Graded Embedding in PI-Algebras

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- $\mathbb{K}$ a field.
- Algebra := associative algebra with 1 over $\mathbb{K}$, unless otherwise stated.

Definition (2-graded algebra)

The algebra A is 2-graded if A can be written as the direct sum of 2 subspaces

$$A = A_0 \oplus A_1$$

such that $A_iA_j \subset A_{i+j(mod2)}$.

The 2×2 matrices have a natural 2-grading.

$$M_2(\mathbb{K}) = M_2(\mathbb{K})_0 \oplus M_2(\mathbb{K})_1$$

where

- $M_2(\mathbb{K})_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} ; a, d \in \mathbb{K} \right\}$,
- $M_2(\mathbb{K})_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} ; b, c \in \mathbb{K} \right\}$.

The 2×2 upper triangular matrices over $\mathbb{K}$, denoted by $U_2(\mathbb{K})$, is a 2-graded algebra too. Its grading is induced by the grading in matrices.

Let $X = \{x_1, x_2, x_3, \dots\}$ and $Y = \{y_1, y_2, y_3, \dots\}$ be countable infinite sets such that $X \cap Y = \emptyset$. The variables from X have degree 0, and those from Y have degree 1. The free algebra

$$\mathbb{K}\langle X \cup Y\rangle = \mathbb{K}\langle X \cup Y\rangle_0 \oplus \mathbb{K}\langle X \cup Y\rangle_1$$

where

$$\begin{aligned} \mathbb{K}\langle X \cup Y\rangle_0 &= \langle \text{monomials of degree } 0 \pmod 2\rangle \\ \mathbb{K}\langle X \cup Y\rangle_1 &= \langle \text{monomials of degree } 1 \pmod 2\rangle \end{aligned}$$

$\mathbb{K}\langle X \cup Y\rangle$ as above is said to be the free 2-graded algebra.

Definition

Let $A = A_0 \oplus A_1$ be a 2-graded algebra and $f = f(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) \in \mathbb{K}\langle X \cup Y \rangle$ be a polynomial. Then f is a 2-graded identity for A if

$$f(a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m) = 0 \ \forall a_i \in A_0, \ \forall b_j \in A_1.$$

$T_2(A) = \{$2-graded polynomial identities of $A\}$ is an ideal, stable under all 2-graded endomorphisms.

If $T_2(A) \neq 0$, A is said to be a 2-graded PI-algebra.

If $\mathbb{K}$ is an infinite field, a basis is known for the T_n-ideal of the $n \times n$ matrices (Vasilovsky in characteristic 0 and Azevedo, in characteristic $p \neq 2$) and the $n \times n$ upper triangular matrices (Koshlukov, Valenti). In particular, for $n = 2$, results due to Di Vincenzo (in characteristic 0), to Azevedo and Koshlukov (in positive characteristic), and to Valenti, give that:

- The set
$$\{[x_1, x_2], \quad y_1 y_2 y_3 - y_3 y_2 y_1\}$$
is a basis of 2-graded polynomial identities of $M_2(\mathbb{K})$.
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Graded embedding in $M_{1,1}$

Theorem (Berele, 1995)

Let $\mathbb{K}$ be an infinite field.

1. *There exists an $A = A_0 \oplus A_1$ such that $T_2(A) \supset T_2(M_{1,1}(E))$ but which does not have a 2-graded embedding into $M_{1,1}(S)$ over any supercommutative algebra S.*
2. *If $A = A_0 \oplus A_1$ is a 2-graded algebra such that $T_2(A) \supset T_2(M_{1,1}(E))$ and $Ann_A A_1 = 0$ then, A has a 2-graded embedding into $M_{1,1}(S)$ for some supercommutative algebra S where $Ann_A A_1 = \{a \in A \ : \ aA_1 = 0 \text{ or } A_1 a = 0\}$ and $E =$ Grassmann algebra.*

2-graded embedding = 2-graded injective homomorphism.

Graded embedding in M_2

Theorem

Let $\mathbb{K}$ be an infinite field.

1. *There exists an $A = A_0 \oplus A_1$ such that $T_2(A) \supset T_2(M_2(\mathbb{K}))$ but which does not have a 2-graded embedding into $M_2(C)$ over any commutative algebra C.*
2. *If $A = A_0 \oplus A_1$ is a 2-graded algebra such that $T_2(A) \supset T_2(M_2(\mathbb{K}))$ and $Ann_A A_1 = 0$ then A has a 2-graded embedding into $M_2(C)$ for some commutative algebra S, where $Ann_A A_1 = \{a \in A \, ; \; aA_1 = 0 \text{ or } A_1 a = 0\}$.*

In other words, Statement 1 says that the necessary condition $T_2(A) \supset T_2(M_2(\mathbb{K}))$ is not sufficient for $A \hookrightarrow M_2(C)$ while Statement 2 gives a sufficient condition for the embedding.

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Proof (Statement 1).

- $A = \dfrac{\mathbb{K}\langle X \cup Y \rangle}{J}$,

  where J is the ideal generated by $T_2(M_2(\mathbb{K}))$ and $y_1 y_2$.
- The grading in A is induced by the grading in $\mathbb{K}\langle X \cup Y \rangle$.
- $a = y_1 + J, b = y_2 + J \in A_1 \Rightarrow ab = 0, ba \neq 0.$
- $M, N \in M_2(C)_1, MN = 0 \Leftrightarrow NM = 0$,
  for any commutative algebra C.
- $A \not\hookrightarrow M_2(C)$, for any commutative algebra C.

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Graded embedding in U_2

Theorem

Let $A = A_0 \oplus A_1$ be 2-graded algebra.
$T_2(A) \supset T_2(U_2(\mathbb{K})) \Leftrightarrow A$ has a 2-graded embedding in $U_2(C)$ for some commutative algebra C.

Proof.

- $T_2(A) \supset T_2(U_2(\mathbb{K})) \Leftrightarrow \begin{cases} A_0 \text{ is commutative} \\ {A_1}^2 = 0 \end{cases}$
- A_1 is an $(A_0 \otimes A_0)$-module under the action

$$a_1(b \otimes c) = (b \otimes c)a_1 = ba_1c, \ \forall\ b,\ c \in A_0,\ a_1 \in A_1.$$

- $C = (A_0 \otimes A_0)[A_1]$
- Consider

$$\begin{array}{cccc} \varphi: & A & \rightarrow & U_2(C) \\ & A_0 \ni a_0 & \longmapsto & \begin{pmatrix} a_0 \otimes 1 & 0 \\ 0 & 1 \otimes a_0 \end{pmatrix} \\ & A_1 \ni a_1 & \longmapsto & \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix} \end{array}$$

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- φ is injective.
- φ is a homomorphism.

$$\varphi(a_0)\varphi(\tilde{a}_0) = \varphi(a_0\tilde{a}_0),\ \varphi(a_1)\varphi(\tilde{a}_1) = \varphi(a_1\tilde{a}_1) = 0$$

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$\forall\ a_0, \tilde{a}_0 \in A_0,\ \forall a_1, \tilde{a}_1 \in A_1.$

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$\forall\ a_0, \tilde{a}_0 \in A_0,\ \forall a_1, \tilde{a}_1 \in A_1.$

- $sl_2(\mathbb{K})$ = the Lie algebra of the traceless 2×2 matrices (the bracket is given by the usual commutator).

$$sl_2(\mathbb{K}) = (sl_2(\mathbb{K}))_0 \oplus (sl_2(\mathbb{K}))_1$$

where $(sl_2(\mathbb{K}))_0$ consists of the diagonal matrices and $(sl_2(\mathbb{K}))_1$ of the off-diagonal ones.
- $L = L_0 \oplus L_1$ is a 2-graded Lie algebra.
- $\mathcal{L} = \mathcal{L}(X \cup Y)$ is the 2-graded free Lie algebra.
- $T_2(L) = \{$2-graded polynomial identities of $L\} \subset \mathcal{L}$

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Theorem (P. Koshlukov, 2008)

If $\mathbb{K}$ is an infinite field of characteristic $\neq 2$, then

$$\{[x_1, x_2]\}$$

is a basis of the 2-graded polynomial identities of $sl_2(\mathbb{K})$.

Graded embedding in sl_2

Theorem

Let $L = L_0 \oplus L_1$ be 2-graded Lie algebra over an infinite field $\mathbb{K}$ of characteristic $\neq 2$. If $T_2(L) \supset T_2(sl_2(\mathbb{K}))$ and $Ann_{L_1} L_1 = 0$ then L has a 2-graded embedding in $sl_2(C)$ for some C commutative algebra, where $Ann_{L_1} L_1 = \{\ell_1 \in L_1 \ : \ [\ell_1, L_1] = 0\}$.

Proof.

- $T_2(L) \supset T_2(sl_2(\mathbb{K})) \Leftrightarrow L_0$ is an abelian Lie algebra.
- $U(L_0)$ is the universal enveloping algebra of L_0. In this case $U(L_0)$ is the tensor algebra $T(L_0)$ such that $a \otimes b = b \otimes a$, $\forall a, b \in L_0$.
- We will define L_1 as $U(L_0)$-module in two different forms.
- $M = L_1$ as $U(L_0)$-module under the action induced by

$$a(m(\ell_1)) = (m(\ell_1))a = \frac{1}{2}m([a, \ell_1]),$$

where $a \in L_0$ and $m(\ell_1)$ is the element of M corresponding to $\ell_1 \in L_1$.
- The action is well defined because

$$(a \otimes b)(m(\ell_1)) = a(b(m(\ell_1))) = b(a(m(\ell_1))) = (b \otimes a)(m(\ell_1)).$$

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- $N = L_1$ as $U(L_0)$-module under the action induced by

$$a(n(\ell_1)) = (n(\ell_1))a = \frac{1}{2}n([\ell_1, a])$$

where $a \in L_0$ and $n(\ell_1)$ is the element of N corresponding to $\ell_1 \in L_1$.

- $C = \dfrac{U(L_0)[M \oplus N]}{J}$,

where J is the ideal generated by all elements

$$[\ell_1, \tilde{\ell_1}] - m(\ell_1)n(\tilde{\ell_1}) + m(\tilde{\ell_1})n(\ell_1)$$

for all $\ell_0, \tilde{\ell_0} \in L_0$.

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- φ is a homomorphism.
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$$\varphi(\ell_0) = 0 \Leftrightarrow \ell_0 = 0$$

$$\varphi(\ell_1) = 0 \Leftrightarrow m(\ell_1), n(\ell_1) \in J \Leftrightarrow [\ell_1, \tilde{\ell_1}] \in J, \ \forall\ \tilde{\ell_1} \in L_1 \Leftrightarrow [\ell_1, \tilde{\ell_1}] = 0, \ \forall\ \tilde{\ell_1} \in L_1 \Leftrightarrow \ell_1 \in Ann_{L_1} L_1 = 0 \Leftrightarrow \ell_1 = 0$$

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Thank you