Computing of Cocharacter Sequences of PI-algebras – 1

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Let K be a field of characteristic 0. We shall consider associative unital K-algebras R only. Let $K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle$ be the free associative algebra (the algebra of polynomials in noncommuting variables). For a PI-algebra R with T-ideal $T(R) \subset K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle$ let

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda(R)\chi_\lambda, \quad n = 0, 1, 2, \ldots,$$

be the cocharacter sequence of R.

This is the sequence of characters of the S_n-modules

$$P_n(R) = P_n/(P_n \cap T(R)), \quad n = 0, 1, 2, \dots,$$

where P_n is the space of multilinear polynomials of degree n in $K\langle X\rangle$. Here χ_λ is the irreducible character of S_n corresponding to the partition

$$\lambda = (\lambda_1, \dots, \lambda_n) \vdash n, \quad \lambda_1 \geq \dots \geq \lambda_n \geq 0, \quad \lambda_1 + \dots + \lambda_n = n,$$

and the nonnegative integers $m_\lambda(R)$ are the multiplicities of χ_λ in $\chi_n(R)$.

The purpose of the two talks is to present some results on the explicit form of the multiplicities $m_\lambda(R)$ for a given algebra R as well as methods to find them.

One of the possible methods to find $m_\lambda(R)$ is to calculate the Hilbert (or Poincaré) series

$$H(F_d(R), T_d) = H(F_d(R), t_1, \dots, t_d)$$

$$= \sum_{n_i \geq 0} \dim F_d(R)^{(n_1,\dots,n_d)} t_1^{n_1} \cdots t_d^{n_d},$$

where $F_d(R)^{(n_1,\dots,n_d)}$ is the multihomogeneous component of degree $(n_1, \dots, n_d)$ of the relatively free algebra $F_d(R) = K\langle x_1, \dots, x_d\rangle/(T(R) \cap K\langle x_1, \dots, x_d\rangle)$.

This Hilbert series is a symmetric function which decomposes as a series of Schur functions

$$H(F_d(R), T_d) = \sum m_\lambda(R) S_\lambda(T_d)$$

and for $\lambda = (\lambda_1, \ldots, \lambda_d)$ the multiplicities $m_\lambda(R)$ are the same as in the cocharacter sequence of R.

Consider the generating function of the multiplicities

$$M(R, T_d) = M(H(F_d(R)), T_d) = \sum m_\lambda(R) T_d^\lambda$$

$$= \sum m_\lambda(R) t_1^{\lambda_1} \cdots t_d^{\lambda_d}$$

which Drensky and Georgi Genov call the multiplicity series of R.

If we know the multiplicity series and if we can expand it as a series we shall know the multiplicities.

Theorem of Belov

For a PI-algebra R the Hilbert series of $F_d(R)$ is of the form

$$H(F_d(R), T_d) = \frac{p(T_d)}{\prod(1 - T^a)^{b_a}} = \frac{p(T_d)}{\prod(1 - t_1^{a_1} \cdots t_d^{a_d})^{b_a}},$$

where $p(T_d)$ is a polynomial, $a_i \geq 0$, $b_a > 0$.

Berele calls such functions nice rational functions.

Theorem of Berele

For a PI-algebra R the multiplicity series $M(R, T_d)$ is a nice rational function.

Simplification of the problem: Proper polynomial identities

Let B be the subalgebra of proper polynomials in $K\langle X\rangle$. It is generated by products of commutators $[x_{i_1},\ldots,x_{i_k}]$, $k\geq 2$. Let

$$B_d(R)=(B\cap K\langle x_1,\ldots,x_d\rangle/(T(R)\cap B\cap K\langle x_1,\ldots,x_d\rangle)$$

be the subalgebra of proper polynomials in d variables in the relatively free algebra $F_d(R)$.

Theorem

The Hilbert series of $F_d(R)$ and $B_d(R)$ are related by

$$H(F_d(R), T_d) = H(K[X_d], T_d)H(B_d(R), T_d)$$

$$= H(B_d(R), T_d)\prod_{i=1}^{d}\frac{1}{1-t_i} = H(B_d(R), T_d)\sum_{n\geq 0} S_{(n)}(T_d).$$

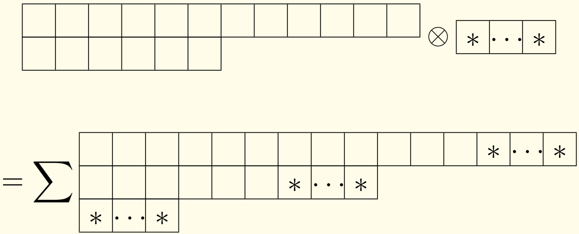
Hence if we know the multiplicities of the Hilbert series of $B_d(R)$ we can find the multiplicities of R by the Young rule.

The Young rule

$$S_\mu(T_d)S_{(n)}(T_d) = \sum S_\lambda(T_d),$$

where the sum runs on all partitions $\lambda \vdash |\mu| + n$ such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_d \geq \mu_d.$$



Regev calls sequences of S_n-characters obtained applying the Young rule to another sequence of S_n-characters in the same way as the cocharacters of a PI-algebra are obtained from the proper cocharacters Young-derived.

Let the Hilbert series and the multiplicity series of $B_d(R)$ be

$$H(B_d(R)) = \sum k_\mu(R) S_\mu(T_d)$$

$$M(H(B_d), T_d) = M(H(B_d), t_1, \ldots, t_d) = \sum k_\mu(R) T_d^\mu,$$

respectively. In the language of multiplicity series the Young rule gives:

Lemma

The multiplicity series of R satisfies

$$M(R,T_d)=\prod_{i=1}^{d}\frac{1}{1-t_i}\sum(-t_2)^{\varepsilon_2}\cdots(-t_d)^{\varepsilon_d}\times$$

$$\times M(H(B_d(R)),t_1t_2^{\varepsilon_2},t_2^{1-\varepsilon_2}t_3^{\varepsilon_3}\dots t_{d-1}^{1-\varepsilon_{d-1}}t_d^{\varepsilon_d},t_d^{1-\varepsilon_d}),$$

where the summation runs on all $\varepsilon_2,\dots,\varepsilon_d=0,1$.

Let $\mathbb{C}[[T_d]]_{\geq}$ be the subspace of the algebra of formal power series $\mathbb{C}[[T_d]]$ consisting of the series

$$\sum_{n_1\geq\cdots\geq n_d} a_n T_d^n = \sum_{n_1\geq\cdots\geq n_d} a_n t_1^{n_1}\cdots t_d^{n_d}.$$

Every element of $\mathbb{C}[[T_d]]_{\geq}$ is a multiplicity series of a symmetric function in $\mathbb{C}[[T_d]]^{S_d}$.

We define an operator Y (“Y” for Young)

$$Y : \mathbb{C}[[T_d]]_{\geq} \to \mathbb{C}[[T_d]]_{\geq}$$

by the following rule. If $g = g(T_d) \in \mathbb{C}[[T_d]]^{S_d}$ is a symmetric function, then

$$Y(M(g), T_d) = M\left(g(T_d)\prod_{i=1}^{d}\frac{1}{1-t_i}\right).$$

In this way

$$M(R, T_d) = Y(M(B_d(R)), T_d).$$

For two variables we obtain: If

$$h = h(T_2) = h(t_1, t_2) \in \mathbb{C}[[t_1, t_2]]_{\geq}$$

is a multiplicity series, then

$$Y(h, T_2) = \frac{h(t_1, t_2) - t_2 h(t_1 t_2, 1)}{(1 - t_1)(1 - t_2)}.$$

In the case of d variables

$$S_{(1^d)}(T_d) = t_1 \cdots t_d,$$

$$S_{(1^d)}(T_d)S_\mu(T_d) = S_\lambda(T_d), \quad \lambda_i = \mu_i + 1, i = 1, \ldots, d.$$

Hence

$$M((t_1 \cdots t_d)f(T_d), T_d) = (t_1 \cdots t_d)M(f(T_d), T_d)$$

and the functions of $t_1 \cdots t_d$ behave like constants.

For two variables and a function $a(t_1t_2) \in \mathbb{C}[[t_1, t_2]]$ we define an operator

$$Y_a(M(h), t_1, t_2) = M\left(h\frac{1}{(1 - at_1)(1 - at_2)}\right),$$

where $h(t_1, t_2) \in \mathbb{C}[[t_1, t_2]]^{S_2}$ is a symmetric function. Then

$$Y_a(h, T_2) = \frac{h(t_1, t_2) - at_2h(at_1t_2, 1)}{(1 - at_1)(1 - at_2)}.$$

This formula works also for symmetric functions of the form

$$h = \sum_{n\geq 0} b_n(t_1t_2)(t_1^n + t_2^n),$$

where $a = a(t_1t_2)$ and $b_n = b_n(t_1t_2)$ belong to the algebraic closure $\overline{\mathbb{C}(t_1t_2)}$ of $\mathbb{C}(t_1t_2)$.

Symmetric nice rational functions in two variables are of the form

$$f(t_1,t_2)=p(t_1,t_2)\prod_{i=1}^{k}\frac{1}{(1-(t_1t_2)^{b_i}t_1^{c_i})(1-(t_1t_2)^{b_i}t_2^{c_i})},$$

$p(t_1,t_2)\in\mathbb{C}[t_1,t_2]^{S_2}$. The symmetric polynomial $p(t_1,t_2)$ can be presented explicitly (and easily) as a linear combination of Schur functions. The factors $(1-(t_1t_2)^{b_i}t_1^{c_i})(1-(t_1t_2)^{b_i}t_2^{c_i})$ of the denominator decompose as products of $(1-\varepsilon^j a_i t_1)(1-\varepsilon^j a_i t_2)$, where $a_i^{c_i k}=(t_1t_2)^{b_i}$ and ε is a primitive c_ith root of 1.

The following corollary solves completely the problem how to compute the multiplicity series of a symmetric nice rational function in two variables.

Corollary

If $f(t_1, t_2)$ is a symmetric nice rational function in two variables with nominator $p(t_1, t_2)$, then its multiplicity series can be presented in the form

$$M(f(t_1, t_2), t_1, t_2) = Y_{a_1} \cdots Y_{a_m}(M(p(t_1, t_2), t_1, t_2)).$$

Applications

Computing of the multiplicities and their asymptotics of the pure and mixed trace algebra generated by two generic 3×3 matrices:

Drensky and Genov for the pure trace algebra (correcting a technical error of Berele - missing summand);

Drensky, Genov and Angela Valenti for the mixed trace algebra.

Computing the Hilbert series of invariants of the group of unitriangular 2×2 matrices (Drensky and Genov).

For symmetric nice rational functions in any number of variables there are algorithms to find the multiplicity series. The methods are based on ideas of Berele combined with classical results of Elliott (1903) developed to solve linear diophantine equations and inequalities in nonnegative integers, generalized later by MacMahon (1916) to Partition Analysis (or Ω-Calculus). Well forgotten, the method of Elliott and MacMahon has his second life in a series of twelve papers on MacMahon's partition analysis by Andrews, alone or jointly with Paule, Riese and Strehl, with further improvements and computer realizations by {Andrews, Paule and Riese}, Han, Xin and {Fu and Lascoux}.

Applications

Computing of the multiplicities and their asymptotics of the pure and mixed trace algebra generated by three generic 3×3 matrices (Francesca Benanti, Silvia Boumova, Drensky).

Computing the Hilbert series of invariants of the group of unitriangular 3×3 matrices (Boumova, Drensky).

Example

The Grassmann algebra E of an infinite dimensional vector space satisfies the identities

$$[x_1, x_2, x_3] = 0, \quad [x_1, x_2][x_2, x_3] = 0$$

and the linearization of the latter identity

$$[x_1, x_2][x_3, x_4] + [x_1, x_3][x_2, x_4] = 0.$$

This implies that $B_d(E)$ is spanned by the products of commutators

$$[x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}], \quad 1 \leq i_1 < i_2 < \cdots < i_{2k-1} < i_{2k} \leq d,$$

and this set is a basis of $B_d(E)$.

Hence the Hilbert series of $B_d(E)$ is

$$H(B_d(E), T_d) = \sum_{k=0}^{\lceil d/2 \rceil} S_{(1^{2k})}(T_d)$$

and the Young rule gives that

$$H(F_d(E), T_d) = 1 + \sum_{m \geq 1} \sum_{k=0}^{d-1} S_{(m,1^k)}(T_d).$$

Therefore the multiplicities in the cocharacter sequence of E are

$$m_\lambda(E) = \begin{cases} 1, & \lambda = (n-k, 1^k), \\ 0, & \text{otherwise.} \end{cases}$$

The proper cocharacter sequence $\xi_n(M_2(K))$, $n = 0, 1, 2, \ldots$, of the 2×2 matrix algebra $M_2(K)$ with entries from K is

$$\xi_n(M_2(K)) = \sum \chi_{(\lambda_1,\lambda_2,\lambda_3)},$$

where the sum runs on all partitions of n different from (n) for $n > 0$ and from (1^3) for $n = 3$. By the Young rule, this sequence is Young-derived from the sequence

$$\zeta_n = \begin{cases} \chi_{(k,k)}, & n = 2k, \\ 0, & n = 2k+1. \end{cases} .$$

If

$$f(T_d) = \sum_{k\geq 0} S_{(k,k)}(T_d),$$

then

$$M(f, T_d) = \sum_{k\geq 0} (t_1 t_2)^k = \frac{1}{1 - t_1 t_2},$$

$$Y(M(f), T_d) = \frac{1}{(1 - t_1)(1 - t_1 t_2)(1 - t_1 t_2 t_3)},$$

$$Y^2(M(f), T_d) = \frac{1}{(1-t_1)^2(1-t_1t_2)^2(1-t_1t_2t_3)^2(1-t_1t_2t_3t_4)}$$

$$= \sum (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1) t_1^{\lambda_1} t_2^{\lambda_2} t_3^{\lambda_3} t_4^{\lambda_4},$$

where the summation runs on all $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Up to the correction

$$Y\left(M(S_{(1^3)}(T_d) + \sum_{n \geq 1} S_{(n)}(T_d) \right) = Y\left(t_1t_2t_3 + \frac{t_1}{1-t_1} \right)$$

this gives the multiplicities $m_\lambda(M_2(K))$.

Applications to algebras with involution

Let $K\langle X \cup X^* \rangle$ be the free associative algebra with involution $*$ which fixes K and is defined by $x_i \to x_i^*$, $x_i^* \to x_i$, $i = 1, 2, \dots$. Let $T(R, *)$ be the ideal of $*$-polynomial identities of the algebra with involution $(R, *)$. It is standard to replace the variables with new symmetric and antisymmetric variables

$$Y = \{y_i = \frac{1}{2}(x_i + x_i^*) \mid i = 1, 2, \dots\},$$

$$Z = \{z_i = \frac{1}{2}(x_i - x_i^*) \mid i = 1, 2, \dots\}.$$

The cocharacter "sequence" of $(R,*)$ is the sequence $\chi_{p,q}(R)$, $p,q=0,1,2,\ldots$, of $S_p\times S_q$-characters, where $S_p\times\langle 1\rangle$ acts on the symmetric variables $y_1,\ldots,y_p$ and $\langle 1\rangle\times S_q$ acts on the skew-symmetric variables $z_1,\ldots,z_q$. It is of the form

$$\chi_{p,q}(R)=\sum_{\lambda\vdash p}\sum_{\mu\vdash q}m_{\lambda,\mu}(R)(\chi_\lambda\otimes\chi_\mu),\quad p,q=0,1,2,\ldots.$$

Let $F_{p,q}(R)$ be the relatively free algebra in the variety of algebras with involutions generated by R. This algebra is generated by p symmetric and q skew-symmetric variables. Its Hilbert series is

$$H(F_{p,q}(R), U_p, V_q) = \sum_{\lambda,\mu} m_{\lambda,\mu}(R) S_\lambda(U_p) S_\mu(V_q)$$

As in the case of ordinary polynomial identities the multiplicities $m_{\lambda,\mu}(R)$ in the Hilbert series and the cocharacter sequence are the same.

We define the multiplicity series

$$M((R,*), U_p, V_q) = \sum_{\lambda,\mu} m_{\lambda,\mu}(R) U_p^\lambda V_q^\mu.$$

If we know it as a function and expand it as a series, we shall find the multiplicities in the cocharacter sequence of $(R,*)$.

Up to $*$-PI-equivalence, finite dimensional simple algebras with involution are

$$(M_n(K), t), \quad (M_{2n}(K), s), \quad (M_n(K) \oplus M_n(K)^{\text{op}}, *), \quad n = 1, 2, \dots,$$

where t is the transpose and s is the symmetric involution. In the third case the involution changes the places of the coordinates:

$$(a, b)^* = (b, a), \quad (a, b) \in M_n(K) \oplus M_n(K)^{\text{op}}.$$

For any PI-algebra R, the $*$-polynomial identities of $(R \oplus R^{\text{op}}, *)$ are

$$T(R \oplus R^{\text{op}}, *) = T(R) \cap T(R^{\text{op}}),$$

where $= T(R)$ and $T(R^{\text{op}})$ are T-ideals of ordinary polynomial identities in $K\langle Y \cup Z \rangle$. Since $M_2(K)^{\text{op}} \cong M_2(K)$, we obtain a basis of the $*$-polynomial identities for $(M_2(K) \oplus M_2(K)^{\text{op}}, *)$. Also, the $*$-polynomial identities are known for $(M_2(K), t)$ and $M_2(K), s)$ (and in the trivial case of 1×1 matrices).

The cocharacter sequences are known for $K = (M_1, t)$ (trivial), $(K \oplus K)$ (easy), $(M_2(K), t)$ and $(M_2(K), s)$.

Cocharacters of $(M_2(K)\oplus M_2(K)^{\rm op},*)$

(Discussion with Antonio Giambruno)
Observation:

$$H_{p,q}=H(F_{p,q}(M_2(K)\oplus M_2(K)^{\rm op},*),U_p,V_q)$$

$$=H(F_{p+q}(M_2(K)),u_1,\ldots,u_p,v_1,\ldots,v_q)$$

$$=\prod_{i=1}^p\frac{1}{(1-u_i)^2}\prod_{j=1}^q\frac{1}{(1-v_j)^2}\sum_{n\geq 0}S_{(n,n)}(U_p,V_q)$$

$$-\prod_{i=1}^p\frac{1}{1-u_i}\prod_{j=1}^q\frac{1}{1-v_j}\left(\sum_{n\geq 1}S_{(n)}(U_p,V_q)+S_{(1^3)}(U_p,V_q)\right).$$

$$\sum_{n\geq 0} S_{(n,n)}(U_p, V_q) = \sum_{a,b,c\geq 0} S_{(a+b,b)}(U_p) S_{(a+c,c)}(V_q)$$

and the corresponding multiplicity series is

$$M\left(\sum_{n\geq 0} S_{(n,n)}(U_p, V_q), U_p, V_q\right) = \sum_{a,b,c\geq 0} u_1^{a+b} u_2^b v_1^{a+c} v_2^c$$

$$= \frac{1}{(1-u_1v_1)(1-u_2)(1-v_2)}.$$

$$\sum_{n\geq 1} S_{(n)}(U_p,V_q) = -1 + \sum_{n\geq 0} S_{(n)}(U_p,V_q) = -1 + \prod_{i=1}^{p}\frac{1}{1-u_i}\prod_{j=1}^{q}\frac{1}{1-v_j},$$

$$S_{(1^3)}(U_p,V_q) = S_{(1^3)}(U_p) + S_{(1^2)}(U_p)S_{(1)}(V_q)$$
$$+S_{(1)}(U_p)S_{(1^2)}(V_q) + S_{(1^3)}(V_q),$$

and the corresponding multiplicity series are

$$-1 + \frac{1}{(1-u_1)(1-v_1)}, \quad u_1u_2u_3 + u_1u_2v_1 + u_1v_1v_2 + v_1v_2v_3.$$

Theorem

Let Y_U and Y_V be the Young operators of $\mathbb{C}[[U_p, V_q]]_{\geq}$ which act on the variables U_p and V_q, respectively. Then the multiplicity series of the $*$-polynomial identities of the algebra $(M_2(K) \oplus M_2(K)^{op}, *)$ is

$$M((M_2(K) \oplus M_2(K)^{op}, *), U_p, V_q)$$

$$= Y_U^2 Y_V^2 \left(\frac{1}{(1 - u_1 v_1)(1 - u_2)(1 - v_2)} - 1 \right)$$

$$+ Y_U Y_V (1 - (u_1 u_2 u_3 + u_1 u_2 v_1 + u_1 v_1 v_2 + v_1 v_2 v_3)).$$

Problem

If we know the polynomial identities, their multiplicities, etc. of two algebras R_1 and R_2, how to obtain information about the polynomial identities of algebras built on R_1 and R_2?

Possible constructions

$T(R_1 \oplus R_2) = T(R_1) \cap T(R_2)$ (difficult);
$T(R_1 \otimes R_2)$ (difficult);

$$R = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix},$$

where M is an R_1-R_2-bimodule. In this case $T(R) \supseteq T(R_1)T(R_2)$. In some special cases we have equality.

Example

(Giambruno and Zaicev, a nontrivial corollary of a result of Lewin)
If

$$R = \begin{pmatrix} M_{p_1}(K) & * & \cdots & * \\ 0 & M_{p_2}(K) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{p_m}(K) \end{pmatrix},$$

then

$$T(R) = T(M_{p_1})T(M_{p_2})\cdots T(M_{p_m}).$$

Theorem of Formanek

If U and V are ideals of $K\langle X_d\rangle = K\langle x_1,\ldots,x_d\rangle$, then the Hilbert series of U, V and UV satisfy

$$H(K\langle X_d\rangle)H(UV) = H(U)H(V)$$

The proof (with minor gaps) is given in a paper by Halpin.

Sketch of the proof

Step 1. *Let U be a right ideal of $K\langle X_d\rangle$ generated by a homogeneous system of polynomials. Then U is a free right $K\langle X_d\rangle$-module.* (This is a special case of the well known theorem that free algebras are FIR (free ideal rings).

Proof. Let $G = \{u_i \mid i \in I\}$ be a minimal homogeneous generating system of U. If U is not free, then

$$\sum_j \left(\sum_{i\in I} \alpha_{ij} u_i\right) x_{j_1} \cdots x_{j_k} = 0, \quad \alpha_{ij} \in K,$$

and all summands are homogeneous of the same degree. Choosing the $x_{j_1} \cdots x_{j_k}$s of minimal length k_0 we obtain

$$\sum_j \left(\sum_{i\in I} \alpha_{ij} u_i + \sum_{i\in I} u_i v_{ij}\right) x_{j_1} \cdots x_{j_{k_0}} = 0$$

for suitable homogeneous polynomials v_{ij} of positive degree. Hence

$$\sum_{i\in I} \alpha_{ij} u_i + \sum_{i\in I} u_i v_{ij} = 0.$$

Hence some u_i belongs to the right ideal generated by the other generators of U.

Step 2. If U, V are ideals of $K\langle X_d\rangle$ we choose a minimal homogeneous system $\{u_i \mid i \in I\}$ of U as a right ideal and a minimal homogeneous system $\{v_j \mid j \in J\}$ of V as a left ideal. Then

$$U = \bigoplus_{i\in I} u_i K\langle X_d\rangle, \quad V = \bigoplus_{j\in J} K\langle X_d\rangle v_j,$$

$$UV = \bigoplus_{\substack{i\in I\\ j\in J}} u_i (K\langle X_d\rangle)^2 v_j = \bigoplus_{\substack{i\in I\\ j\in J}} u_i K\langle X_d\rangle v_j,$$

The Hilbert series of U, V and UV are

$$H(U)=\sum_{i\in I} T_d^{\deg u_i} H(K\langle X_d\rangle),\quad H(V)=\sum_{j\in J} H(K\langle X_d\rangle)\, T_d^{\deg v_j},$$

$$H(UV)=\sum_{\substack{i\in I\\ j\in J}} T_d^{\deg u_i} H(K\langle X_d\rangle)\, T_d^{\deg v_j}$$

$$=\frac{1}{H(K\langle X_d\rangle)} H(U)H(V).$$

Corollary

If R_1, R_2 and R are PI-algebras such that $T(R) = T(R_1)T(R_2)$, then

$$H(F_d(R)) = H(F_d(R_1)) + H(F_d(R_2))$$

$$+(t_1 + \cdots + t_d - 1)H(F_d(R_1))H(F_d(R_2)).$$

Proof. The formula follows by direct calculations using that

$$H(K\langle X_d\rangle) = H(F_d(R)) + H(T(R) \cap K\langle X_d\rangle),$$

similarly for $F_d(R_1)$, $F_d(R_2)$, and

$$H(K\langle X_d\rangle) = \frac{1}{1 - (t_1 + \cdots t_d)}.$$

In the language of Schur functions and their multiplicities the corollary gives

$$\sum m_\lambda(R)S_\lambda(T_d) = \sum m_\lambda(R_1)S_\lambda(T_d) + \sum m_\lambda(R_2)S_\lambda(T_d)$$

$$+(S_{(1)}(T_d) - 1)\sum m_\mu(R_1)m_\nu(R_2)S_\mu(T_d)S_\nu(T_d)$$

and the product $S_\mu(T_d)S_\nu(T_d)$ can be decomposed by the Littlewood-Richardson rule

$$S_\mu(T_d)S_\nu(T_d) = \sum_{\lambda\vdash|\mu|+|\nu|} c_{\mu\nu}^{\lambda} S_\lambda(T_d).$$

The translation in cocharacter sequences is the formula of Berele and Regev

$$\chi_n(R) = \chi_n(R_1) + \chi_n(R_2) + \chi_{(1)} \hat{\otimes} \sum_{i=0}^{n-1} \chi_i(R_1) \hat{\otimes} \chi_{n-1-i}(R_2)$$

$$- \sum_{i=0}^{n} \chi_i(R_1) \hat{\otimes} \chi_{n-i}(R_2),$$

where $\hat{\otimes}$ denotes the outer tensor product of S_n-characters: If χ_i and χ_j are S_i- and S_j-characters and S_i and S_j act, respectively, on $\{1, \ldots, i\}$ and $\{i+1, \ldots, i+j\}$, then $S_i \times S_j \subset S_{i+j}$ and $\chi_i \hat{\otimes} \chi_j = (\chi_i \otimes \chi_j) \uparrow S_{i+j}$ is the induced S_{i+j}-character.