Group actions and identities for the simple Lie algebra $sl_2(\mathbb{C})$

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Beginning:

- A. Giambruno , A. Regev, Wreath products and PI algebras, J. Pure Appl. Algebra 35 (1985) 133-145.

Problem

To have a description of G-identities of $sl_2(\mathbb{C})$, where G is a finite group that acts faithfully on $sl_2(\mathbb{C})$.

- Berele in 2004 in the paper "Polynomial identities for 2×2 matrices with finite group actions, J. Algebra 274 (1) (2004) 202-214", described bases of the G-identities for the matrix algebra of order two $M_2(\mathbb{C})$, where G is a finite group acting faithfully on $M_2(\mathbb{C})$.
- It is well known that if G is a finite group that acts faithfully on $M_2(\mathbb{C})$, then it must be one of the groups $\mathbb{Z}_n$, D_n, A_4, A_5 and S_4.
- His proofs rely on the concrete basis of the 2-graded identities for $M_2(\mathbb{C})$ found by Di Vincenzo in 1993 and on computations in the group algebras of the corresponding groups.

- Our problem is analogous to Berele's problem for the simple three-dimensional Lie algebra $sl_2(\mathbb{C})$.
- Fortunately, in this case it is well known that the finite groups that act faithfully on $sl_2(\mathbb{C})$ are the same as those for $M_2(\mathbb{C})$, and they act on $sl_2(\mathbb{C})$ in the same way that they act on $M_2(\mathbb{C})$, by conjugation of matrices.
- We use the concrete form of the 2-graded identities for $sl_2(\mathbb{C})$, and moreover, some methods and techniques developed by Berele. We exhibit bases of the corresponding G-identities for $sl_2(\mathbb{C})$.

How we approach the problem:

How we approach the problem: The same way that Berele did.

- $\mathbb{Z}_2$ is the easiest case, because of that we discuss it separately.
- We treat each group family in turn, so we divide the problem in three cases.
- In the first two cases, where the groups are $\mathbb{Z}_n$ and D_n, we consider computations in the group algebras of the corresponding groups.
- In the last case, where the groups are A_4, A_5 and S_4 we consider $sl_2(\mathbb{C})$ as a G-module.

Notation:

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- $X = \{x_1, x_2, x_3, \ldots\}$ will denote an infinite countable set.
- G will denote a group.
- By a *G-Lie Algebra* we understand a Lie Algebra together with a group G that acts on it.
- $L\mathbb{C}\langle X; G\rangle$ will denote the $\mathbb{C}$-free Lie algebra of symbols $g(x)$, where $g \in G$ and $x \in X$, that we will refer to as *G-free Lie algebra*.

Definition

Given $f \in L\mathbb{C}\langle X; G\rangle$ and a G- Lie algebra A, f is called a G-polynomial identity of A, if for every G-homomorphism $\varphi : L\mathbb{C}\langle X; G\rangle \to A$, $\varphi(f) = 0$.

Remark

- *The set of all G-polynomial identities of A is a G-ideal of $L\mathbb{C}\langle X; G\rangle$, that is invariant under all G-endomorphisms. This set will be denoted by $T_G(A)$.*
- *The relatively free algebra of the G-Lie algebra A is the algebra* $\dfrac{L\mathbb{C}\langle X; G\rangle}{T_G(A)}$.

$\mathbf{G} = \mathbb{Z}_2$:

Let $G = \{e, g\} = \mathbb{Z}_2$ and its action on $sl_2(\mathbb{C})$ be given by:

$$e\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \text{ and } g\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}.$$

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- G-identity:

$$[e(x) + g(x), e(y) + g(y)] = 0.$$

- The algebra $sl_2(\mathbb{K})$, where $\mathbb{K}$ is an infinite field of characteristic different from 2, admits a natural $\mathbb{Z}_2$-grading:

$$sl_2(\mathbb{K}) = \mathbb{K}(e_{11} - e_{22}) \oplus (\mathbb{K}e_{12} + \mathbb{K}e_{21}).$$

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Theorem (Koshlukov)

The graded identities for $sl_2(\mathbb{K})$ with the above grading follow from the identity $[x_1, x_2] = 0$ where x_1 and x_2 are even (neutral) variables.

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For all $x \in sl_2(\mathbb{C})$ define

$$\pi_0(x) = e(x) + g(x).$$

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- g acts on $sl_2(\mathbb{C})$ by conjugation by a matrix A of order n.
- We can assume that A is a diagonal matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$, where ω is a n-th primitive root of unity.
- Then $g \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & \omega^{-1}b \\ \omega c & -a \end{pmatrix}$

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- Then $g \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & \omega^{-1}b \\ \omega c & -a \end{pmatrix}$
- $\mathbb{C}G = \oplus_{i=0}^{n-1} \mathbb{C}e_i$, where each e_i, $i \in \{0, \ldots, n-1\}$, is a primitive idempotent and $e_i = (1/n) \sum_{j=0}^{n-1} \omega^{-ji} g^j$.
- Since G is abelian we can use this decomposition to obtain an equivalence between the G-actions and the G-gradings.
- $sl_2(\mathbb{C}) = \sum_{i=0}^{n-1} e_i sl_2(\mathbb{C})$. If $i \neq 0, 1, n-1$, then $e_i sl_2(\mathbb{C}) = 0$.

- Denote e_{n-1} by e_{-1}, then instead of considering the $\mathbb{Z}_n$ grading on $sl_2(\mathbb{C})$, we can consider a $\mathbb{Z}$-grading concentrated in $\{-1, 0, 1\}$.

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- Thus we have that

$$e_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \; e_1 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \text{ and}$$

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- G-identities: $[e_i(x_1), e_i(x_2)] = 0$, for all $i \in \{-1, 0, 1\}$.

Theorem

Let $G = \mathbb{Z}_n$*, where* $n \geq 3$*. Then all the* G*-identities of* $sl_2(\mathbb{C})$ *are consequences of the following:*

1. $e_0(x) + e_1(x) + e_{-1}(x) = x$;
2. *For* $\alpha \in \{-1, 0, 1\}$, $[e_\alpha(x_1), e_\alpha(x_2)] = 0$.

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- To prove this theorem we construct a basis for the relatively free algebra $\dfrac{L\mathbb{C}\langle X; G\rangle}{T_G(sl_2(\mathbb{C}))}$.

Theorem

The following set is a basis for the relatively free algebra $\dfrac{L\mathbb{C}\langle X; G\rangle}{T_G(sl_2(\mathbb{C}))}$:

1. $[e_1(x_{i_1}), e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}), \ldots, e_1(x_{i_\alpha}), e_{-1}(x_{j_\alpha})]$;
2. $[e_1(x_{i_1}), e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}), \ldots, e_1(x_{i_\alpha})]$;
3. $[e_1(x_{i_1}), e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}), \ldots, e_1(x_{i_\alpha}), e_{-1}(x_{j_\alpha}), e_{-1}(x_{j_{\alpha+1}})]$;
4. $[e_{-1}(x_{j_1}), e_0(x)^{\underline{m}}]$;
5. $e_0(x_{i_1})$;

where $\underline{m} \geq 0$, $\alpha \geq 1$ *and* $[e_k(x_{i_1}), e_0(x)^{\underline{m}}]$, *for all* $k \in \{-1, 1\}$ *means* $[e_k(x_{i_1}), e_0(x^{(1)}), \ldots, e_0(x^{(m)})]$.

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- It is well known that:
- If $n = 2m$, for some $m \in \mathbb{N}$, then

$$\mathbb{C}D_n = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \underbrace{M_2(\mathbb{C}) \oplus \ldots \oplus M_2(\mathbb{C})}_{m-1}.$$

- If $n = 2m + 1$, for some $m \in \mathbb{N}$, then

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- Let e_i be as before, then for each i, $he_i = e_{-i}h$.
- The two copies of $\mathbb{C}$ in $\mathbb{C}D_n$ are generated by the idempotents $(1/2)(1 - h)e_0$ and $(1/2)(1 + h)e_0$, and for $i > 0$ there is a copy of $M_2(\mathbb{C})$ with basis given by e_i, e_{-i}, he_i, and he_{-i}.

- Embedding D_n into $PGL_2(\mathbb{C})$, we can take g as before, $g = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$, and h as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

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- So the action by h is

$$h \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} -a & c \\ b & a \end{pmatrix}.$$

- As before, if $i \geq 2$ then each $e_{\pm i}$ and $he_{\pm i}$ acts as zero on $sl_2(\mathbb{C})$.

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$$(1-h)e_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & -2a \end{pmatrix},$$

$$e_1 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \; he_1 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix},$$

$$e_{-1} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ and } he_{-1} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

Theorem

Let $G = D_n$, where $n \geq 1$. Then, all G-identities of $sl_2(\mathbb{C})$ are consequences of the following:

1. $\dfrac{(1-h)e_0}{2}(x) + e_1(x) + e_{-1}(x) = x;$
2. *For* $\alpha \in \{-1, 1\}$, $[e_\alpha(x_1), e_\alpha(x_2)] = 0;$
3. *For* $\alpha \in \{-1, 1\}$, $[he_\alpha(x_1), he_\alpha(x_2)] = 0;$
4. *For* $\alpha \in \{-1, 1\}$, $[e_\alpha(x_1), he_{-\alpha}(x_2)] = 0.$

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3. *For* $\alpha \in \{-1, 1\}$, $[he_\alpha(x_1), he_\alpha(x_2)] = 0;$
4. *For* $\alpha \in \{-1, 1\}$, $[e_\alpha(x_1), he_{-\alpha}(x_2)] = 0.$

- Again, to prove this theorem we construct a basis for the relatively free algebra $\dfrac{L\mathbb{C}\langle X; G\rangle}{T_G(sl_2(\mathbb{C}))}$.

- Note that this case is very similar to the previous one, the difference being that we have two elements in the components 1 and -1.

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- **Component 0:** $(1-h)e_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & -2a \end{pmatrix}$.
- **Component 1:**

$$e_1 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \text{ and } he_{-1} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

- **Component -1:**

$$e_{-1} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ and } he_1 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}.$$

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1. e_0 is replaced by $(1 - h)e_0$;
2. Wherever e_1 appeared, we have two possibilites: e_1 or he_{-1};
3. Wherever e_{-1} appeared, we have two possibilites: e_{-1} or he_1.

Theorem

The following set is a basis for the relatively free algebra $\dfrac{L\mathbb{C}\langle X;G\rangle}{T_G(sl_2(\mathbb{C}))}$:

1. $[e_1(x_{i_1}),(1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}),\dots, e_1(x_{i_\alpha}), e_{-1}(x_{j_\alpha})]$;
2. $[e_1(x_{i_1}),(1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}),\dots, e_1(x_{i_\alpha})]$;
3. $[e_1(x_{i_1}),(1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}),\dots, e_1(x_{i_\alpha}), e_{-1}(x_{j_\alpha}), e_{-1}(x_{j_{\alpha+1}})]$;
4. $[e_{-1}(x_{j_1}),(1-h)e_0(x)^{\underline{m}}]$;
5. $(1-h)e_0(x_{i_1})$;

6. $[e_1(x_{i_1}),(1-h)e_0(x)^{\underline{m}}, he_1(x_{j_1}), e_1(x_{i_2}), he_1(x_{j_2}),\dots, e_1(x_{i_\alpha}), he_1(x_{j_\alpha})]$;
7. $[e_1(x_{i_1}),(1-h)e_0(x)^{\underline{m}}, he_1(x_{j_1}), e_1(x_{i_2}), he_1(x_{j_2}),\dots, e_1(x_{i_\alpha})]$;
8. $[e_1(x_{i_1}),(1-h)e_0(x)^{\underline{m}}, he_1(x_{j_1}), e_1(x_{i_2}), he_1(x_{j_2}),\dots, e_1(x_{i_\alpha}), he_1(x_{j_\alpha}), he_1(x_{j_{\alpha+1}})]$;
9. $[he_1(x_{j_1}),(1-h)e_0(x)^{\underline{m}}]$;

Theorem

10. $[he_{-1}(x_{i_1}), (1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), he_{-1}(x_{i_2}), e_{-1}(x_{j_2}), \dots, he_{-1}(x_{i_\alpha}), e_{-1}(x_{j_\alpha})]$;
11. $[he_{-1}(x_{i_1}), (1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), he_{-1}(x_{i_2}), e_{-1}(x_{j_2}), \dots, he_{-1}(x_{i_\alpha})]$;
12. $[he_{-1}(x_{i_1}), (1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), he_{-1}(x_{i_2}), e_{-1}(x_{j_2}), \dots, he_{-1}(x_{i_\alpha}), e_{-1}(x_{j_\alpha}), e_{-1}(x_{j_{\alpha+1}})]$;

13. $[e_1(x_{i_1}), (1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}), \dots, e_1(x_{i_k}), e_{-1}(x_{j_k}),$ $he_{-1}(x_{i_{k+1}}), e_{-1}(x_{j_{k+1}}), \dots, he_{-1}(x_{i_\alpha}), e_{-1}(x_{j_\alpha})]$;
14. $[e_1(x_{i_1}), (1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}), \dots, e_1(x_{i_k}), e_{-1}(x_{j_k}),$ $he_{-1}(x_{i_{k+1}}), e_{-1}(x_{j_{k+1}}), \dots, he_{-1}(x_{i_\alpha})]$;
15. $[e_1(x_{i_1}), (1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}), \dots, e_1(x_{i_k}), e_{-1}(x_{j_k}),$ $he_{-1}(x_{i_{k+1}}), e_{-1}(x_{j_{k+1}}), \dots, he_{-1}(x_{i_\alpha}), e_{-1}(x_{j_\alpha}), e_{-1}(x_{j_{\alpha+1}})]$;

16. $[e_1(x_{i_1}), (1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}), \dots, e_1(x_{i_k}), e_{-1}(x_{j_k}),$ $e_1(x_{i_{k+1}}), he_1(x_{j_{k+1}}), \dots, e_1(x_{i_\alpha}), he_1(x_{j_\alpha})]$;
17. $[e_1(x_{i_1}), (1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}), \dots, e_1(x_{i_k}), e_{-1}(x_{j_k}),$ $e_1(x_{i_{k+1}}), he_1(x_{j_{k+1}}), \dots, e_1(x_{i_\alpha})]$;
18. $[e_1(x_{i_1}), (1-h)e_0(x)^{\underline{m}}, e_{-1}(x_{j_1}), e_1(x_{i_2}), e_{-1}(x_{j_2}), \dots, e_1(x_{i_k}), e_{-1}(x_{j_k}),$ $e_1(x_{i_{k+1}}), he_1(x_{j_{k+1}}), \dots, e_1(x_{i_\alpha}), he_1(x_{j_\alpha}), he_1(x_{j_{\alpha+1}})]$;

where $\underline{m} \geq 0$, $\alpha \geq 1$ *and* $(1-h)e_0(x)^{\underline{m}}$ *is analogous to* $e_0(x)^{\underline{m}}$.

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- The span of the image of G in the group of automorphisms of $sl_2(\mathbb{C})$ is isomorphic to $\mathbb{C} \oplus sl_3(\mathbb{C})$.

$\mathbf{G = A_4, A_5}$ or $\mathbf{S_4}$:

- All these cases are similar.
- The span of the image of G in the group of automorphisms of $sl_2(\mathbb{C})$ is isomorphic to $\mathbb{C} \oplus sl_3(\mathbb{C})$.
- Thus it contains elements ϵ_{ij}, where $i, j \in \{1, 2, 3\}$, that act on $sl_2(\mathbb{C})$ in the following way: $\epsilon_{ij} v_\alpha = \delta_{j,\alpha} v_i$, where $v_1 = e_{11} - e_{22}$, $v_2 = e_{12}$ and $v_3 = e_{21}$ form a basis for $sl_2(\mathbb{C})$.

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- If $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ we have:
- **Component 0:**

$$\epsilon_{11}(A) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = e_0(A),\ \epsilon_{12}(A) = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix},\ \epsilon_{13}(A) = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}.$$

- **Component 1:**

$$\epsilon_{21}(A) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix},\ \epsilon_{22}(A) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = e_{-1}(A) \text{ and } \epsilon_{23}(A) = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}.$$

- **Component -1:**

$$\epsilon_{31}(A) = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix},\ \epsilon_{32}(A) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \text{ and } \epsilon_{33}(A) = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} = e_1(A).$$

Theorem

Let $G = A_4, A_5, S_4$. Then all G-identities of $sl_2(\mathbb{C})$ are consequences of the following:

1. $\epsilon_{11}(x) + \epsilon_{22}(x) + \epsilon_{33}(x) = x$;
2. *For* $\alpha, \beta \in \{1,2,3\}$, $[\epsilon_{1\alpha}(x), \epsilon_{1\beta}(y)] = 0$;
3. *For* $\alpha, \beta \in \{1,2,3\}$, $[\epsilon_{2\alpha}(x), \epsilon_{2\beta}(y)] = 0$;
4. *For* $\alpha, \beta \in \{1,2,3\}$, $[\epsilon_{3\alpha}(x), \epsilon_{3\beta}(y)] = 0$.

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- Again, to prove this theorem we construct a basis for the relatively free algebra $\dfrac{L\mathbb{C}\langle X; G\rangle}{T_G(sl_2(\mathbb{C}))}$.

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  2. Wherever e_{-1} appeared, we have three possibilites: ϵ_{21}, $e_{-1} = \epsilon_{22}$, or ϵ_{23};
  3. Wherever e_1 appeared, we have three possibilites: ϵ_{31}, ϵ_{32}, or $e_1 = \epsilon_{33}$.

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  3. Wherever e_1 appeared, we have three possibilites: ϵ_{31}, ϵ_{32}, or $e_1 = \epsilon_{33}$.
- Thus, this basis is worse than in the previous case, and will not be displayed here.

With similar but more complicated calculations we can show that the bases for the G-identities for $gl_2(\mathbb{C})$ are the same as those for $sl_2(\mathbb{C})$.