Identities of matrix-like algebras

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Polynomial Identities in Algebras. II
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Outline

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1. The fundamental example: The algebra $M_2(K)$
   - Characteristic zero case
   - Weak identities
   - The identities of sl_2
   - The identities of M_2
2. The identities of M_2 when char$K = p \neq 2$
   - Back to M_2
3. Other applications of the method
   - Identities with involution
   - More (on) weak identities
4. Graded identities for sl_2
   - Graded Lie identities
5. Graded identities for Jordan algebras
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As dim $M_2 = 4$ one has the standard identity

$$s_4 = \sum (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0.$$

Technical matters

It can be shown that modulo s_4, the polynomial h_5 is equivalent, as a PI, to

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More precisely ($w = [x_1, x_2]$)

$$\begin{aligned}(u \circ v)w &= (1/8)([x_1, u, v, x_2] + [x_1, v, u, x_2] \\ &\quad -[x_2, u, x_1, v] - [x_2, v, x_1, u]).\end{aligned}$$

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The latter is not so technical...

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Weak (Lie) identities

Let L be a Lie algebra and A an associative envelope of A. We denote $K(X)$ the free associative algebra and by $L(X)$ the free Lie algebra, $L(X) \subseteq K(X)^{-}$.

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Definition

1. A polynomial $f \in K(X)$ is a weak identity for the pair (A, L) if $f = 0$ when evaluated on L.
2. $T(A, L) = \{f \in K(X) \mid f \text{ is a weak identity for } (A, L)\}$.

Clearly $T(A, L)$ is an ideal; it is closed under Lie substitutions.

Weak identities (cont'd)

One may consider weak identities in a more general setting. Suppose A is an algebra and V a vector subspace of A such that V generates A as an algebra.

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The polynomial $f(x_1, \ldots, x_n)$ is a weak identity for the pair (A, V) if $f(v_1, \ldots, v_n) = 0$ for all $v_i \in V$.

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Clearly the set $T(A, V)$ of all weak identities for (A, V) is an ideal. It is closed under linear substitutions of the variables. Depending on the properties of A and of V one defines rules for taking consequences and thus different types of weak identities.

Weak identities (cont'd)

Assume $P \subseteq K(X)$ is a (nonempty) set of polynomials such that $p(v_1, \ldots, v_n) \in V$ for every $p \in P$ and $v_i \in V$.

Definition

The polynomial g is a P-consequence of $f \in T(A, V)$ if g lies in the ideal generated by all $f(p_1, \ldots, p_n)$, $p_i \in P$.

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Depending on the set P (and on the properties of A and V) we obtain different types of weak identities.

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- $P = SJ(X)$, the free special Jordan algebra ($SJ(X) \subseteq K(X)$). If $V \circ V \subseteq V$ we have the weak Jordan identities.
- $P = sp(X)$ If (A, V) is a pair "an associative algebra — a vector space" then the ideal $T(A, V)$ is P–closed. In this way we obtain the weakest PI's.

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That is why the latter (weakest) PI's are called sometimes GL-identities.
One defines variety of pairs in the usual way.
Assuming K infinite then $T(A, V)$ is generated by its multihomogeneous elements; if $\operatorname{char} K = 0$ then it is generated by its multilinear elements.

Back to M_2 and sl_2

Fix $V = sl_2$, $A = M_2$, $\mathrm{char}\, K = 0$.

Theorem (Razmyslov)

The weak Lie identities for (A, V) follow from

$$[x \circ y, z].$$

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One shows that for each $f(x_1, \ldots, x_n) \in K(X)$

$$f(x_1, \ldots, x_n) - f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in I, \sigma \in S_n.$$

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The identities of sl_2

These are much trickier than the weak ones.

Theorem (Razmyslov)

Let char $K = 0$. The identities of sl_2 admit a finite basis.

Recall that Razmyslov proved the identities of sl_2 follow from those of small degree.

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Let $\mathrm{char}\, K = 0$. The identities of sl_2 admit a finite basis.

Recall that Razmyslov proved the identities of sl_2 follow from those of small degree.
Later on an analysis on the identities of degree up to 6 proved that:

The identities of sl_2

Corollary (Drensky)

The polynomials

$$\begin{array}{rcll} f & = & \sum(-1)^{\sigma}[[x_{\sigma(1)}, x_1, x_1], [x_{\sigma(2)}, x_{\sigma(3)}]] & (3,1,1) \\ g & = & \sum(-1)\sigma[x_1, x_{\sigma(1)}, x_{\sigma(2)}], [x_{\sigma(3)}, x_{\sigma(4)}]], & (2,1,1,1) \end{array}$$

form a basis of the identities of sl_2.

The linearizations of these polynomials generate irreducible S_5-modules hence the above system is minimal.

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Corollary (Filippov)

A basis of the identities of sl_2 is given by

$$v_5 = [x_2, x_3, [x_4, x_1], x_1] + [x_2, x_1, [x_3, x_1], x_4].$$

From sl_2 to M_2

One studies proper polynomials only; that is linear combinations of products of commutators = Lie elements.

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One studies proper polynomials only; that is linear combinations of products of commutators = Lie elements. Denote by I the ideal of identities in $K(X)$ generated by

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\begin{aligned}
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s_4 &= \sum (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} \\
r_6 &= (u \circ v)w - (1/8)([x_1, u, v, x_2] + [x_1, v, u, x_2] \\
&\qquad - [x_2, u, x_1, v] - [x_2, v, x_1, u])
\end{aligned}
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where u, v, w are commutators, $w = [x_1, x_2]$.

Proper elements

Let $f(x_1, \ldots, x_n)$ be a proper multilinear polynomial.

- Step one: f can be written, modulo I, as

$$f = \ell + p.$$

Here ℓ is a Lie element;
p is a linear combination of

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  where p_i are commutators.
- Step two: f is an identity for M_2 if and only if both p and ℓ are identities for M_2.

Proper elements (cont'd)

Let $p = \sum a_i(c_i \circ [d_i, x_1])$ be multilinear.
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The polynomial p is an identity for M_2 if and only if $\sum a_i[c_i, d_i]$ is an identity for sl_2 (and for M_2 as well).

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Now add to the ideal I the basis of the identities of sl_2.

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The identities of M_2

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is an identity for M_2.
Add w_6 to the T-ideal I. Thus

$$I = \langle s_4, h_5, r_6, v_5, w_6 \rangle^T$$

The basis for $T(M_2)$

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Corollary (Drensky)

The basis of the identities of M_2 in characteristic 0 consists of

$$s_4, \qquad h_5.$$

The scheme of proof

From now on assume K is infinite field, $\operatorname{char} K \neq 2$.
The main steps in the proof are the following.

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The main steps in the proof are the following.

1. Describe a basis of the identities of sl_2.
2. Describe a basis of the weak identities of (M_2, sl_2).
3. Transfer all this to a basis of the identities of M_2.

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2. You have to deal with multihomogeneous polynomials instead.
3. You cannot use the representations of S_n (neither those of GL_m).

The specifics of M_2

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$$u \circ v = \lambda I, \quad u, v \in sl_2, \lambda \in K.$$

One may consider the orthogonal group of such a form. Its invariants are well known over infinite fields.

The invariants of O_n

Let $x_i = (x_{i1}, \dots, x_{in})$ be "vectors" whose coordinates x_{ij} are commuting variables, and define the symmetric, nondegenerate bilinear form

$$x_i \circ x_j = x_{i1}x_{j1} + \cdots + x_{in}x_{jn}$$

on the span of these vectors.

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The algebra of O_n-invariants in $K[x_{ij}]$ is generated by all $x_i \circ x_j$:

$$K[x_{ij}]^{O_n} = K[x_i \circ x_j].$$

Invariants

Define the double tableau

$$T = \left(\begin{array}{cccc|cccc} p_{11} & p_{12} & \cdots & p_{1m_1} & q_{11} & q_{12} & \cdots & q_{1m_1} \\ p_{21} & p_{22} & \cdots & p_{2m_2} & q_{21} & q_{22} & \cdots & q_{2m_2} \\ \hdashline p_{k1} & p_{k2} & \cdots & p_{km_k} & q_{k1} & q_{k2} & \cdots & q_{km_k} \end{array}\right).$$

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increase strictly along the rows;
increase (with possible repetitions) along the columns.

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For $T = (p_1 p_2 \dots p_m \mid q_1 q_2 \dots q_m)$ define the polynomial

$$\tilde{\varphi}(T) = \sum (-1)^{\sigma} (x_{p_1} \circ x_{q_{\sigma(1)}})(x_{p_2} \circ x_{q_{\sigma(2)}}) \dots (x_{p_m} \circ x_{q_{\sigma(m)}})$$

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One has $\tilde{\varphi}(T) = \det((x_{p_i} \circ x_{q_j}))$.

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One has $\tilde{\varphi}(T) = \det((x_{p_i} \circ x_{q_j}))$.
If $T^1, \dots, T^k$ are the rows of T then

$$\tilde{\varphi}(T) = \tilde{\varphi}(T^1) \cdots \tilde{\varphi}(T^k)$$

Theorem (De Concini, Procesi)

The algebra $K[x_{ij}]^{O_n}$ admits a basis indexed by all doubly standard tableaux with $m_1 \leq n$.

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Corollary

The polynomials $\tilde{\varphi}(T)$ form a basis for $K[x_{ij}]^{O_n}$.

Invariants for SO_n

Let

$$\langle i_1, i_2, \ldots, i_n \rangle = \det(x_{i_1}, \ldots, x_{i_n})$$

be the determinant formed by the coordinates of the vectors above.

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The invariants of SO_n are generated by those of O_n plus the above determinant.

Formally one puts an initial row on top of T whose first half is void.
One usually assigns values $(-n, -(n-1), \dots, -1)$ to this first half. Then a basis of $K[x_{ij}]^{SO_n}$ is given by all standard tableaux T.

Straightening rules

Every double tableaux is a linear combination of standard ones.

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Such rules were described in the general case by De Concini and Procesi.

Outline

The scheme of proof

Recall that K is an infinite field, $\operatorname{char} K \neq 2$.
The main steps in the proof are the following.

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The scheme of proof

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The main steps in the proof are the following.

1. Describe a basis of the identities of sl_2.
2. Describe a basis of the weak identities of (M_2, sl_2).
3. Transfer all this to a basis of the identities of M_2.

The identities of sl_2

A basis of the identities of sl_2 was described by Vasilovsky.

Theorem (Vasilovsky)

Let K be infinite, $\mathrm{char} K \neq 2$. The ideal of identities of sl_2 is generated by the single identity

$$v_5 = [x_2, x_3, [x_4, x_1], x_1] + [x_2, x_1, [x_3, x_1], x_4].$$

The weak identities of (M_2, sl_2)

Theorem (PK)

Let K be infinite, char$K \neq 2$. The ideal of the weak Lie identities for (M_2, sl_2) is generated by $[x^2, y]$.

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Theorem (Drensky, PK)

The GL-ideal of (M_2, sl_2) is generated by $[x^2, y]$ and by s_4.

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The result in the latter paper was much more general.
It was transferred to positive characteristic by PK as well.

The identities of M_2

Theorem (PK)

Let K be infinite, char$K \neq 2$. Then $T(M_2)$ is generated by

$$
\begin{aligned}
h_5 &= [[x_1, x_2] \circ [x_3, x_4], x_5] \\
s_4 &= \sum (-1)^{\sigma} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} \\
r_6 &= (u \circ v)w - (1/8)([x_1, u, v, x_2] + [x_1, v, u, x_2] \\
&\qquad - [x_2, u, x_1, v] - [x_2, v, x_1, u])
\end{aligned}
$$

and by w_6.

In r_6 in the above theorem $w = [x_1, x_2]$.

The identities of M_2

Corollary (PK)

If char$K > 5$ then the basis consists of s_4 and h_5.
If char$K = 3$ then s_4, h_5 and r_6 are independent.

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If char$K = 3$ then s_4, h_5 and r_6 are independent.

Corollary (Colombo, PK)

If char$K = 3$ then s_4, h_5 and r_6 form a basis of $T(M_2)$.
If char$K = 5$ then s_4 and h_5 form a basis of $T(M_2)$.

The recipe

The identity

$$
\begin{aligned}
[x_1, x_2](u \circ v) \quad - \quad & (1/8)([x_1, u, v, x_2] + [x_1, v, u, x_2] \\
- \quad & [x_2, u, x_1, v] - [x_2, v, x_1, u])
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holds for M_2.

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The identity

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\begin{aligned}
[x_1, x_2](u \circ v) &- (1/8)([x_1, u, v, x_2] + [x_1, v, u, x_2] \\
&- [x_2, u, x_1, v] - [x_2, v, x_1, u])
\end{aligned}
$$

holds for M_2.
Define a transformation $L(u, v)$ on M_2 (or sl_2) as follows

$$
\begin{aligned}
[x_1, x_2]L(u, v) &= (1/8)([x_1, u, v, x_2] + [x_1, v, u, x_2] \\
&\quad + [x_2, u, x_1, v] + [x_2, v, x_1, u])
\end{aligned}
$$

for $u, v \in sl_2$.

The recipe

Thus the following is a weak identity for M_2

$$[x_1, x_2]L(u, v) = [x_1, x_2] \circ (u \circ v)$$

u, $v \in sl_2$.

The recipe

Thus the following is a weak identity for M_2

$$[x_1, x_2]L(u, v) = [x_1, x_2] \circ (u \circ v)$$

$u, v \in sl_2$.
This was first observed by Iltyakov for nonassociative algebras (the expressions involved are more complicated).

The recipe, sl_2

One considers separately polynomials of even and of odd degree.

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- When $\deg f$ is even you write it as

$$\sum a_i[x_i, x_j]P(L)$$

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Here $P(L)$ is a polynomial in the operators $L(u, v)$.

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Here $P(L)$ is a polynomial in the operators $L(u, v)$.
In this way you split the relatively free algebra. In a sense you may consider it as a module over the (commutative) algebra of operators generated by the $L(u, v)$.

The recipe, sl_2

Afterwards one associates to each element as before, a double tableau.
Using the straightening rules one gets linear combinations of standard tableaux.

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Afterwards one associates to each element as before, a double tableau.
Using the straightening rules one gets linear combinations of standard tableaux.
As the straightening rules are implied by identities for sl_2 one gets a (finite) basis of its identities.

The difficulties, sl_2

Here we list some of the critical points (the case of sl_2).

1. The transformations L are well defined.
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1. The transformations L are well defined.
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3. The straightening rules follow from known identities of sl_2.
4. Reduce the identities to one of them.

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Sort of similar to that of sl_2. One deals with proper polynomials only.

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1. Work modulo s_4, h_5, v_5.
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Sort of similar to that of sl_2. One deals with proper polynomials only.

1. Work modulo s_4, h_5, v_5.
2. Consider only proper non-Lie polynomials.
3. If f is proper then it is a combination of $g_1 \circ g_2$, both commutators.
4. Define $(g_1 \circ g_2)L(u, v) = g_1 \circ (g_2 L(u, v))$.
5. Follow the steps from above.

The difficulties, M_2

Here the main problems are as follows.

1. $L(u, v)$ are well defined (quite technical but as often happens lengthy).
2. Separating even and odd degree proper elements gives considerably more work. One considers

$$[x_a, x_b] \circ [x_c, x_d]; \quad [x_a, x_b, x_c] \circ [x_d, x_e]$$

with suitable orders for the indices.

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with suitable orders for the indices.

3. In odd degree one alternates on a, b, d. And all is acted on by some $P(L)$.
4. The straightening procedure yields a long list of identities.

Outline

Involutions on M_2

By extending the field K one may assume it algebraically closed.
Moreover one considers only involutions of first kind.

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By extending the field K one may assume it algebraically closed.
Moreover one considers only involutions of first kind.
Hence only two involutions to consider.

1. The transpose t: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$
2. The symplectic s: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Identities in characteristic 0

We denote by x symmetric and by y skew-symmetric variables.
$(x \leftrightarrow (a + a^*)/2, \quad y \leftrightarrow (a - a^*)/2)$

Theorem (Levchenko)

Let char$K = 0$.
(a) A basis of the identities for (M_2, t) *consists of*

$$[y_1 y_2, x], \qquad [y_1, y_2],$$
$$[x_1, x_2][x_3, x_4] - [x_1, x_3][x_2, x_4] + [x_1, x_4][x_2, x_3],$$
$$[y_1 x_1 y_2, x_2] - y_1 y_2 [x_1, x_2].$$

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$$[y_1 x_1 y_2, x_2] - y_1 y_2 [x_1, x_2].$$

(b) A basis for the identities for (M_2, s) consists of

$$[x_1, x_2], \qquad [x, y].$$

Characteristic $p \neq 2$

Theorem (Colombo, PK)

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Consider first s, the symplectic involution.

1. The symmetric elements are just scalar matrices.
2. The PI's depend only on skew variables (that is sl_2).
3. Hence s-identities are weak identities for (M_2, sl_2).

The transpose involution

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Lemma

Every identity is equivalent to one in symmetric variables only.

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Every identity is equivalent to one in symmetric variables only.

1. Define an analog to the transformation $L(u, v)$ (for symmetric variables only).
2. Consider proper elements and split them into three as follows:
3. Products of two skew commutators; skew commutators, symmetric commutators.
4. Follow the recipe for the ordinary PI's (not much easier).

Outline

GL-identities

Let V be a vector space with a nondegenerate symmetric bilinear form, and denote by C its Clifford algebra.

Theorem

The GL-identities for (C, V) are generated by $[x^2, y]$ if $\dim V = \infty$
If $\dim V = n$ *one adds an analog of the standard polynomial,* w_{n+1}. *It is skew symmetric and multilinear,* $\deg w_{n+1} = n + 1$.

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The polynomials w_n are as follows: $w_1 = x_1$, $w_2 = [x_1, x_2]$,

$$w_{n+1} = [w_n, x_{n+1}] \text{ or } w_{n+1} = w_n \circ x_{n+1}$$

depending on whether n is odd or even.

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The theorem was first proved by Drensky and PK in characteristic 0.
The proof relies on the representations of the symmetric and general linear groups.

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GL-identities

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The proof relies on the representations of the symmetric and general linear groups.
Later on it was extended to any characteristic $p \neq 2$. It is a sort of simplified model for applying the invariants of the orthogonal group.
It is necessary to split the relatively free algebra in various parts in order to consider it as a "good" module over the algebra of operators $L(u, v)$.

Outline

1. The fundamental example: The algebra $M_2(K)$
   - Characteristic zero case
   - Weak identities
   - The identities of sl_2
   - The identities of M_2
2. The identities of M_2 when char$K = p \neq 2$
   - Back to M_2
3. Other applications of the method
   - Identities with involution
   - More (on) weak identities
4. Graded identities for sl_2
   - Graded Lie identities
5. Graded identities for Jordan algebras
   - Graded Jordan identities

Gradings on sl_2

Assume char$K = 0$ and K algebraically closed. The following fact is well known.

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Let sl_2 be G-graded where G is a group. Then the grading is equivalent to one of the following.

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2. $G = Z$, $sl_2 = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ -0 & -a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$;

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3. $G = C_2 \times C_2$,
   $(0,0) \to 0$, $(1,1) \to K(e_{11} - e_{22})$,
   $(0,1) \to K(e_{12} + e_{21})$, $(1,0) \to K(e_{12} - e_{21})$.

The grading by C_2

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Let $f(y_1, \ldots, y_k, z_1, \ldots, z_{n-k})$ be multilinear, depending on k even (degree 0) variables y and
$n - k$ odd (degree 1) variables z.
Then $P_{k,n-k}$ is an $S_k \times S_{n-k}$-module.

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Then $P_{k,n-k}$ is an $S_k \times S_{n-k}$-module.
Denote $Q_{k,n-k}$ the multilinear part of the relatively free algebra of sl_2 with the 2-grading, and $\chi_{k,n-k}$ its character. Then

$$\chi_{k,n-k} = \sum m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu)$$

Here $\chi_\lambda \otimes \chi_\mu$ is the irreducible $S_{k,n-k}$ character for the pair of partitions (λ, μ).

The grading by C_2

Lemma (Repin)

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Theorem (Repin)

Let $G = C_2$, and let $\lambda = (k)$, $\mu = (q + r, q)$. Then $m_{\lambda,\mu} = 1$ if and only if

$$n \neq k, \quad n \neq r, \quad r \equiv_2 1 \quad or \quad k + q \equiv_2 1.$$

In all remaining cases $m_{\lambda,\mu} = 0$.

The grading by $C_2 \times C_2$

Analogously for the $C_2 \times C_2$-grading one has four sets of variables, and $S_p \times S_q \times S_r \times S_t$-module and character, $n = p + q + r + t$.

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Theorem (Repin)

$m_{\lambda,\mu,\nu,\pi} = 1$ if and only if

$$\lambda = (p), \quad \mu = (q), \quad \nu = (r), \quad \pi = (t),$$

and $p = 0$, $q \neq n$, $r \neq n$, $t \neq n$, and

$$q - r \equiv_2 1, \quad or \quad r - t \equiv_2 1$$

In all remaining cases $m_{\lambda,\mu,\nu,\pi} = 0$.

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Hence one considers $S_p \times S_q \times S_r$-modules and characters.

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Theorem (Repin)

Let $n = p + q + r > 1$. Then $m_{\lambda,\mu,\nu} = 1$ if and only if

$$\lambda = (p), \quad \mu = (q), \quad \nu = (r),$$

and $p \neq n$, $q \neq n$, $r \neq n$, and $p = r$ or $p = r \pm 1$.
In the remaining cases $m_{\lambda,\mu,\nu} = 0$.

Basis of the graded identities

We consider the grading by C_2, assuming K infinite, char$K \neq 2$.
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The graded identities for sl_2 with the C_2-grading follow from

$$[y_1, y_2].$$

Recall that y are even, and z are odd variables.

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The graded identities for sl_2 with the C_2-grading follow from

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Recall that y are even, and z are odd variables.
We shall give a (very brief) sketch of the proof.

Some graded identities

Denote $I = \langle [y_1, y_2] \rangle$.

Lemma

The following polynomials lie in I.

1. $[z_1, y_1, \ldots, y_t, z_2] - (-1)^{t-1}[z_2, y_1, \ldots, y_t, z_1]$.
2. $[y, z_1, z_2, z_3, z_4] - [y, z_3, z_4, z_1, z_2]$.

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2. $[y, z_1, z_2, z_3, z_4] - [y, z_3, z_4, z_1, z_2]$.
3. $[z_1, z_2, z_3, y_1, y_2] - [z_1, y_1, y_2, z_2, z_3]$.
4. $[y_1, z_1, z_2, z_3, y_2] = [y_2, z_1, z_2, z_3, y_1]$.

The operators $L(a, b)$

Now one defines the transformation $L(a, b)$ as usual. If w_1, w_2, a, b are elements of the Lie algebra $L(Y, Z)/I$ then $[w_1, w_2]L(a, b)$ equals

$$(1/8)([w_1, a, b, w_2] + [w_1, b, a, w_2] - [w_2, a, w_1, b] - [w_2, b, w_1, a]).$$

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Lemma

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Now one applies the invariants of the orthogonal group as done several times earlier, and obtains the theorem.

The remaining gradings

These follow from the description of the 2-graded identities.

Theorem

1. *If $G = Z$ then the graded identities follow from*

$$[x_1, x_2], \deg x_i = 0, \qquad y = 0, |\deg y| > 1.$$

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Theorem

1. *If $G = Z$ then the graded identities follow from*

$$[x_1, x_2], \deg x_i = 0, \qquad y = 0, |\deg y| > 1.$$

2. *If $G = C_2 \times C_2$ then the graded identities follow from*

$$t = 0, \qquad \deg t = (0, 0).$$

Outline

1. The fundamental example: The algebra $M_2(K)$
   - Characteristic zero case
   - Weak identities
   - The identities of sl_2
   - The identities of M_2
2. The identities of M_2 when char$K = p \neq 2$
   - Back to M_2
3. Other applications of the method
   - Identities with involution
   - More (on) weak identities
4. Graded identities for sl_2
   - Graded Lie identities
5. Graded identities for Jordan algebras
   - Graded Jordan identities

The algebra J

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Definition

Let V be a vector space with a symmetric bilinear form f, $\dim V = \infty$. Define on $B = K \oplus V$ a product

$$(\alpha + u) \circ (\beta + v) = (\alpha\beta + f(u, v)) + (\alpha u + \beta v),$$

$\alpha, \beta \in K$, $u, v \in V$. When $\dim V = n$ denote it V_n, resp. B_n.

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The Jordan algebras B, B_n are simple (as long as f is nondegenerate).
They are special ($B \subseteq C$ where C is the Clifford algebra of V).

Gradings on B and on B_n

The gradings on B and B_n by a group G are known.

Theorem (Bahturin, Shestakov)

Let $B = \oplus B_g$ be G-graded.
Then there exists a homogeneous basis T of V such that

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Let $B = \oplus B_g$ be G-graded.
Then there exists a homogeneous basis T of V such that $T = E \cup E' \cup F$, a disjoint union where

1. *There is a 1–1 correspondence $E \leftrightarrow E'$ with $e \leftrightarrow e'$ and $|e| = |e'|^{-1} \neq e \in G$; $|f|^2 = e$ for all $e \in E$, $f \in F$.*
2. *F is orthonormal and $F \perp E$, $F \perp E'$.*

The converse also holds.
Analogously for B_n.

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Recall V can have several nonequivalent bilinear forms that define nonisomorphic Jordan algebras J.
Fortunately their (graded) identities are the same.

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Then

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Hence if $V_2 = sp(a, b)$ then $J = K \oplus V_2$ is a Jordan algebra of a symmetric bilinear form on V_2.

Gradings on B by C_2

Now fix $G = C_2$, the cyclic group of order 2. It follows

Theorem

Let $B = K \oplus V$ be a Jordan algebra of a bilinear form. Every G-grading on B is defined by splitting V into a direct sum of two orthogonal subspaces.
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The above theorem is a particular case of the above cited classification of the gradings on B and B_n given by Bahturin and Shestakov.
On the other hand it can be obtained directly (and quite easily).

Gradings on J

Corollary

Up to a graded isomorphism there are two nontrivial gradings on $J = J_0 \oplus J_1$.

- *the nonscalar grading*

$$J_0 = sp(1, a), \qquad J_1 = sp(b).$$

- *the scalar grading*

$$J_0 = K, \qquad J_1 = sp(a, b).$$

The nonscalar grading

We denote

$$(x, y, z) = (xy)z - x(yz)$$

the associator of x, y, z in a nonassociative algebra.

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Recall we use y and z for even, respectively odd variables.
Letters x denote any of these.

The nonscalar grading

Denote by I the ideal of graded identities generated by the polynomials

$$
\begin{aligned}
& x_1(x_2x_3) - x_2(x_1x_3), \qquad |x_1| = |x_2| \\
& (y_1y_2, z_1, z_2) - (y_1(y_2, z_1, z_2) + y_2(y_1, z_1, z_2) - 2z_1(z_2, y_1, y_2)) \\
& (y_1y_2, y_3, z_1) - (y_1(y_2, y_3, z_1) + y_2(y_1, y_3, z_1)) \\
& (z_1z_2, x_1, x_2) \\
& (y_1, y_2, z_1, x, y_3) - (y_1, y_3, z_1, x, y_2)
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Graded identities

Let T be the ideal of graded identities of J (with the nonscalar grading).

Lemma

$I \subseteq T$.

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Lemma

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Hence we can work modulo I.
Denote $L = J(X)/I$, L is a graded Jordan algebra.

Lemma

Let $L = L_0 \oplus L_1$. Then

1. *The subalgebra L_0 is associative.*
2. *The subalgebra of L generated by L_1 is associative.*

A technical lemma

We give several (quite a lot) consequences of the identities in I.

Lemma

The following polynomials lie in I.

(a) (x_1, x_2, x_3), $|x_1| = |x_3|$;

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(c) $(z_1, y_1, \ldots, y_{2k}) - (z_1, y_{\sigma(1)}, \ldots, y_{\sigma(2k)})$, $\sigma \in S_{2k}$;

(d) $(z_1, y_1, \ldots, y_{2k}, z_2, y_{2k+1}) - (z_{\tau(1)}, y_{\sigma(1)}, \ldots, y_{\sigma(2k)}, z_{\tau(2)}, y_{\sigma(2k+1)})$, $\sigma \in S_{2k+1}$, $\tau \in S_2$.

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The following polynomials lie in I.

(i) $(y_1, z_2, (y_2 z_1)) - (y_2(y_1, z_1, z_2) + z_1(y_1, y_2, z_2))$;

(ii) $z_1(z_2, z_3, y_1)$;

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(iii) $(z_1 z_2)(z_3, x, y_1) - (z_1, z_2, y_1, x, z_3)$;

(iv) $(y_1, z_1, z_2)(y_2, z_3, z_4) - z_1(y_1, z_2, z_3, y_2, z_4)$;

(v) $(y_1, y_2, z_1)(y_3, y_4, z_2) - z_1(z_2, y_1, y_2, y_3, y_4)$.

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(iv) $(y_1, z_1, z_2)(y_2, z_3, z_4) - z_1(y_1, z_2, z_3, y_2, z_4)$;

(v) $(y_1, y_2, z_1)(y_3, y_4, z_2) - z_1(z_2, y_1, y_2, y_3, y_4)$.

Corollary

If u_1 and u_2 are two nonzero associators of the same multidegree in L. Then $u_1 = \pm u_2$.

The basis of graded identities

Here it should go a handful of slides containing technical details and computations. These occupy some 5 pages of computations.

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Here it should go a handful of slides containing technical details and computations. These occupy some 5 pages of computations.
Instead we omit these altogether and state the main result.

Theorem (D. Silva, PK)

The ideal of graded identities of J with the nonscalar grading coincides with I.

The scalar grading

The scalar grading: $J = K \oplus V_2$.
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We describe the graded identities of the Jordan algebras B and B_n of a nondegenerate symmetric bilinear form on the vector spaces V and V_n, respectively.
We start with $B = K \oplus V$.
Here $B^{(0)} = K$, and $B^{(1)} = V$ (using upper indices).

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Lemma

The associator

$$(y, x_1, x_2)$$

is a graded identity for B.

A reduction

Denote by I the ideal of graded identities generated by this associator, and set $L = J(X)/I$.

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Let $f(y_1, \ldots, y_p, z_1, \ldots, z_q)$ be multihomogeneous.

Lemma

In L one has

$$f(y_1, \ldots, y_p, z_1, \ldots, z_q) = y_1^{n_1} \ldots y_p^{n_p} g(z_1, \ldots, z_q)$$

where g depends on the variables z only.

Weak identities and graded identities

Recall what a weak Jordan identity is.

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Let f be as above. Then the following are equivalent.

1. *f is a graded identity for B.*

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Corollary

Let f be as above. Then the following are equivalent.

1. *f is a graded identity for B.*
2. *g is a graded identity for B.*
3. *g is a weak Jordan identity for the pair (B, V).*

Another reduction

Let M be the subalgebra of $L = J(X)/I$ generated by all variables Z.
Then $M = M^{(0)} \oplus M^{(1)}$ is C_2-graded (induced by the grading on L).

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Lemma

1. *The subalgebra $M^{(0)}$ is spanned by all products $(z_{i_1} z_{j_1}) \dots (z_{i_k} z_{j_k})$.*
2. *The vector space $M^{(1)}$ is spanned by all $z_{i_0}(z_{i_1} z_{j_1}) \dots (z_{i_k} z_{j_k})$.*

Weak identities

Recall B is a special Jordan algebra and the Clifford algebra C is its associative envelope.

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The weak (associative) identities for (C, V) and for (C_n, V_n) were described as follows.

- If char$K = 0$, by Drensky and PK.
- If K is infinite of characteristic $p \neq 2$, by PK.

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- If K is infinite of characteristic $p \neq 2$, by PK.

The latter result relied heavily on the invariants of the orthogonal group O_n.

Basis of the weak Jordan identities

We use again Invariant theory, and obtain the following theorem.

Theorem (D. Silva, PK)

1. *The weak Jordan identities for the pair (B, V) are consequences of the polynomial $(x_1 x_2, x_3, x_4)$.*

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Theorem (D. Silva, PK)

1. *The weak Jordan identities for the pair* (B, V) *are consequences of the polynomial* $(x_1 x_2, x_3, x_4)$.
2. *The weak Jordan identities for the pair* (B_n, V_n) *follow from* $(x_1 x_2, x_3, x_4)$ *and*

$$f_n = \sum (-1)^\sigma x_{\sigma(1)} (x_{n+2} x_{\sigma(2)}) \dots (x_{2n+1} x_{\sigma(n+1)})$$

The basis of graded identities

Recall that $I = \langle (y, x_1, x_2) \rangle$.

Corollary

1. The ideal of the graded identities for B with the scalar grading coincides with the ideal I.

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Corollary

1. The ideal of the graded identities for B with the scalar grading coincides with the ideal I.
2. The ideal of the graded identities for B_n with the scalar grading is generated by (y, x_1, x_2) and by the identity

$$g_n = \sum (-1)^{\sigma} z_{\sigma(1)} (z_{n+2} z_{\sigma(2)}) \ldots (z_{2n+1} z_{\sigma(n+1)})$$

where $\sigma \in S_{n+1}$.

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where $\sigma \in S_{n+1}$.
3. The graded identities for the Jordan algebra of the symmetric 2×2 matrices (with the scalar grading) follow from (y, x_1, x_2) and

$$\sum (-1)^\sigma z_{\sigma(1)}(z_4 z_{\sigma(2)})(z_5 z_{\sigma(3)}), \quad \sigma \in S_3.$$