Twisted Weyl algebras and their representations

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Definition (Mazorchuk & Turowska 1999): Given

- R = a commutative algebra over a field $\mathbb{k}$,
- $t = (t_1, \ldots, t_n) \in (R\backslash\{0\})^n$,
- $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathrm{Aut}_{\mathbb{k}}(R)^n$ pairwise commuting,
- $\mu = (\mu_{ij})_{i,j=1}^n$ matrix with $\mu_{ij} \in \mathbb{k}\backslash\{0\}$,

such that

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the twisted generalized Weyl construction, $A' = A'(R, \sigma, t, \mu)$, is the R-ring generated by $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ modulo

$$X_i Y_j = \mu_{ij} Y_j X_i \quad (i \neq j) \tag{1.1}$$

$$Y_i X_i = t_i, \qquad X_i Y_i = \sigma_i(t_i), \tag{1.2}$$

$$X_i r = \sigma_i(r) X_i, \qquad r Y_i = Y_i \sigma_i(r) \quad \forall r \in R. \tag{1.3}$$

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$\mathbb{Z}^n$-gradation on A':

$$\deg(r) = 0 \; \forall r \in R,$$

$$\deg(X_i) = (0, \ldots, 1^i, \ldots, 0), \quad \deg(Y_i) = (0, \ldots, -1^i, \ldots, 0).$$

Let $I :=$ sum of all graded ideals J in A' with $J \cap (A'_0) = 0$.

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Let $I :=$ sum of all graded ideals J in A' with $J \cap (A'_0) = 0$.
The twisted generalized Weyl algebra $A = A_\mu(R,\sigma,t)$ is defined as

$$A = A'/I.$$

Theorem

1. *The construction of a twisted generalized Weyl algebra* $A_\mu(R,\sigma,t)$ *from a TGW datum* (R,σ,t) *defines a functor* $\mathcal{A}_\mu : \mathrm{TGW}_n(\Bbbk) \to \mathbb{Z}^n\text{-}\mathrm{GrAlg}_\Bbbk$. *That is, for any morphism* $\varphi : (R,\sigma,t) \to (R',\sigma',t')$ *in* $\mathrm{TGW}_n(\Bbbk)$ *there is a morphism* $\mathcal{A}_\mu(\varphi) : A_\mu(R,\sigma,t) \to A_\mu(R',\sigma',t')$ *in* $\mathbb{Z}^n\text{-}\mathrm{GrAlg}_\Bbbk$ *such that* $\mathcal{A}_\mu$ *preserves compositions and identity morphisms.*
2. *Given any morphism* $\varphi : (R,\sigma,t) \to (R',\sigma',t')$ *in* $\mathrm{TGW}_n(\Bbbk)$, *we have the following commutative diagram in* $\mathbb{Z}^n\text{-}\mathrm{GrAlg}_\Bbbk$:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ \downarrow{\scriptstyle \rho_{\mu,(R,\sigma,t)}} & & \downarrow{\scriptstyle \rho_{\mu,(R',\sigma',t')}} \\ A_\mu(R,\sigma,t) & \xrightarrow{\mathcal{A}_\mu(\varphi)} & A_\mu(R',\sigma',t') \end{array} \tag{1.4}$$

The n:th Weyl algebra:

$$R = \mathbb{k}[t_1, \ldots, t_n],$$

$$\sigma_i(t_j) = t_j + \delta_{ij} \quad \forall i, j$$

$$\mu_{ij} = 1 \quad \forall i, j$$

Then

$$I = \langle X_i X_j - X_j X_i, Y_i Y_j - Y_j Y_i \mid \forall i \neq j \rangle$$

and $A_\mu(R, \sigma, t) \simeq A_n(\mathbb{k})$, the n:th Weyl algebra over $\mathbb{k}$.

If $\sigma_i(t_j) = t_j \ \forall i \neq j$ and t_i are not zero-divisors then

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- Noncommutative Type-A Kleinian singularities. (studied by Bavula, Hodges, Rosenberg, Bavula & Jordan): Let $n = 1$, $R = \mathbb{k}[u]$, $\sigma(u) = u - 1$, $t \in R$ arbitrary.
  - If $t = u$, then $R(\sigma, t) \simeq A_1(\mathbb{k})$.
  - If $t = -u^2 - u - \frac{\lambda}{4}$, $\lambda \in \mathbb{k}$, then $R(\sigma, t) \simeq U(sl_2(\mathbb{k}))/\langle C - \lambda \rangle$.

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- $U(sl_2(\mathbb{k}))$, $U_q(sl_2(\mathbb{k}))$, $U(sl_3^+(\mathbb{k}))$, $U_q(sl_3^+(\mathbb{k}))$, (generalized) down-up algebras (Benkart & Roby 1998, Cassidy & Shelton 2004), Witten-Woronowicz deformations of $U(sl_2)$, ...

- A TGWA of "type A_2" (Mazorchuk and Turowska 1999): $R = \mathbb{C}[H]$, $t_1 = H$, $t_2 = H + 1$, $\sigma_1(H) = H + 1, \sigma_2(H) = H - 1$, all $\mu_{ij} = 1$. Then

$$\sigma_1^{-1}(t_2)\sigma_2^{-1}(t_1) = H(H+1) = t_1 t_2$$

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- Quantized Weyl algebras $A_n^{q,\Lambda}$,
- Mickelsson step algebras $Z(\mathfrak{gl}_n, \mathfrak{gl}_{n-1} \times \mathfrak{gl}_1)$.

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A regular TGW datum $D = (R, \sigma, t) \in \mathrm{TGW}_n(\Bbbk)$ is μ-consistent iff

$$\sigma_i\sigma_j(t_it_j) = \mu_{ij}\mu_{ji}\sigma_i(t_i)\sigma_j(t_j), \qquad \forall i, j = 1, \dots, n,\ i \neq j, \tag{1.5}$$

$$t_j\sigma_i\sigma_k(t_j) = \sigma_i(t_j)\sigma_k(t_j), \qquad \forall i, j, k = 1, \dots, n,\ i \neq j \neq k \neq i. \tag{1.6}$$

Theorem A

Assume that $D = (R, \sigma, t)$ is a μ-consistent data. If $t_1, \ldots, t_n$ are invertible in R, then $\mathcal{A}_\mu(R, \sigma, t)$ is graded isomorphic to a $\mathbb{Z}^n$-crossed product algebra over R: there is a unique σ-twisted 2-cocycle $\alpha : \mathbb{Z}^n \times \mathbb{Z}^n \to R^\times$ for which there exists a graded isomorphism

$$\xi_{\mu,D} : R \rtimes_\alpha^\sigma \mathbb{Z}^n \to \mathcal{A}_\mu(R, \sigma, t)$$

satisfying

$$\xi(ru_g) = rX_1^{g_1} \cdots X_n^{g_n},$$

$$\alpha(g, h)\alpha(gh, k) = \sigma_g(\alpha(h, k))\alpha(g, hk); \alpha(g, e) = \alpha(e, g) = 1$$

Remark

If $t_1, \ldots, t_n$ are regular in R, then $\mathcal{A}_\mu(R, \sigma, t)$ can be embedded into a $\mathbb{Z}^n$-crossed product algebra over a localization of R.

Definition

A twisted generalized Weyl algebra $A = A_\mu(R, \sigma, t)$ is *locally finite over* $\Bbbk$ if

$$\dim_\Bbbk \left(\mathrm{Span}_\Bbbk\{\sigma_i^k(t_j) \mid i, j = 1, \ldots, n, k \in \mathbb{Z}\}\right) < \infty.$$

If A is locally finite over $\Bbbk$, let p_{ij} be the minimal polynomial for σ_i acting on the space

$$\mathrm{Span}_\Bbbk\{\sigma_i^k(t_j) \mid k \in \mathbb{Z}\}.$$

Theorem

The matrix $C_A = (a_{ij})_{i,j=1}^n$ *defined by*

$$a_{ij} = \begin{cases} 2, & \textit{if } i = j, \\ 1 - \deg(p_{ij}), & \textit{otherwise.} \end{cases}$$

is a generalized Cartan matrix. That is,

$$\deg(p_{ij}) \geq 1 \quad \forall i \neq j,$$

and

$$\deg(p_{ij}) = 1 \Longleftrightarrow \deg(p_{ji}) = 1.$$

Theorem

Let $A = A_\mu(R, \sigma, t)$ be a locally finite TGWA, where R has no zero-divisors and $\mu_{ij} = 1 \forall i, j$. Then for all $i, j = 1, \ldots, n$, $i \neq j$, there is a unique minimal $m_{ij} \in \mathbb{Z}_{>0}$ and unique $\lambda_{ij}^{(1)}, \ldots \lambda_{ij}^{(m_{ij})} \in \Bbbk$, $\lambda_{ij}^{(m_{ij})} \neq 0$ such that

$$X_i^{m_{ij}} X_j + \lambda_{ij}^{(1)} X_i^{m_{ij}-1} X_j X_i + \cdots + \lambda_{ij}^{(m_{ij})} X_j X_i^{m_{ij}} = 0.$$

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They are given by the polynomials p_{ij}:

$$p_{ij}(x) = x^{m_{ij}} + \lambda_{ij}^{(1)} x^{m_{ij}-1} + \cdots + \lambda_{ij}^{(m_{ij})}.$$

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Also,

$$Y_j Y_i^{m_{ij}} + \lambda_{ij}^{(1)} Y_i Y_j Y_i^{m_{ij}-1} + \cdots + \lambda_{ij}^{(m_{ij})} Y_i^{m_{ij}} Y_j = 0.$$

Example:
Let $A = A(R, \sigma, t, \mu)$ be the TGWA of "type A_2": $R = \mathbb{C}[H]$, $t_1 = H$, $t_2 = H + 1$, $\sigma_1(H) = H + 1, \sigma_2(H) = H - 1$, all $\mu_{ij} = 1$.
Then

$$\sigma_2^2(t_1) - 2\sigma_2(t_1) + t_1 = 0$$

and $\{t_1, \sigma_2(t_1)\}$ is linearly independent.

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and $\{t_1, \sigma_2(t_1)\}$ is linearly independent. So $1 - \deg(p_{21}) = -1$. Similarly $1 - \deg(p_{12}) = -1$. Thus

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really motivating us to say A is of type A_2! Moreover, in A,

$$X_i^2 X_j - 2X_i X_j X_i + X_j X_i^2 = 0, \quad \{i, j\} = \{1, 2\},$$

$$Y_i^2 Y_j - 2Y_i Y_j Y_i + Y_j Y_i^2 = 0, \quad \{i, j\} = \{1, 2\}.$$

To any generalized Cartan matrix $C = (a_{ij})$ we can associate the polynomials $p_{ij}(x) = (x-1)^{1-a_{ij}}$, $i \neq j$. Then the above relations look like ordinary Serre relations for the Kac-Moody algebra $g(C)$. If we instead take $p_{ij}(x) = (x - q^{a_{ij}})(x - q^{a_{ij}+2}) \cdots (x - q^{-a_{ij}})$ we get quantum Serre relations.

Theorem

Let $A = \mathcal{A}_\mu(R,\sigma,t)$ is a locally finite TGW algebra of type $(A_1)^n$, where (R,σ,t) is regular and μ-consistent. Then A is isomorphic to the R-ring generated by $X_1,\ldots,X_n,Y_1,\ldots,Y_n$ modulo the relations

$$\begin{aligned} X_i r &= \sigma_i(r)X_i, \qquad Y_i r = \sigma_i^{-1}(r)Y_i \quad \forall r\in R, \forall i,\\ Y_iX_i &= t_i, \qquad\qquad X_iY_i = \sigma_i(t_i), \qquad \forall i, \end{aligned} \tag{2.1}$$

$$X_iY_j = \mu_{ij}Y_jX_i, \quad X_iX_j = \gamma_{ij}\mu_{ij}^{-1}X_jX_i, \quad Y_jY_i = \gamma_{ij}\mu_{ji}^{-1}Y_iY_j, \quad i\neq j, \tag{2.2}$$

where $\sigma_i(t_j) = \gamma_{ij}t_j$, $i\neq j$.

Moreover, A is simple if and only if R is $\mathbb{Z}^n$-simple and $Rt_i + R\sigma_i^d(t_i) = R$ for all $d \in \mathbb{Z}_{>0}$ and $i = 1, \ldots, n$.

Let $n \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}\setminus\{0\}$ and let $\Lambda = (\lambda_{ij})$, $r = (r_{ij})$ and $s = (s_{ij})$ be three $n \times n$-matrix with entries from $\mathbb{k}\setminus\{0\}$, such that

$$\lambda_{ii} = 1 \ \forall i \text{ and } \lambda_{ij}\lambda_{ji} = 1 \ \forall i \neq j, \tag{3.1}$$

$$r_{ii}/s_{ii} \text{ is a nonroot of unity } \forall i, \tag{3.2}$$

$$r_{ij}^k = s_{ij}^k \quad i \neq j. \tag{3.3}$$

Let

$$R = \mathbb{k}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}, v_1^{\pm 1}, \ldots, v_n^{\pm 1}], \tag{3.4}$$

$\sigma_1, \ldots, \sigma_n \in \mathrm{Aut}_{\mathbb{k}}(R)$ as follows:

$$\sigma_i(u_j) = r_{ij}^{-1} u_j, \qquad \sigma_i(v_j) = s_{ij}^{-1} v_j, \tag{3.5}$$

for all $i, j \in \{1, \ldots, n\}$, and

$$t_i = \frac{(r_{ii}u_i)^k - (s_{ii}v_i)^k}{r_{ii}^k - s_{ii}^k}. \tag{3.6}$$

Put

$$\mu_{ij} = r_{ji}^{-k} \lambda_{ji}, i \neq j. \tag{3.7}$$

Then (R, σ, t) is μ-consistent and $\mathcal{A}_\mu(R, \sigma, t) = A_n^k(r, s, \Lambda)$ is a *multiparameter twisted Weyl algebra*.

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Let $R^{\mathbb{Z}^n} = \{r \in R \mid \sigma_i(r) = r \; \forall i = 1, \ldots, n\}$,
$G = \{d \in \mathbb{Z}^{2n} \mid u^d \in R^{\mathbb{Z}^n}\}$ the gradation group of $R^{\mathbb{Z}^n}$.

Theorem B

Let $A = A_n^k(r,s,\Lambda)$ be a multiparameter twisted Weyl algebra.

(a) *The assignment*

$$\mathfrak{n} \mapsto A/\langle \mathfrak{n} \rangle \tag{3.8}$$

where $\langle \mathfrak{n} \rangle$ denotes the ideal in A generated by $\mathfrak{n}$, is a bijection between the set of maximal ideals in the invariant subring $R^{\mathbb{Z}^n}$ and the set of simple quotients of A in which all $X_i,\ Y_i\,(i = 1,\dots,n)$ are regular.

(b) *For any $\mathfrak{n} \in \mathrm{Specm}(R^{\mathbb{Z}^n})$, the quotient $A/\langle \mathfrak{n} \rangle$ is isomorphic to the twisted generalized Weyl algebra $\mathcal{A}_\mu(R/R\mathfrak{n}, \bar{\sigma}, \bar{t})$, where $\bar{\sigma}_g(r + R\mathfrak{n}) = \sigma_g(r) + R\mathfrak{n}$, $\forall g \in \mathbb{Z}^n, r \in R$ and $\bar{t}_i = t_i + R\mathfrak{n}$, $\forall i$.*

(c) *$A/\langle \mathfrak{n} \rangle$ is a domain for all $\mathfrak{n} \in \mathrm{Specm}(R^{\mathbb{Z}^n})$ if and only if $\mathbb{Z}^{2n}/G$ is torsion-free.*

Let $\underline{r} = (r_1, \ldots, r_n)$, $\underline{s} = (s_1, \ldots, s_n)$ are such that $(r_i s_i^{-1})^2 \neq 1$ for each i. Let $A_{\underline{r},\underline{s}}(n)$ be the unital associative algebra generated by $\rho_i, \rho_i^{-1}, \sigma_i, \sigma_i^{-1}, x_i,\ y_i,\ \ i = 1, \ldots, n$, subject to the following relations:

(R1) The $\rho_i^{\pm 1}$, $\sigma_j^{\pm 1}$ all commute with one another and $\rho_i \rho_i^{-1} = \sigma_i \sigma_i^{-1} = 1$;

(R2) $\rho_i x_j = r_i^{\delta_{i,j}} x_j \rho_i \qquad \rho_i y_j = r_i^{-\delta_{i,j}} y_j \rho_i \qquad 1 \leq i,j \leq n$;

(R3) $\sigma_i x_j = s_i^{\delta_{i,j}} x_j \sigma_i \qquad \sigma_i y_j = s_i^{-\delta_{i,j}} y_j \sigma_i \qquad 1 \leq i,j \leq n$;

(R4) $x_i x_j = x_j x_i, \qquad y_i y_j = y_j y_i, \qquad 1 \leq i,j \leq n$;
$y_i x_j = x_j y_i, \qquad 1 \leq i \neq j \leq n$;

(R5) $y_i x_i - r_i^2 x_i y_i = \sigma_i^2$ and $y_i x_i - s_i^2 x_i y_i = \rho_i^2, \qquad 1 \leq i \leq n$,
or equivalently

(R5') $y_i x_i = \dfrac{r_i^2 \rho_i^2 - s_i^2 \sigma_i^2}{r_i^2 - s_i^2}$ and $x_i y_i = \dfrac{\rho_i^2 - \sigma_i^2}{r_i^2 - s_i^2} \qquad 1 \leq i \leq n.$

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Take $k = 2$, and for all i, j put $\lambda_{ij} = 1$, $r_{ij} = r_i^{\delta_{ij}}$, $s_{ij} = s_i^{\delta_{ij}}$, where $r_i, s_i \in \mathbb{k}\backslash\{0\}$, $i = 1, \ldots, n$. Then $A_n^k(r, s, \Lambda)$ is isomorphic to $A_{\underline{r},\underline{s}}(n)$.

Proposition

(Benkart) When the parameters r_i, s_i are generic, the multiparameter Weyl algebra $A_{\underline{r},\underline{s}}(n)$ is isomorphic to the degree n generalized Weyl algebra over D, where D is the $\mathbb{k}$-algebra generated by the elements $\rho_i, \rho_i^{-1}, \sigma_i, \sigma_i^{-1}$, $i = 1, \ldots, n$, subject to the relations in (R1). Thus, $A_{\underline{r},\underline{s}}(n)$ is Noetherian domain.

Let $\bar{q} = (q_1, \dots, q_n)$ be an n-tuple of elements of $\Bbbk \backslash \{0\}$. Let $\Lambda = (\lambda_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with $\lambda_{ij} \in \Bbbk \backslash \{0\}$, such that: $\lambda_{ij}\lambda_{ji} = 1$ for all i, j.
The *multiparameter quantized Weyl algebra of degree n over* $\Bbbk$, denoted $A_n^{\bar{q},\Lambda}(\Bbbk)$, is defined as the unital $\Bbbk$-algebra generated by x_i, y_i, $1 \le i \le n$ subject to the following defining relations:

$$y_i y_j = \lambda_{ij} y_j y_i, \qquad \forall i, j, \tag{3.9}$$

$$x_i x_j = q_i \lambda_{ij} x_j x_i, \qquad i < j, \tag{3.10}$$

$$x_i y_j = \lambda_{ji} y_j x_i, \qquad i < j, \tag{3.11}$$

$$x_i y_j = q_j \lambda_{ji} y_j x_i, \qquad i > j, \tag{3.12}$$

$$x_i y_i - q_i y_i x_i = 1 + \sum_{k=1}^{i-1} (q_k - 1) y_k x_k, \qquad \forall i. \tag{3.13}$$

For $\mathbb{k} = \mathbb{C}$ and $q_1 = \cdots = q_n = \mu^2$, $\lambda_{ji} = \mu \ \forall j < i$, where $\mu \in \mathbb{k} \backslash \{0\}$, the algebra $A_n^{\bar{q},\Lambda}(\mathbb{k})$ is isomorphic to the quantized Weyl algebra introduced by Pusz and Woronowicz.

MQWA can be realized as a TGWA. Let $P = \mathbb{k}[s_1, \ldots, s_n]$ and $\tau_i \in \operatorname{Aut} P$ defined by

$$\tau_i(s_j) = \begin{cases} s_j, & j < i, \\ 1 + q_i s_i + \sum_{k=1}^{i-1}(q_k - 1)s_k, & j = i, \\ q_i s_j, & j > i. \end{cases} \tag{3.14}$$

Let $\mu = (\mu_{ij})_{i,j=1}^n$ be such that

$$\mu_{ij} = \begin{cases} \lambda_{ji}, & i < j, \\ q_j \lambda_{ji}, & i > j. \end{cases} \tag{3.15}$$

Put $\tau = (\tau_1, \ldots, \tau_n)$ and $s = (s_1, \ldots, s_n)$. Then $\mathcal{A}_\mu(P, \tau, s)$ is $\mathbb{k}$-finitistic, $p_{ij}(x) = x - 1$ for $i < j$ and $p_{ij}(x) = x - q_i$ for $j > i$, so it is of type $(A_1)^n$, and $\mathcal{A}_\mu(P, \tau, s)$ is isomorphic to $A_n^{\bar{q},\Lambda}(\mathbb{k})$ via $X_i \mapsto x_i$, $Y_i \mapsto y_i$ and $s_i \mapsto y_i x_i$.

Identify P with its image in $A_n^{\bar{q},\Lambda}$ via $s_i \mapsto y_i x_i$. Consider

$$z_i = 1 + \sum_{k \le i} (q_k - 1) s_k, \qquad i = 1, \ldots, n. \tag{3.16}$$

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The set $S := \{z_1^{k_1} \cdots z_n^{k_n} \mid k_1, \ldots, k_n \in \mathbb{Z}\} \setminus \{0\}$ is an Ore set, and provided that none of the q_i is a root of unity, the algebra

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Then we have

$$B_n^{\bar{q},\Lambda} \simeq S^{-1} A_n^{\bar{q},\Lambda} \simeq S^{-1} \mathcal{A}_\mu(P, \tau, s) \simeq \mathcal{A}_\mu(S^{-1}P, \widetilde{\tau}, s).$$

Algebra $B_n^{\bar{q},\Lambda}$ fits into the framework of multiparameter twisted Weyl algebras. Let $\bar{q} = (q_1, \ldots, q_n) \in (\mathbb{k}\backslash\{0\})^n$, $\Lambda = (\lambda_{ij})_{i,j=1}^n$ with $\lambda_{ij} \in \mathbb{k}\backslash\{0\}$, $\lambda_{ii} = 1$, $\lambda_{ij}\lambda_{ji} = 1$ for all i, j (none of the q_i is a root of unity). Let

$$r_{ij} = \begin{cases} 1, & j \le i \\ q_i^{-1}, & j > i \end{cases} \qquad s_{ij} = \begin{cases} 1, & j < i \\ q_i^{-1}, & j \ge i \end{cases} \tag{3.17}$$

Let $A_n^k(r, s, \Lambda) = \mathcal{A}_\mu(R, \sigma, t)$ where

$$R = \mathbb{k}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}, v_1^{\pm 1}, \ldots, v_n^{\pm 1}], \tag{3.18}$$

$$\sigma_i(u_j) = r_{ij}^{-1} u_j, \qquad \sigma_i(v_j) = s_{ij}^{-1} v_j, \tag{3.19}$$

$$t_i = \frac{u_i - q_i^{-1} v_i}{1 - q_i^{-1}}, \tag{3.20}$$

$$\mu_{ij} = r_{ji}^{-1} \lambda_{ji}, \forall i, j. \tag{3.21}$$

$$A_n^k(r,s,\Lambda) = \mathcal{A}_\mu(R,\sigma,t)$$

$$\frac{A_n^k(r,s,\Lambda)}{\langle J \rangle} \simeq \mathcal{A}_\mu(R/J,\bar{\sigma},\bar{t}) \underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}} \mathcal{A}_\mu(S^{-1}P,\widetilde{\tau},s) \simeq B_n^{\bar{q},\Lambda}$$

$$\mathcal{A}_\mu(P,\tau,s) \simeq A_n^{\bar{q},\Lambda}$$

THANK YOU!