A SHORT PROOF OF ZELMANOV'S THEOREM ON LIE ALGEBRAS WITH AN ALGEBRAIC ADJOINT REPRESENTATION

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Abstract. Zelmanov's theorem on Lie PI-algebras with an algebraic adjoint representation is here revisited. New tools and recent results on Jordan structures in Lie algebras are used to shorten and simplify the proof.

1. Introduction

A celebrated theorem due to Zelmanov [16] proves that a Lie algebra over a field of characteristic zero with an algebraic adjoint representation and satisfying a polynomial identity is locally finite-dimensional. In this note we shorten and simplify the proof of this result by using new tools. Thus, we replace Jordan pairs by Jordan algebras at a Jordan element [5], which fit in much better with the transference between Lie and Jordan properties. Also, we use the socle theory [4], the structure theorem of simple finitary Lie algebras [1], and the fact, proved in [8], that any nondegenerate Lie algebra is a subdirect product of strongly prime Lie algebras, which allows us to reduce the proof to the strongly prime case, thus avoiding the difficulty in the original Zelmanov's proof (see at the bottom of page 550 of [16]) of transferring primitive ideals of the semiprimitive Jordan pair defined by a finite grading in the Lie algebra $\overline{L}$ to the whole $\overline{L}$. Thus, the deep results on Lie algebras with finite gradings [16, pages 543-548] needn't be used in this new approach.

2. Lie algebras and Jordan algebras

1. Throughout this note, and unless specified otherwise, we will be dealing with Lie algebras L [10], with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x, and with Jordan algebras J [11], with Jordan product $x \cdot y$, multiplication operators $\lambda_x : y \mapsto x \cdot y$, and quadratic operators $U_x = 2\lambda_x^2 - \lambda_{x^2}$, over a field Φ of characteristic 0. We set

$$[x_1] := x_1 \quad \text{and} \quad [x_1, x_2, \ldots, x_n] := [x_1, [x_2, \ldots, x_n]]$$

for $n > 1$ and $x_1, \ldots x_n \in L$. Similarly, we set

$$x_1 \cdot x_2 \cdots x_n := x_1 \cdot (x_2 \cdots x_n)$$

for $n > 1$ and $x_1, \ldots, x_n \in J$.

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Lemma 2.1. *For each positive integer n there exists a function $f_n : S_n \to \{0, 1, -1\}$ such that, for any $x_1, x_2, \dots x_n, y$ in any Lie algebra L,*

$$\mathrm{ad}_{[x_1,\dots,x_n]} y = \sum_{\sigma \in S_n} f_n(\sigma)[x_{\sigma(1)}, \dots, x_{\sigma(n)}, y].$$

Proof. By induction on n. The case $n = 1$ is trivial. Now

$$\mathrm{ad}_{[x_1,x_2,\dots,x_{n+1}]} y = \mathrm{ad}_{[x_1,[x_2,\dots,x_{n+1}]]} y = \mathrm{ad}_{x_1} \mathrm{ad}_{[x_2,\dots,x_{n+1}]} y - \mathrm{ad}_{[x_2,\dots,x_{n+1}]} \mathrm{ad}_{x_1} y.$$

Hence, by the induction hypothesis,

$$\begin{aligned} \mathrm{ad}_{[x_1,\dots,x_{n+1}]} y &= \sum_{\sigma \in S_n} f_n(\sigma)[x_1, x_{\sigma(2)}, \dots, x_{\sigma(n+1)}, y] \\ &- \sum_{\sigma \in S_n} f_n(\sigma)[x_{\sigma(2)}, \dots, x_{\sigma(n+1)}, x_1, y] \\ &= \sum_{\tau \in S_{n+1}} f_{n+1}(\tau)[x_{\tau(1)}, \dots, x_{\tau(n+1)}, y]. \end{aligned}$$

□

2. An *inner ideal* of J is a vector subspace B of J such that $U_B J \subseteq B$. Similarly, an *inner ideal* of L is a vector subspace B of L such that $[B, [B, L]] \subseteq B$. An *abelian inner ideal* of L is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$. Natural examples of abelian inner ideals occur in finite gradings: The extreme subspaces L_α and $L_{-\alpha}$ of a finite grading in L (see [16, page 543]) are abelian inner ideals of L.

3. An element $x \in L$ is called *Engel* if ad_x is a nilpotent operator. In this case, the nilpotence index of ad_x is called the *index* of x. Engel elements of index at most 3 are called *Jordan elements.* Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, by [3, Lemma 1.8], any Jordan element x generates the abelian inner ideal $\mathrm{ad}_x^2 L$. A good reason for this terminology is the following analogue of the fundamental identity for Jordan algebras:

$$\mathrm{ad}^2_{\mathrm{ad}_x^2 y} = \mathrm{ad}_x^2 \mathrm{ad}_y^2 \mathrm{ad}_x^2$$

which holds for any Jordan element x and any $y \in L$ [3, Lemma 1.7(iii)]. Another reason is given in the next proposition [5, Theorem 2.4].

Proposition 2.2. *Let $a \in L$ be a Jordan element. Then L with the new product defined by $x \cdot_a y := \frac{1}{2}[[x, a], y]$ is a nonassociative algebra denoted by $L^{(a)}$, such that*

(i) $\mathrm{Ker}_L a := \{x \in L | [a, [a, x]] = 0\}$ *is an ideal of $L^{(a)}$.*
(ii) $L_a := L^{(a)}/\mathrm{Ker}_L a$ *is a Jordan algebra, called* the Jordan algebra of L at a.

4. A well-known lemma due to Kostrikin [12, Lemma 2.1.1] provides a method to construct Jordan elements by means of Engel elements, namely, if $x \in L$ is an Engel element of index n then, for any $a \in L$, $\mathrm{ad}_x^{n-1} a$ is Engel of index $\leq n - 1$. Recently, García and Gómez have given the following refinement of this result [7, Theorem 2.3 and Corollary 2.4].

Lemma 2.3. *If $x \in L$ is an Engel element of index n, then $\mathrm{ad}_x^{n-1} L$ is an abelian inner ideal of L. Hence, $\mathrm{ad}_x^{n-1} a$ is a Jordan element for any $a \in L$.*

5. An element $x \in J$ is called an *absolute zero divisor* if $U_x = 0$. Thus J is said to be *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $U_B B = 0$ implies $B = 0$, and *prime* if $U_B C = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of J. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[B, B] = 0$ implies $B = 0$, and *prime* if $[B, C] = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of L. A Jordan or Lie algebra is *strongly prime* if it is prime and nondegenerate. *Simplicity,* for both Jordan and Lie algebras, means nonzero product and the absence of nonzero proper ideals.

6. The adjoint representation of a Lie algebra L is said to be *algebraic* if ad_x is an algebraic operator for each x in L. It has proved in [14] that a Lie algebra whose adjoint representation is algebraic contains a maximal locally finite-dimensional ideal and the quotient algebra over this ideal has no nonzero locally finite-dimensional ideals.

7. Following [12, Definition 5.4.1], the least ideal of a Lie algebra L whose associated quotient algebra is nondegenerate is called the *Kostrikin radical* of L. We denote it by $K(L)$. Put $K_0(L) = 0$ and let $K_1(L)$ the ideal generated by all absolute zero divisors. Using transfinite induction we define a nondecreasing chain of ideals $K_\alpha(L)$ by putting $K_\alpha(L) = \bigcup_{\beta<\alpha} K_\beta(L)$ if α is a limit ordinal, and $K_\alpha(L)/K_{\alpha-1}(L) = K_1(L/K_{\alpha-1}(L))$ otherwise. It is obvious that $K(L) = \bigcup_\alpha K_\alpha(L)$.

The following result, proved by Grishkov in [9], can be found translated to English in [12, Theorem 5.4.2].

Theorem 2.4. *Let L be a Lie algebra over a field of characteristic zero. Then $K_1(L)$ is locally nilpotent. Hence, simple Lie algebras over a field of characteristic zero are nondegenerate.*

The following characterization of the Kostrikin radical was proved in [8, Theorem 3.10].

Theorem 2.5. *The Kostrikin radical $K(L)$ of a Lie algebra L over a field of characteristic zero is the intersection of all strongly prime ideals of L. Therefore, L is nondegenerate if, and only if, it is a subdirect product of strongly prime Lie algebras.*

8. The *socle* of a nondegenerate Jordan algebra is the sum of all its minimal inner ideals [13]. The *socle* of a nondegenerate Lie algebra L, $\mathrm{Soc}\, L$, is defined as the sum of all minimal inner ideals of L [4]. By [13, Theorem 17] (for Jordan algebras) and [4, Theorem 2.5] (for Lie algebras), the socle of a nondegenerate Jordan algebra or Lie algebra is the direct sum of its minimal ideals, each of which is a simple Jordan or Lie algebra.

9. Let L be a Lie algebra over a field Φ.

(i) L is said to be *finitary* (over Φ) if it is a subalgebra of the Lie algebra $\mathfrak{f}gl(X)$ consisting of all finite rank operators on a vector space X over Φ.

(ii) A nonzero element $x \in L$ is said to be *extremal* if $\mathrm{ad}_x^2 L = \Phi x$, that is, if it generates a one-dimensional inner ideal.

Infinite dimensional simple finitary Lie algebras over a field Φ of characteristic zero were described by Baranov in [1, Theorem 1.1]. If, additionally, Φ is algebraically closed, we have the following elementary characterization [4, Corollary 5.5].

Theorem 2.6. *Let L be a simple Lie algebra over an algebraically closed field Φ of characteristic zero. Then L is finitary if, and only if, it contains an extremal element.*

10. Let A any nonassociative algebra over a field F. Then F is said to be a *large field* for A if card $F > \dim_F A + 1$.

The proof of the following useful result, known as Amitsur's cardinality trick, can be found in [11, Proposition 4.5.9].

Proposition 2.7. *Let J be a division Jordan algebra over a large algebraically closed field F. Then $J = F1$.*

Corollary 2.8. *Let L be a nondegenerate Lie algebra over a large algebraically closed field F of characteristic zero. Then any abelian minimal inner ideal B of L is one-dimensional, so any nonzero element of B is extremal.*

Proof. Let x be a nonzero element of B. Then $\mathrm{ad}_x^2 L = B$ and x is a Jordan element. By [5, (2.14)] together with the minimality of B, the Jordan algebra L_x of L at x has no nonzero proper ideals, that is, it is a division Jordan algebra, and by [5, Proposition 2.15(ii)], any $y \in L$ such that $[[x, y], x] = 2x$ yields the identity element $\overline{y}$ of L_x. By Proposition 2.7, $L_x = F\overline{y}$, and hence $B = \mathrm{ad}_x^2 L = Fx$. □

3. The theorem

Every finitary Lie algebra $L \leq \mathfrak{f}gl(X)$ (over a field Φ) has an algebraic adjoint representation. Indeed, any finite rank operator $a \in \mathcal{F}(X)$ is algebraic and hence the left multiplication $\lambda_a : b \mapsto ab$ and the right multiplication $\rho_a : b \mapsto ba$, $b \in \mathcal{F}(X)$ are algebraic operators, which implies that $\mathrm{ad}_a = \lambda_a - \rho_a$ is algebraic, since $[\lambda_a, \rho_a] = 0$. Moreover, if Φ has characteristic zero and L is simple, then we can use [1, Theorem 1.1] to prove that L is finite-dimensional whenever it satisfies a polynomial identity. As will be seen below, the converse is true for strongly prime Lie algebras over a large algebraically closed field of characteristic zero.

Lemma 3.1. *Assume that Φ is an algebraically closed field of characteristic zero and let L be a Lie algebra over Φ. If there exists a nonzero element in L whose adjoint is algebraic, then L contains a nonzero Jordan element.*

Proof. Let x be a nonzero element of L such that ad_x is algebraic. Then ad_x has a Jordan-Chevalley decomposition in $\mathrm{End}_\Phi L$ [10, Section 2.4.2]. If x is Engel of index, say n, we have by Lemma 2.3 together with the nondegeneracy of L that $\mathrm{ad}_x^{n-1} a$ is a nonzero Jordan element for some $a \in L$. Otherwise, the semisimple part of ad_x, which is a derivation of L, is nonzero and hence it yields a nontrivial finite grading on L. As previously noted, any element in any of the extreme subspaces of a finite grading is actually a Jordan element. □

Proposition 3.2. *Let $L \neq 0$ be a strongly prime Lie algebra over an algebraically closed field F of characteristic zero which is large for L. If L contains a nonzero Jordan element and satisfies a polynomial identity of degree n, then L is isomorphic to one of the algebras $G_2, F_4, E_6, E_7, E_8, A_r$, B_r, C_r, or D_r, $r \leq [n/2]$.*

Proof. Let $a \in L$ be a nonzero Jordan element. By [6, Theorem 2.2(i)], the Jordan algebra L_a of L at a inherits strong primeness of L. Moreover, if

$$p(x_{11}, \dots, x_{m,r_m}) = \alpha_1[x_{11}, \dots, x_{1r_1}] + \dots + \alpha_m[x_{m1}, \dots, x_{mr_m}]$$

is a polynomial identity for L, then

$$\mathrm{ad}_{p(x_{11},\dots,x_{m,r_m})}y$$

is also a polynomial identity for L, where y is a new variable, $y \neq x_{ij_i}$ for all $1 \leq i \leq m$ and $1 \leq j_i \leq r_i$. Then, by Lemma 2.1, L satisfies a polynomial identity which is a linear combination of monomials of the form $[x_{j_1}, \dots x_{j_r}, y]$. Replacing in each of these monomials, x_{j_k} by $[x_{j_k}, a]$ yields a polynomial identity for the Jordan algebra L_a which is a linear combination of the Jordan monomials $x_{j_1} \cdots x_{j_r} \cdot y$. Thus L_a is a strongly prime Jordan PI-algebra. Let $Z(L_a)$ denote the center of L_a. By [15, Theorems 5 and 7], L_a has nonzero center and $Z(L_a)^{-1}L_a$ is a simple Jordan algebra containing minimal inner ideal. Since F is large for L, it is also a large field for the Jordan algebra L_a. Hence, by Atmisur's cardinality trick, Proposition 2.7, the field of fractions $Z(L_a)^{-1}Z(L_a)$ is equal to F, and L_a itself is a simple Jordan algebra with nonzero socle. Since minimal inner ideals of L_a give rise to abelian minimal inner ideals of L [5, Proposition 2.15(iii)], L contains abelian minimal inner ideals, and hence extremal elements by Corollary 2.8. Since L is strongly prime, $\mathrm{Soc}\, L$ is a simple Lie algebra, [4, Theorem 2.5(i)], containing extemal elements, so $\mathrm{Soc}\, L$ is finitary by Theorem 2.6. We claim that $\mathrm{Soc}\, L$ is actually finite-dimensional. Otherwise, by Baranov's structure theorem for infinite-dimensional simple finitary Lie algebras over an algebraically closed field of characteristic zero [1, Corollary 1.2], for any positive interger m, $\mathrm{Soc}\, L$ contains a subalgebra isomorphic to either A_m, C_m or D_m, which leads to a contradiction, since no matrix algebra $M_r(F)$ satisfies an identity of degree less than $2r$ and the Lie algebra $M_r(F)^-$ can be embedded in each one of the simple Lie algebras A_r, C_r, or D_r. Thus $\mathrm{Soc}\, L$ is isomorphic to one of the simple finite-dimensional Lie algebra listed in the claim of the proposition. Finally, L can be embedded in $\mathrm{Der}(\mathrm{Soc}\, L)$ via the adjoint representation, and hence $L = \mathrm{Soc}\, L$ since derivations of simple finite-dimensional Lie algebras over a field of characteristic zero are inner. □

Remark 3.3. It is possible to prove that $\mathrm{Soc}\, L$ is finite-dimensional, c.f. Proposition 3.2, without dealing with finitary algebras. The mere existence of an extremal element implies by [16, Lemma 15] that $\mathrm{Soc}\, L$ is locally finite-dimensional, and Bakthturin proves in [2] that a simple Lie algebra (over a field of characteristic zero) which is locally finite-dimensional and satisfies a nontrivial identity over its centroid is finite-dimensional over its centroid. By Proposition 2.7, the centroid of $\mathrm{Soc}\, L$ coincides with F; hence $\mathrm{Soc}\, L$ is finite-dimensional.

Proposition 3.4. *Let L be a nondegenerate Lie algebra over an algebraically closed field F of characteristic zero which is large for L. If L satisfies a polynomial identity of degree n and it is spanned by elements with an algebraic adjoint, then L is a subdirect product of finite-dimensional simple Lie algebras each of which isomorphic to one of the algebras $G_2, F_4, E_6, E_7, E_8, A_r$, B_r, C_r, or D_r, where $r \leq [n/2]$.*

Proof. By Theorem 2.5, L is a subdirect product of a family $\{L_i\}_{i\in I}$ of strongly prime Lie algebras L_i. Moreover, each L_i satisfies a polynomial identity of degree n and it is spanned by elements with an algebraic adjoint. Hence, by Proposition 3.2 together with Lemma 3.1, L_i is isomorphic to either $G_2, F_4, E_6, E_7, E_8, A_r$, B_r, C_r, or D_r, where $r \leq [n/2]$. □

Theorem 3.5. (Zelmanov) *Let L be a Lie algebra over a field Φ of characteristic zero. If L has an algebraic adjoint representation and satisfies a polynomial identity, then L is locally finite-dimensional.*

Proof. As in the original Zelmanov's proof, we make two reductions and take an algebraically closed extension of Φ of sufficiently large cardinality:

(i) By factorizing by the largest locally finite-dimensional ideal (6), it suffices to prove that L contains a nonzero locally finite-dimensional ideal. (ii) Since $K_1(L)$ is locally nilpotent (see Theorem 2.4), it is locally finite-dimensional, so we may suppose that L is nondegenerate. Let F be an algebraically closed extension of Φ of sufficiently large cardinality: card $F > \dim_\Phi L + 1$ and set $\tilde{L} = L \otimes_\Phi F$. By [16, Propositon 2], $K(\tilde{L}) \cap L \subseteq K(L) = 0$, and hence we may assume that L is embedded in the quotient algebra $\overline{L} = \tilde{L}/K(\tilde{L})$. Then $\overline{L}$ is a nondegenerate Lie algebra (over F). Moreover, $\overline{L}$ satisfies a polynomial identity (of degree, say n) and it is spanned by elements with an algebraic adjoint. The desired conclusion is now achieved in a quite simple way:

By Proposition 3.4, $\overline{L}$ is a subdirect product of finite-dimensional simple Lie algebras each of which isomorphic to either $G_2, F_4, E_6, E_7, E_8, A_r,\ B_r,\ C_r,$ or $D_r,\ r \leq [n/2]$. Since L is embedded in $\overline{L}$, it satisfies all the identities of a finite-dimensional Lie algebra $F_4 \oplus E_8 \oplus A_m$. Hence L is locally-finite dimensional by [12, Theorem 5.4.6]. □

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