ is PI and then, by Passman and Petrogradsky's theorem, there exists a nilpotent restricted ideal $N$ of $L$ of finite codimension. Thus we can replace $L$ by $L/N$ and assume that $L$ is finite-dimensional.
- Suppose that $L$ is a counterexample of minimal dimension.

# Proofs

- **Proposition:** Let $L$ be a restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L$ is solvable.
- *Proof.* We can assume that $\mathbb{F}$ is algebraically closed. By Amitsur's theorem, $u(L)$ is PI and then, by Passman and Petrogradsky's theorem, there exists a nilpotent restricted ideal $N$ of $L$ of finite codimension. Thus we can replace $L$ by $L/N$ and assume that $L$ is finite-dimensional.
- Suppose that $L$ is a counterexample of minimal dimension.
- Now, if $M$ is a non-trivial restricted ideal of $L$, then $u(M)^+$ and $u(L/M)^+$ are Lie solvable. Hence the minimality of $L$ forces that both $M$ and $L/M$ are solvable, so that $L$ is solvable, a contradiction. Consequently, $L$ has no non-trivial restricted ideal.

# Proofs

- Now, we claim that $L'$ is a simple Lie algebra.
- Let $I$ be a nonzero ideal of $L'$. As $(L')_p$ is a nonzero restricted ideal of $L$ we have $(L')_p = L$ and then $I$ is an ideal of $L$. It follows that $I_p = L$ and then $L' = [I_p, I_p] = [I, I] \subseteq I$, so that $I = L'$, as claimed. In particular, $L'$ is a simple restricted $L$-module.
- Denote by $\mathcal{J}$ the Jacobson radical of $u(L)$. Note that the simple modules of $u(L)$ and $u(L)/\mathcal{J}$ are the same. Since $\mathcal{J}$ is $\top$-invariant we have that $(u(L)/\mathcal{J})^+$ under the induced involution is Lie solvable. Let $u(L)/\mathcal{J} \cong \oplus_{i=1}^r M_{n_i}(\mathbb{F})$ be the the Wedderburn decomposition of $u(L)/\mathcal{J}$.
- Then by a result of Lee et al. we have that $n_i \leq 2$ for every $i$. Therefore every irreducible representation of $u(L)$ has degree 1 or 2. In particular $L'$ has dimension less than 3, so that $L$ is solvable, a contradiction. □

# Proofs

- **Proposition:** Let $L$ be a finite-dimensional restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L'$ is $p$-nilpotent.

# Proofs

- **Proposition:** Let $L$ be a finite-dimensional restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L'$ is $p$-nilpotent.
- *Proof.* We know that $L$ is solvable. By induction on the derived length of $L$, we can assume $L$ is metabelian. Furthermore, we can suppose that the ground field $\mathbb{F}$ is algebraically closed.

# Proofs

- **Proposition:** Let $L$ be a finite-dimensional restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L'$ is $p$-nilpotent.
- *Proof.* We know that $L$ is solvable. By induction on the derived length of $L$, we can assume $L$ is metabelian. Furthermore, we can suppose that the ground field $\mathbb{F}$ is algebraically closed.
- Let $\mathcal{J}$ be the Jacobson radical of $u(L)$ and consider the restricted Lie algebra $H = L + \mathcal{J}/\mathcal{J}$.

- **Proposition:** Let $L$ be a finite-dimensional restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L'$ is $p$-nilpotent.
- *Proof.* We know that $L$ is solvable. By induction on the derived length of $L$, we can assume $L$ is metabelian. Furthermore, we can suppose that the ground field $\mathbb{F}$ is algebraically closed.
- Let $\mathcal{J}$ be the Jacobson radical of $u(L)$ and consider the restricted Lie algebra $H = L + \mathcal{J}/\mathcal{J}$.
- Since $\mathcal{J}$ is a nilpotent ideal of $u(L)$, it suffices to show that $H'$ is $p$-nilpotent. Thus, we may assume that $H$ is not abelian.

# Proofs

- **Proposition:** Let $L$ be a finite-dimensional restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L'$ is $p$-nilpotent.
- *Proof.* We know that $L$ is solvable. By induction on the derived length of $L$, we can assume $L$ is metabelian. Furthermore, we can suppose that the ground field $\mathbb{F}$ is algebraically closed.
- Let $\mathcal{J}$ be the Jacobson radical of $u(L)$ and consider the restricted Lie algebra $H = L + \mathcal{J}/\mathcal{J}$.
- Since $\mathcal{J}$ is a nilpotent ideal of $u(L)$, it suffices to show that $H'$ is $p$-nilpotent. Thus, we may assume that $H$ is not abelian.
- Note that $u(L)/\mathcal{J} \cong \mathbb{F} \oplus \cdots \mathbb{F} \oplus M_2(\mathbb{F}) \oplus \cdots \oplus M_2(\mathbb{F})$.

# Proofs

- **Proposition:** Let $L$ be a finite-dimensional restricted Lie algebra such that $u(L)^+$ is Lie solvable. Then $L'$ is $p$-nilpotent.
- *Proof.* We know that $L$ is solvable. By induction on the derived length of $L$, we can assume $L$ is metabelian. Furthermore, we can suppose that the ground field $\mathbb{F}$ is algebraically closed.
- Let $\mathcal{J}$ be the Jacobson radical of $u(L)$ and consider the restricted Lie algebra $H = L + \mathcal{J}/\mathcal{J}$.
- Since $\mathcal{J}$ is a nilpotent ideal of $u(L)$, it suffices to show that $H'$ is $p$-nilpotent. Thus, we may assume that $H$ is not abelian.
- Note that $u(L)/\mathcal{J} \cong \mathbb{F} \oplus \cdots \mathbb{F} \oplus M_2(\mathbb{F}) \oplus \cdots \oplus M_2(\mathbb{F})$.
- Without loss of generality we can assume that $H$ is a restricted Lie subalgebra of $M_2(\mathbb{F})$.

# Proofs

- Since $M_2(\mathbb{F})$ is not Lie metabelian, we have $\dim H \leq 3$. If $\dim H = 2$, then there exists $x, y \in H$ such that with $[x, y] = x$. In this case we must have $x^p = 0$ and $y^p = y$, so we are done.

# Proofs

- Since $M_2(\mathbb{F})$ is not Lie metabelian, we have $\dim H \leq 3$. If $\dim H = 2$, then there exists $x, y \in H$ such that with $[x, y] = x$. In this case we must have $x^p = 0$ and $y^p = y$, so we are done.
- Assume then $\dim H = 3$. It is clear that the identity matrix $I_2$ must be in $H$, otherwise $M_2(\mathbb{F})$ is spanned by $H$ and $I_2$ so that $M_2(\mathbb{F})$ is Lie metabelian, which is not possible.

# Proofs

- Since $M_2(\mathbb{F})$ is not Lie metabelian, we have $\dim H \leq 3$. If $\dim H = 2$, then there exists $x, y \in H$ such that with $[x, y] = x$. In this case we must have $x^p = 0$ and $y^p = y$, so we are done.
- Assume then $\dim H = 3$. It is clear that the identity matrix $I_2$ must be in $H$, otherwise $M_2(\mathbb{F})$ is spanned by $H$ and $I_2$ so that $M_2(\mathbb{F})$ is Lie metabelian, which is not possible.
- Note that $H' \subseteq [M_2(\mathbb{F}), M_2(\mathbb{F})] = \mathrm{sl}_2(\mathbb{F})$. Since $H'$ is abelian and $\mathrm{sl}_2(\mathbb{F})$ has no 2-dimensional abelian subalgebra, it follows that $\dim H' = 1$.

# Proofs

- Since $M_2(\mathbb{F})$ is not Lie metabelian, we have $\dim H \leq 3$. If $\dim H = 2$, then there exists $x, y \in H$ such that with $[x, y] = x$. In this case we must have $x^p = 0$ and $y^p = y$, so we are done.
- Assume then $\dim H = 3$. It is clear that the identity matrix $I_2$ must be in $H$, otherwise $M_2(\mathbb{F})$ is spanned by $H$ and $I_2$ so that $M_2(\mathbb{F})$ is Lie metabelian, which is not possible.
- Note that $H' \subseteq [M_2(\mathbb{F}), M_2(\mathbb{F})] = \mathrm{sl}_2(\mathbb{F})$. Since $H'$ is abelian and $\mathrm{sl}_2(\mathbb{F})$ has no 2-dimensional abelian subalgebra, it follows that $\dim H' = 1$.
- Let $x \in H' \subseteq \mathrm{sl}_2(\mathbb{F})$. Since $\mathrm{tr}(x) = 0$, it follows that $x$ and $I_2$ are linearly independent. Let $y \in H$ so that $x, y$ and $I_2$ span $H$.

- Since $M_2(\mathbb{F})$ is not Lie metabelian, we have $\dim H \leq 3$. If $\dim H = 2$, then there exists $x, y \in H$ such that with $[x, y] = x$. In this case we must have $x^p = 0$ and $y^p = y$, so we are done.
- Assume then $\dim H = 3$. It is clear that the identity matrix $I_2$ must be in $H$, otherwise $M_2(\mathbb{F})$ is spanned by $H$ and $I_2$ so that $M_2(\mathbb{F})$ is Lie metabelian, which is not possible.
- Note that $H' \subseteq [M_2(\mathbb{F}), M_2(\mathbb{F})] = \mathrm{sl}_2(\mathbb{F})$. Since $H'$ is abelian and $\mathrm{sl}_2(\mathbb{F})$ has no 2-dimensional abelian subalgebra, it follows that $\dim H' = 1$.
- Let $x \in H' \subseteq \mathrm{sl}_2(\mathbb{F})$. Since $\mathrm{tr}(x) = 0$, it follows that $x$ and $I_2$ are linearly independent. Let $y \in H$ so that $x, y$ and $I_2$ span $H$.
- Since $H$ is not abelian, we can assume $[x, y] = x$. But then $x^p$ is a central element in $H$ and so $x^p = \alpha I_2$, for some $\alpha \in \mathbb{F}$. Since $x \in \mathrm{sl}_2(\mathbb{F})$ and $p \geq 3$, $\mathrm{tr}(x^p) = 0$. Thus, $\alpha = 0$ and so $x^p = 0$.

**Thank you!**