On the centre of the generic algebra of $M_{1,1}$

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- F is a field of characteristic zero;

- $F\langle X\rangle = F\langle x_1, x_2, \ldots\rangle$ is the free associative algebra, freely generated by the set $X = \{x_1, x_2, \ldots\}$;

- $E = E_0 \oplus E_1$ is the Grassmann algebra of an infinite countable dimensional vector space over F.

- If A is a PI-algebra, we denote by $T(A)$, its ideal of polynomial identities and we use the following notations for its relatively free algebras

$$F(A) = \frac{F\langle x_1, x_2, \ldots\rangle}{T(A)} \quad F_k(A) = \frac{F\langle x_1, \ldots, x_k\rangle}{T(A) \cap F\langle x_1, \ldots, x_k\rangle}$$

A T-ideal P of $F\langle X\rangle$ is T-prime if for any T-ideals I and J of $F\langle X\rangle$ such that $IJ \subseteq P$, we have $I \subseteq P$ or $J \subseteq P$ (or both inclusions).

A PI-algebra A is called a T-prime algebra if $T(A)$ is T-prime.

Kemer has proved that if P is a nontrivial T-prime T-ideal then P is the ideal of identities of one of the following algebras:

$$(i)\ M_n(F) \qquad (ii)\ M_n(E) \qquad (iii)\ M_{a,b}$$

where $M_{a,b} = M_{a,b}(E)$ and for any 2-graded algebra $A = A_0 \oplus A_1$, $M_{a,b}(A)$ is defined by

$$M_{a,b}(A) = \begin{pmatrix} M_a(A_0) & M_{a\times b}(A_1) \\ M_{b\times a}(A_1) & M_b(A_0) \end{pmatrix}$$

Berele [*Generic verbally prime algebras and their GK dimensions*, Commun. Algebra 21, 1487-1504 (1993)] has constructed models for the relatively free algebras of the T-prime algebras, using matrices over the free supercommutative algebra.

To construct the free supercommutative algebra, we consider the free associative algebra $F\langle X \cup Y\rangle$ and we induce on it a $\mathbb{Z}_2$-grading by setting $\deg x = 0$ and $\deg y = 1$ if $x \in X$ and $y \in Y$. Finally, we ensure supercommutativity by modding out by all $ab - (-1)^{\deg a \deg b} ba$, for all $a, b \in X \cup Y$. The resulting algebra is the free supercommutative algebra, denoted by $F[X; Y]$.

Now we consider

$$X = \{X_{ij}^{(r)}; i,j \in \{1,\dots,n\}, r \in \mathbb{N}\} \text{ and}$$
$$Y = \{Y_{ij}^{(r)}; i,j \in \{1,\dots,n\}, r \in \mathbb{N}\}.$$

If $a+b=n$ and $r \in \mathbb{N}$, we construct the generic matrices over $F[X;Y]$:

$$A_r := (X_{ij}^{(r)})$$

$$B_r := (X_{ij}^{(r)} + Y_{ij}^{(r)})$$

$$C_r := \begin{pmatrix} (X_{ij}^{(r)})_{a\times a} & (Y_{ij}^{(r)})_{a\times b} \\ (Y_{ij}^{(r)})_{b\times a} & (X_{ij}^{(r)})_{b\times b} \end{pmatrix}$$

And we have

Theorem (Berele)

For any $k \geq 2$, the following isomorphisms hold:

$$\begin{array}{rcl} F_k(M_n(F)) & \cong & F[A_1, \ldots, A_k] \\ F_k(M_n(E)) & \cong & F[B_1, \ldots, B_k] \\ F_k(M_{a,b}) & \cong & F[C_1, \ldots, C_k] \end{array}$$

In the same paper, Berele proved that the centre of $F_k(M_{a,b})$ is the sum of the field and a nilpotent ideal of the centre. He also asks whether or not such centre contains non-scalar elements.

In this talk, we work on the generic algebra of rank 2 of $M_{1,1}$, $F[C_1, C_2] \cong F_2(M_{1,1})$. For this algebra we will:

- Answer Berele's question;
- Give a description of the centre of such algebra;
- Find a basis for its ideal of polynomial identities.

Popov [Identities of the tensor square of a Grassmann algebra. Algebra and Logic 21 (4) 296-316 (1982)] has proved that the T-ideal of polynomial identities of $M_{1,1}$ is generated by the polynomials:

$$[[x_1, x_2]^2, x_1] = 0 \qquad \text{and} \qquad [[[x_1, x_2], [x_3, x_4]], x_5] = 0$$

Since $F_2(M_{1,1})$ is the free algebra of rank 2 in the variety of the identities of $M_{1,1}$ it also satisfies the identities above.
In particular, the element $[C_1, C_2]^2$ is central in $F[C_1, C_2]$.

In order to simplify the notations, we consider

$$C_1 = \begin{pmatrix} X_1 & Y_1 \\ Y_1' & X_1' \end{pmatrix} \quad C_2 = \begin{pmatrix} X_2 & Y_2 \\ Y_2' & X_2' \end{pmatrix}$$

Since

$$[C_1, C_2]^2 = \begin{pmatrix} -2h_1 - h_4 & 2h_2 \\ -2h_3 & -2h_1 + h_4 \end{pmatrix}$$

where

$$\begin{aligned} h_1 &= Y_1 Y_2 Y_1' Y_2' \\ h_2 &= Y_1 Y_2 (Y_1'(X_2' - X_2) - Y_2'(X_1' - X_1)) \\ h_3 &= Y_1' Y_2' (Y_1(X_2' - X_2) - Y_2(X_1' - X_1)) \\ h_4 &= (Y_1'(X_2' - X_2) - Y_2'(X_1' - X_1))(Y_1(X_2' - X_2) - Y_2(X_1' - X_1)), \end{aligned}$$

we have that $[C_1, C_2]^2$ is a non-scalar central element of $F[C_1, C_2]$, answering positively Berele's question for $F_2(M_{1,1})$.

Proposition

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F[C_1, C_2]$.

If A *is central in* $F[C_1, C_2]$ *there exist* f_1 *and* $f_2 \in F[X]$ *such that*

$$b = f_2 h_2 \qquad c = -f_2 h_3 \qquad d - a = f_1 h_1 + f_2 h_4$$

As a consequence, there exist f_1 *and* $f_2 \in F[X]$ *such that*

$$A = aI_2 + f_1 \begin{pmatrix} 0 & 0 \\ 0 & h_1 \end{pmatrix} + f_2 \begin{pmatrix} 0 & h_2 \\ -h_3 & h_4 \end{pmatrix}$$

It is well known that any element of $F\langle x_1, x_2\rangle$ can be written as a linear combination of monomials of the type

$$x_1^n x_2^m u_1^{k_1} \dots u_r^{k_r}$$

where $u_i \in L(x_1, x_2)$ are left-normed commutators, that is commutators of the form

$$[x_{i_1}, \dots, x_{i_r}] = [[x_{i_1}, \dots, x_{i_{r-1}}], x_{i_r}]$$

Lemma

Let $f(x_1, x_2) \in L(x_1, x_2)$ be a left-normed commutator. If $f \neq 0$ then $f(C_1, C_2)$ is not central in $F[C_1, C_2]$.

We say that an element A of $F[C_1, C_2]$ is **central ideal** if A is central in $F[C_1, C_2]$ and the same holds for AB, for any $B \in F[C_1, C_2]$.

Lemma

Let $f_1, f_2, f_3 \in L(x_1, x_2)$ be left-normed commutators, $g = f_1 f_2$ and $h = f_1 f_2 f_3$. Then $g(C_1, C_2)$ is a nonzero central ideal element and $h(C_1, C_2) = 0$.

Proposition

Let $f(C_1, C_2) = C_1^n C_2^m u_1^{k_1} \dots u_r^{k_r} \in F[C_1, C_2]$, with $u_i \in L[C_1, C_2]$ left-normed commutators. Then, f is central if and only if $k_1 + \dots + k_r \geq 2$.

Lemma

Let $u_1,\ldots,u_r \in L(x_1,x_2)$ be left-normed commutators with the same degrees in x_1 and in x_2, and let $f(C_1,C_2)=\sum_i \alpha_i u_i(C_1,C_2) \in F[C_1,C_2]$. Then $f(C_1,C_2)$ is central if and only if it is central ideal, and if and only if

$$\sum_i \alpha_i = 0$$

Proposition

Let $f(C_1, C_2) \in F[C_1, C_2]$. We know that

$$f(C_1, C_2) = \alpha I_2 + \sum_j \left(C_1^{n_j} C_2^{m_j} (\sum_i \alpha_i^{(j)} u_i^{(j)}) \right) + g(C_1, C_2)$$

with $g(C_1, C_2) = \sum C_1^n C_2^m u_1 \dots u_k$, $k \geq 2$ and u_i left normed commutators. Then, $f(C_1, C_2)$ is central in $F[C_1, C_2]$ if and only if, for all j, $\sum_{i \in \Lambda} \alpha_i^{(j)} = 0$, where Λ is the set of i such that the corresponding u_i have the same degrees in x_1 and in x_2.

Corollary

$$Z(F[C_1, C_2]) = F \oplus I$$

Where I is a nilpotent ideal of $F[C_1, C_2]$.

With the previous results, we are able to determine the polynomial identities of $F[C_1, C_2]$.

Proposition

Let $f(C_1, C_2) = C_1^n C_2^m u_1^{k_1} \dots u_r^{k_r} \in F[C_1, C_2]$, with $u_i \in L[C_1, C_2]$ left-normed commutators. Then, $f(C_1, C_2) = 0$ if and only if $k_1 + \cdots + k_r \geq 3$.

Theorem

The algebra $F[C_1, C_2]$ satisfies the identities

$$[[x_1, x_2][x_3, x_4], x_5] \quad \textit{and} \quad [x_1, x_2][x_3, x_4][x_5, x_6]$$

As $\mathrm{char} F = 0$, we know that all polynomial identities of $F[C_1, C_2]$ follows from its proper multilinear polynomial identities.

The vector space of all proper polynomials of degree n is an S_n-module, denoted by Γ_n. It induces an S_n-module structure on

$$\Gamma_n(M_{1,1}) = \frac{\Gamma_n}{T(M_{1,1}) \cap \Gamma_n}$$

In the same paper mentioned before, Popov describes the S_n-module structure of $\Gamma_n(M_{1,1})$.

He showed that

$$\Gamma_n(M_{1,1}) = \left(\bigoplus_{p+s=n} M_p^{(s)}\right) \bigoplus \left(\bigoplus_{\substack{p+q+s=n \\ p,q\geq 2}} M_{p,q}^{(s)}\right)$$

where $M_p^{(s)}$ and $M_{p,q}^{(s)}$ are the irreducible S_n-submodules of $\Gamma_n(M_{1,1})$ generated, respectively by $\varphi_p^{(s)}(x_1,\ldots,x_p)$ and $\varphi_{p,q}^{(s)}(x_1,\ldots,x_p)$, given by:

$$\varphi_p^{(s)} = \begin{cases} \sum_{\sigma \in S_p} \pm [x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(p)}, x_1^{(s)}], & p \equiv 1 \ (2) \\ \sum_{\sigma \in S_p} \pm [x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(p-1)}, x_{\sigma(p)}, x_1^{(s)}], & p \equiv 0 \ (2) \end{cases}$$

$$\varphi_{p,q}^{(s)} = \begin{cases} \sum_{\tau \in S_q} \pm [x_{\tau(1)}, x_{\tau(2)}] \cdots [x_{\tau(q-1)}, x_{\tau(q)}] \varphi_p^{(s)}, & q \equiv 0 \ (2) \\ \sum_{\sigma \in S_p} \pm [x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(p-1)}, x_{\sigma(p)}] \varphi_q^{(s)}, & q \equiv 1, \ p \equiv 0 \ (2) \\ \sum_{\substack{\tau \in S_q \\ \sigma \in S_p}} \pm [x_{\tau(1)}, x_{\tau(2)}] \cdots [x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(p)}, x_1^{(s)}, x_{\tau(q)}], & q \equiv p \equiv 1 \ (2) \end{cases}$$

Now, we observe that for any s:

- $\varphi_p^{(s)}$ is a consequence of $[x_1, x_2][x_3, x_4][x_5, x_6]$, if $p \geq 5$;
- $\varphi_{p,q}^{(s)}$ is a consequence of $[x_1, x_2][x_3, x_4][x_5, x_6]$, if $p + q \geq 5$;
- $\varphi_4^{(s)}$ is a consequence of $\varphi_4^{(0)}$;
- $\varphi_4^{(0)} = 2s_4$.

But we have:

Theorem

The polynomial s_4 is an identity for $F[C_1, C_2]$

Since $F[C_1, C_2]$ satisfies all identities of $M_{1,1}$, we obtain that $\Gamma_n(F[C_1, C_2])$ is a factor module of $\Gamma_n(M_{1,1})$. Then, we remove from the decomposition of $\Gamma_n(M_{1,1})$ the submodules generated by identities of $F[C_1, C_2]$, and we obtain:

$$\Gamma_n(F[C_1, C_2]) = M_2^{(n-2)} \oplus M_3^{(n-3)} \oplus M_{2,2}^{(n-4)}$$

and we have

Lemma

For every n, the polynomials $\varphi_2^{(n-2)}$, $\varphi_3^{(n-3)}$ and $\varphi_{2,2}^{(n-4)}$ are not polynomial identities for $F[C_1, C_2]$.

Finally,

Corollary

The T-ideal of polynomial identities of $F_2(M_{1,1})$ is generated by the identities s_4, $[[x_1, x_2][x_3, x_4], x_5]$,and $[x_1, x_2][x_3, x_4][x_5, x_6]$.

Thank you!