Borel and parabolic subalgebras of some locally finite Lie algebras

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Outline

- Toral and Borel subalgebras of $gl(\infty)$
- Relationship to generalized flags
- Main theorems and examples on Borel subalgebras of $gl(\infty)$
- Parabolic subalgebras
- Comparison with subalgebras of $gl(2^\infty)$

Basic definitions

- Let V and V_* be countable-dimensional vector spaces over $\mathbb{C}$. Let
$$\langle \cdot, \cdot \rangle : V \times V_* \to \mathbb{C}$$
be a nondegenerate pairing. Then $V \otimes V_*$ is an associative algebra such that
$$(v_1 \otimes w_1)(v_2 \otimes w_2) = \langle v_2, w_1 \rangle \, v_1 \otimes w_2$$
where $v_1, v_2 \in V$ and $w_1, w_2 \in V_*$.
Then $gl(V, V_*)$ is the Lie algebra associated to $V \otimes V_*$, and $sl(V, V_*)$ is the commutator subalgebra of $gl(V, V_*)$.
- $gl(V, V_*)$ does not depend on the pairing and
$$gl(V, V_*) \cong gl(\infty).$$
- If $W \subset V$ then $\overline{W} := (W^\perp)^\perp$ is the **closure** of W. A subspace W is **closed** if $W = \overline{W}$.

Toral subalgebras

Definition

An element of $gl(\infty)$ is **semisimple** *if it is semisimple as a linear operator on the natural representation of $gl(\infty)$.*
A subalgebra $\mathfrak{t} \subset gl(\infty)$ is **toral** *if all its elements are semisimple.*

Proposition

(i) *Every maximal toral subalgebra $\mathfrak{t}$ of $gl(\infty)$ has the form*

$$\mathfrak{t} = \bigoplus_{\alpha \in A} (\mathbb{C}u_\alpha) \otimes (\mathbb{C}u_\alpha^*)$$

where $\{u_\alpha\}$ and $\{u_\alpha^*\}$ are maximal sets of vectors in V and V_* with the property that $< u_\alpha, u_\beta^* >= \delta_{\alpha,\beta}$. Conversely, every such expression defines a maximal toral subalgebra of $gl(\infty)$.

(ii) *The centralizer $C(\mathfrak{t})$ of $\mathfrak{t}$ has the form*

$$C(\mathfrak{t}) = \mathfrak{t} \oplus (span\{u_\alpha\})^\perp \otimes (span\{u_\alpha^*\})^\perp$$

Toral subalgebras

Proposition

Let $\mathfrak{t}$ be a maximal toral subalgebra of $gl(\infty)$. The following are equivalent:

(i) *There is an exhaustion $\bigcup_i \mathfrak{g}_i$ of $gl(\infty)$ such that $\mathfrak{t} \cap \mathfrak{g}_i$ is a maximal toral subalgebra of $\mathfrak{g}_i$.*

(ii) $\mathfrak{t} = \bigoplus_{\alpha \in A} (\mathbb{C} u_\alpha) \otimes (\mathbb{C} u_\alpha^*)$, *where $\{u_\alpha\}$ and $\{u_\alpha^*\}$ is a pair of dual bases in V and in V_*.*

- A maximal toral subalgebra as above is **splitting**.
- **Example.** The following is a non-splitting maximal toral subalgebra:

$$\mathfrak{t} = \bigoplus_{n \geq 2} \mathbb{C}(e_1 + e_n) \otimes \mathbb{C}(e_n^*)$$

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Borel subalgebras

Definition

(i) *A locally finite Lie algebra* $\mathfrak{g}$ *is* **locally solvable** *if every finite subset of* $\mathfrak{g}$ *is contained in a solvable subalgebra.*

(ii) *A* **Borel subalgebra** *of* $\mathfrak{g}$ *is a maximal locally solvable subalgebra.*

Proposition

Let $\mathfrak{b}$ *be a Borel subalgebra of* $gl(\infty)$*. The following are equivalent:*

(i) $\mathfrak{b}$ *contains a splitting maximal toral subalgebra.*

(ii) *There exists an exhaustion* $\bigcup_i \mathfrak{g}_i$ *of* $gl(\infty)$ *such that* $\mathfrak{b} \cap \mathfrak{g}_i$ *is a Borel subalgebra of* $\mathfrak{g}_i$*.*

- A Borel subalgebra as above is **splitting**.

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Generalized flags

Definition. Let X be a vector space. A *chain* of subspaces in X is a set $\mathcal{C}$ of subspaces in X linearly ordered by inclusion.
A **generalized flag** in X is a chain of subspaces $\mathfrak{F}$ in X satisfying the following properties:

(i) each space $F \in \mathfrak{F}$ has an immediate predecessor or an immediate successor;

(ii) for every $0 \neq x \in X$ there exists a pair $F', F'' \in \mathfrak{F}$, such that $x \in F'' \setminus F'$ and F'' is the immediate successor of F'.

- A generalized flag $\mathfrak{F}$ is *semiclosed* if $\overline{F'} \in \{F', F''\}$ for every predecessor-successor pair (F', F'').
- A generalized flag $\mathfrak{F}$ is *closed* if it is semiclosed and F'' is closed for every pair (F', F'').
- A generalized flag $\mathfrak{F}$ is *strongly closed* if $\overline{F} = F$ for every $F \in \mathfrak{F}$.

Generalized flags

Example:
Let $\dim X = \aleph_0$ and let $\{x_q\}_{q\in\mathbb{Q}}$ be a basis of X enumerated by $\mathbb{Q}$.
Let $\mathfrak{F} = \{F'_q, F''_q\}_{q\in\mathbb{Q}}$ be the following generalized flag:

- $F'_q = \mathrm{span}\{x_s : s < q\}$
- $F''_q = \mathrm{span}\{x_s : s \leq q\}$

Properties of $\mathfrak{F}$:

- No subspace F in $\mathfrak{F}$ has both an immediate predecessor and an immediate successor.
- $\mathfrak{F}$ is a maximal generalized flag but not a maximal chain.
- The unique maximal chain $\mathcal{C}$ which contains $\mathfrak{F}$ is the chain $\{F'_r : r \in \mathbb{R}\} \cup \{F''_q : q \in \mathbb{Q}\} \cup \{0, X\}$, where $F'_r = \mathrm{span}\{x_s : s < r\}$.

Main theorems

Theorem (I.Dimitrov, I.Penkov)

Let $\mathfrak{g}$ be one of $gl(\infty)$ and $sl(\infty)$. Every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is the stabilizer of a unique maximal (semi-)closed generalized flag $\mathfrak{F}_{\mathfrak{b}}$ in V, and the correspondence

$$\mathfrak{b} \mapsto \mathfrak{F}_{\mathfrak{b}}$$

is a bijection between the set of Borel subalgebras in $\mathfrak{g}$ and the set of maximal (semi-)closed generalized flags in V.

Theorem (I.Dimitrov, I.Penkov)

Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. The following are equivalent:

(i) *$\mathfrak{b}$ is splitting.*

(ii) *The unique $\mathfrak{b}$-stable maximal (semi-)closed generalized flag in V is strongly closed.*

Examples

(1) Let $\mathfrak{F}$ be the generalized flag

$$0 \subset U_1 \subset U_2 \subset \cdots \subset U_n \subset \cdots \subset U \subset V$$

where $U_n = \mathrm{span}\{e_1 + e_2, \ldots, e_1 + e_n\}$ for each n and $U = \bigcup_n U_n$. Then each U_n is closed and $\overline{U} = V$. Hence, $\mathfrak{F}$ is closed but not strongly closed.

(2) Let $V = \mathrm{span}\{\tilde{x}_q\}_{q \in \mathbb{Q}}$ and $V_* = \mathrm{span}\{x_q^*\}_{q \in \mathbb{Q}}$ where

$$< \tilde{x}_q, x_s^* >= 1 \text{ if } q > s \text{ and } 0 \text{ otherwise.}$$

Then $V \otimes V_* \cong gl(\infty)$. Let $\mathfrak{F} = \{F_q', F_q''\}_{q \in \mathbb{Q}}$ be:

- $F_q' = \mathrm{span}\{\tilde{x}_s : s < q\}$
- $F_q'' = \mathrm{span}\{\tilde{x}_s : s \leq q\}$

Then $\mathfrak{F}$ is a maximal closed generalized flag for which $\overline{F_q'} = F_q''$ for each q. Moreover, $\mathfrak{b} = St_{\mathfrak{F}}$ coincides with its nilradical. Hence, $\mathfrak{b}$ contains no nontrivial toral subalgebras.

Parabolic subalgebras

Definition

- *A subalgebra of a locally finite Lie algebra is* **parabolic** *if it contains a Borel subalgebra.*
- *Two semiclosed generalized flags $\mathfrak{F}$ in V and $\mathfrak{G}$ in V_* form a* **taut couple** *if the chain $\mathfrak{F}^{\perp}$ is stable under $St_{\mathfrak{G}}$ and the chain $\mathfrak{G}^{\perp}$ is stable under $St_{\mathfrak{F}}$.*

Theorem (E.Dan-Cohen, I.Penkov)

Let $\mathfrak{g}$ be one of $gl(\infty)$ and $sl(\infty)$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Then there exists a unique taut couple $\mathfrak{F}, \mathfrak{G}$ such that

$$\mathfrak{p}_- \subset \mathfrak{p} \subset \mathfrak{p}_+$$

where $\mathfrak{p}_+ = St_{\mathfrak{F}} \cap St_{\mathfrak{G}}$, $\mathfrak{p}_- = \mathfrak{n}_{\mathfrak{p}_+} + [\mathfrak{p}_+, \mathfrak{p}_+]$, and $\mathfrak{n}_{\mathfrak{p}_+}$ is the linear nilradical of $\mathfrak{p}_+$, i.e. the set of all nilpotent elements in the maximal locally solvable ideal of $\mathfrak{p}_+$. Moreover, $N_{\mathfrak{g}}(\mathfrak{p}) = N_{\mathfrak{g}}(\mathfrak{p}_+) = \mathfrak{p}_+$.

Comparison with $gl(2^\infty)$

- Similarities
  - There are maximal toral subalgebras of $gl(2^\infty)$ which cannot be exhausted by finite-dimensional maximal toral subalgebras.
  - There are Borel subalgebras which cannot be exhausted by finite-dimensional Borel subalgebras.
  - For every Borel subalgebra $\mathfrak{b}$ of $gl(2^\infty)$ there exists a maximal generalized flag $\mathfrak{F}$ such that $\mathfrak{b} = St_{\mathfrak{F}}$.
- Differences
  - There are Borel subagebras which contain maximal splitting toral subalgebras but cannot be exhausted by finite-dimensional Borel subalgebras.
  - There are maximal strongly closed generalized flags whose stabilizers are not Borel subalgebras.

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