

A characterization of the Kostrikin radical of a Lie algebra.

International Workshop: "Infinite-Dimensional Lie Algebras"

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(joint work with E. García)

History

Theorem

Let R associative. Then

- *semiprime and d.c.c. on left ideals.*
- $R \cong \bigoplus_{i=1}^k \mathcal{M}_{n_i}(\Delta_{n_i})$

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$$I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$$

$$\exists m \mid I_m = I_{m+s} \text{ for all } s \in \mathbb{N}$$

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- semiprime and d.c.c. on inner ideals.
 - I inner: if $IRI \subset I$.

Theorem

Let L be a simple Lie algebra over a field F of characteristic 0 or greater than 7. Then L is Artinian and nondegenerate if and only if it is one of the following:

- 1 *A division Lie algebra.*
- 2 *A (finite dimensional over its centroid) simple exceptional Lie algebra.*
- 3 *$[R, R]/[R, R] \cap Z(R)$, where R is a simple Artinian associative algebra.*
- 4 *$[K, K]/[K, K] \cap Z(R)$, where $K = \text{Skew}(R, *)$ and R is a simple associative algebra with involution $*$ which coincides with its socle, such that...*

History

Theorem

Let L nondegenerate. $\text{Soc}(L) := \sum I_\alpha$, I_α minimal inner ideal of L

- *$\text{Soc}(L) = \bigoplus M_i$ with M_i simple nondegenerate coinciding with its socle.*

Theorem

L Lie simple, nondegenerate, char = 0 or > 7 with I minimal, then:

- 1 A (finite dimensional over its centroid) simple exceptional Lie algebra with Jordan elements.*
- 2 $L \cong [R, R]/Z(R) \cap [R, R]$, R simple associative with socle not a division algebra.*
- 3 $L \cong [K, K]/Z(R) \cap [K, K]$ for $K = \text{Skew}(R, *)$, R simple associative algebra with isotropic involution $*$, with socle and where either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16.*

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$$L \hookrightarrow \prod_{\alpha \in \Delta} L_{\alpha} \rightarrow L_{\alpha} \text{ onto for all } \alpha$$

$$\exists \{I_{\alpha}\}_{\alpha \in \Delta} \text{ ideals of } L \text{ such that } L/I_{\alpha} \text{ is s.p. and } \bigcap_{\alpha \in \Delta} I_{\alpha} = 0$$

R associative

- $B(R)$ Baer radical (prime) de R .
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- Semiprime, $I \triangleleft R, I^2 = 0 \implies I = 0$
- Prime, $I, J \triangleleft R, IJ = 0 \implies I = 0$ or $J = 0$

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- $\{x_n\}_{n \in \mathbb{N}}$, $x_{n+1} := x_n a_n x_n$,

$$x_1 \in B(R) \iff \exists k | x_k = 0$$

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Kostrikin radical

- $x \in L$ **absolute zero divisor**, $[x, [x, L]] = 0$.
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- $K(L) = \bigcup_{\alpha} K_{\alpha}(L)$ donde $K_{\alpha}(L)$: (Kostrikin radical)
 - $K_1(L) \triangleleft L$ ideal generate by A.Z.D L .
 - $K_{\alpha}(L) = \bigcup_{\beta < \alpha} K_{\beta}(L)$ with α ordinal limit, and
 - $K_{\alpha}(L)/K_{\alpha-1}(L) = K_1(L/K_{\alpha-1}(L))$ other case.

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 - $K_{\alpha}(L)/K_{\alpha-1}(L) = K_1(L/K_{\alpha-1}(L))$ other case.
 - $I \triangleleft L$ non degenerate (s.p.) if L/I is non-degenerate (s.p.).
- 1 L non-degenerate $\iff L$ is a S.P of strongly prime?
 - 2 $\{x_n\}_{n \in \mathbb{N}}$, $x_{n+1} := [x_n, [a_n, x_n]]$. If $x_1 \in K(L)$, finite m-sequence?

Theorem

R associative, Φ with no 2-torsion, $K(R^-)$

- 1 R prime, $R^-/Z(R)$ strongly prime.*
- 2 $K(R^-)$ is the intersection of all s.p. ideals of R^- .*
- 3 $K(R^-) = \pi^{-1}(Z(R/r(R)))$, $r(R)$ Baer radical of R and $\pi : R \rightarrow R/r(R)$ the canonical projection.*
- 4 $K(R^-) = \{x \in R \mid m\text{-sequence in } x \text{ has finite length}\}$.*

Theorem

$(R, *)$ associative with involution, Φ with no 2-torsion,
 $L := \text{Skew}(R, *)$ and $K(L)$ Kostrikin radical.

- 1 R prime, $L/(Z(R) \cap \text{Skew}(R, *))$ strongly prime??
- 2 $K(L)$ is the intersection of all s.p. ideals of L .
- 3 $K(L) = \pi^{-1}(Z(L/(r(R) \cap L)))$, $r(R)$ Baer radical of R and $\pi : R \rightarrow R/r(R)$ the canonical projection.
- 4 $K(L) = \{x \in R \mid m\text{-sequence in } x \text{ has finite length}\}$.

J Jordan algebra

- $Mc(J)$ McCrimmon radical.
 - $J/Mc(J)$ nondegenerate.
 - $I \triangleleft J$ nondegenerate if J/I nondegenerate.
 - $I \triangleleft J$ strongly prime if J/I prime and nondegenerate.
 - $Mc(J) = \bigcap_{I \triangleleft_{f.p.} J} I$.
- 1 J nondegenerate $\iff J$ subdirect product of s.p.
 - 2 m-sequences: $\{x_n\}_{n \in \mathbb{N}}$, $x_{n+1} := U_{x_n} a_n$,

$$x_1 \in Mc(J) \iff \exists k | x_k = 0$$

$$J \text{ nondegenerate: } U_x J := 0 \implies x = 0$$

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 - $L_x := (L/\ker(x), \bullet)$.

$$U_{\bar{a}}\bar{b} = \frac{1}{8}\overline{\text{ad}_a^2 \text{ad}_x^2 b}, \quad \text{for all } a, b \in L, \quad \text{and}$$

$$\{\bar{a}, \bar{b}, \bar{c}\} = -\frac{1}{4}\overline{[a, [\text{ad}_x^2 b, c]]} \quad \text{for all } a, b, c \in L$$

- $x \in L$ Jordan element, if $\text{ad}_x^3 = 0$.
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- **Theorem:** $L, x \in L$ Jordan element

$$Mc(L_x) \subset \{\bar{a} \in L_x \mid [x, [x, a]] \subset K(L)\}.$$

Definition

- α an ordinal. L satisfies \mathcal{H}_α if $\beta \leq \alpha$ not ordinal limit every submodule $L_\beta := K_\beta(L)/K_{\beta-1}(L)$ invariant by inner automorphisms of L_β is an ideal of L_β .
- L satisfies \mathcal{H} if satisfies $\mathcal{H}_\alpha, \forall \alpha$.

Examples

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- L Generate by ad-nilpotent elements index $\leq k$. $p > 2k - 2$
- $L = \bigoplus_{i=1}^n L_i, \mathbb{Z}$ -graded $L_0 = \sum_{i=1}^n [L_i, L_{-i}]$. $p > 4n$

L Lie algebra with \mathcal{H}

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Let $x \in L$ Jordan element. Then any m -sequence on x has finite length. In particular, $Mc(L_x) = L_x$.

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Proposition

Let M an m -sequence of Jordan elements. If $P \triangleleft L$ maximal
 $M \cap P = \emptyset$, Then P is s.p. ideal of $L.$

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- 1 A construction of gradings of Lie Algebras. **Int. Math. Res. Not.** (2007)

Authors: E. Neher, A. Fernández López, E. García, — 

Papers

- 1 A construction of gradings of Lie Algebras. **Int. Math. Res. Not.** (2007)
- 2 The Jordan Algebras of a Lie Algebra. **J. Algebra** (2007)

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- 2 The Jordan Algebras of a Lie Algebra. **J. Algebra** (2007)
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- 6 A characterization of the Kostrikin radical of a Lie algebra. **preprint** (???)

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