Finite-dimensional Lie algebras arising from a Nichols algebras of diagonal type

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Plan of the talk.

I. Preliminaries.

II. Nichols algebras of diagonal type.

III. Pre-Nichols and Post-Nichols algebras.

IV. An exact sequence and Lie algebras.

I. Preliminaries.

Definitions. (V, c) braided vector space: V is a vector space and $c \in GL(V \otimes V)$ satisfies the braid equation,

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$

Γ **abelian group**; ${}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD} =$ Yetter-Drinfeld modules over $\mathbb{C}\Gamma$:

- $V = \oplus_{g \in \Gamma} V_g$ is a Γ-graded vector space;

- V is a left Γ-module such that $g \cdot V_h = V_h$ (compatibility).

$V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD} \implies V$ braided vector space:

$$c(v \otimes w) = g \cdot w \otimes v, \qquad v \in V_g, w \in V.$$

Given $V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$, the **Nichols algebra** of V is the graded Hopf algebra $\mathcal{B}(V) = \oplus_{n\geq 0}\mathcal{B}^n(V)$ in ${}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ such that

1. $\mathcal{B}^0(V) = \mathbb{C}$, $\qquad \mathcal{B}^1(V) = P(V) := \{x : \Delta(x) = x \otimes 1 + 1 \otimes x\}$,

2. $\mathcal{B}(V) = \mathbb{C}\langle \mathcal{B}^1(V) \rangle$.

$\mathcal{B}(V) \simeq T(V)/\mathcal{J}(V)$, where $\mathcal{J}(V)$ maximal graded Hopf ideal generated by elements in $\oplus_{n\geq 2}\mathcal{B}^n(V)$.

Problem: Determine when $\dim \mathcal{B}(V) < \infty$, or $\mathrm{GKdim}\, \mathcal{B}(V) < \infty$; and the structure of the ideal $\mathcal{J}(V)$ (it depends crucially on the braiding c on V).

II. Nichols algebras of diagonal type.

In this talk, we are interested in the following class:

(V, c) is of **diagonal type**: $\exists$ a basis $(x_i)_{i\in\mathbb{I}_\theta}$ and $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$, $q_{ii} \neq 1$ for all i, with connected diagram, such that

$$c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i, \qquad\qquad i, j \in \mathbb{I}_\theta.$$

$V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ semisimple (and f. d.) $\implies$ (V, c) is of diagonal type.

Theorem (Heckenberger). V of diagonal type, $\dim \mathcal{B}(V) < \infty$ classified.

Remark. [AAH] $\dim \mathcal{B}(V) < \infty \implies V$ of diagonal type (Γ abelian).

$$\begin{tikzcd}
& \text{Cartan type} & \\
\text{List of Heckenberger} \arrow[ur] \arrow[rr] \arrow[d] \arrow[dr] & & \text{super type} \\
\text{11 UFO's} & \text{(super) modular type} &
\end{tikzcd}$$

$(q_{ij})_{i,j\in\mathbb{I}} \rightsquigarrow \mathcal{B}(V)$

$(q_{ij})_{i,j\in\mathbb{I}}$ in $\mathbb{G}_\infty^{\mathbb{I}\times\mathbb{I}}$, $1 \neq q_{ii}$, is of *Cartan type* if for all $i \neq j \in \mathbb{I}$, there exists $a_{ij} \in \mathbb{Z}$, $-\operatorname{ord} q_{ii} < a_{ij} \leq 0$ such that

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}.$$

Set $a_{ii} = 2$. Then (a_{ij}) is a generalized Cartan matrix.

Theorem. (N. A.–H.-J. Schneider). (V, c) Cartan type, with matrix $(q_{ij})_{i,j\in\mathbb{I}}$. Assume $(\operatorname{ord} q_{ij}, 210) \overset{(\#)}{=} 1$ for all i, j. Then

$$\dim \mathcal{B}(V) < \infty \overset{(*)}{\Longleftrightarrow} (a_{ij}) \text{ of finite type.}$$

If this is the case, $\mathcal{B}(V) = \mathbb{C}\langle x_1, \dots, x_\theta \rangle$ with relations:

(Serre)	$\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0$, $i \neq j$
(Powers root vect.)	$x_\alpha^{N_J} = 0$.

- Original proof uses Lusztig's results.
- The conclusion $(*)$ holds without the assumption $(\#)$ (Heckenberger) but the defining relations are more involved (Angiono).
- If $(|\mathbb{G}|, 210) = 1$, then the possible V are of Cartan type (Heckenberger).

Fix $\theta \in \mathbb{N}$, $\mathbb{I} = \{1, 2, \dots, \theta\}$.

Basic datum: $(\mathcal{X}, \rho)$: a set $\mathcal{X} \neq \emptyset$, together with $\rho : \mathbb{I} \to \mathbb{S}_{\mathcal{X}}$ such that $\rho_i^2 = \mathrm{id}$ for all $i \in \mathbb{I}$.

The set $\mathcal{X}$ will be the basis of various sorts of bundles and ρ commands the symmetries that they may have:

$$\begin{array}{c} \mathcal{E} \\ \big\downarrow \\ \mathcal{X} \end{array}$$

$C = (c_{ij})_{i,j\in\mathbb{I}} \in \mathbb{Z}^{\theta\times\theta}$ is a generalized Cartan matrix (GCM) if

$$c_{ii} = 2, \quad c_{ij} = 0 \iff c_{ji} = 0, \quad c_{ij} \le 0 \quad \text{for all } i \neq j \in \mathbb{I}.$$

A **semi-Cartan graph** is a family of GCM $\mathcal{C} = (C^x)_{x\in\mathcal{X}}$, $C^x = (c^x_{ij})_{i,j\in\mathbb{I}}$, $x \in \mathcal{X}$, such that $c^x_{ij} = c^{\rho_i(x)}_{ij}$ for all $x \in \mathcal{X}$, $i, j \in \mathbb{I}$.

Let $s^x_i \in GL(\mathbb{Z}^\theta)$: $s^x_i(\alpha_j) = \alpha_j - c^x_{ij}\alpha_i$, $x \in \mathcal{X}$, $i, j \in \mathbb{I}$.

The **Weyl groupoid** $\mathcal{W}(\mathcal{C}) \rightrightarrows \mathcal{X}$ is generated by $s^x_i \in \mathrm{Hom}(x, \rho_i(x))$, $i \in I$, $x \in \mathcal{X}$. So $w \in \mathrm{Hom}(x_m, x_1)$ is a product

$$s^{x_1}_{i_1} s^{x_2}_{i_2} \cdots s^{x_m}_{i_m} \in GL(\mathbb{Z}^\theta),$$

where $x_m = \rho_{i_{m-1}} \cdots \rho_{i_1}(x_1)$.

A **generalized root system** over $\mathcal{C}$ is a collection

$\mathcal{R} = (R^x)_{x\in\mathcal{X}}$, $R^x \subset \mathbb{Z}^\theta$, such that $\forall x \in \mathcal{X}, i \neq j \in \mathbb{I}$

- $R^x = R^x_+ \cup R^x_-$, where $R^x_\pm := \pm(R^x \cap \mathbb{N}_0^\theta)$.
- $R^x \cap \mathbb{Z}\alpha_i = \{\pm\alpha_i\}$,
- $s_i^x(R^x) = R^{\rho_i(x)}$,
- $(\rho_i\rho_j)^{m_{ij}^x}(x) = (x)$, if $m_{ij}^x := |R^x \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)| < \infty$.

$R^{\mathrm{re}\, x} := \{w(\alpha_i) : i \in \mathbb{I}, w \in \cup_{y\in\mathcal{X}} \mathrm{Hom}_{\mathcal{W}}(y,x)\}$ (*real roots* at x).

Theorem. (Heckenberger). If $\dim \mathcal{B}(V) < \infty$, then it has an associated finite Generalized Root System.

Sketch: The semi-Cartan graph is defined by (Rosso)

$$c_{ij}^{\mathfrak{q}} := \begin{cases} 2 & \text{si } i = j; \\ -\min\{n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij} q_{ji}) = 0\} & \text{si } i \neq j. \end{cases}$$

Let $\mathbf{q} : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbb{C}^\times$ be the bilinear form $\mathbf{q}(\alpha_j, \alpha_k) = q_{jk}$ $j, k \in \mathbb{I}$.

$$\rho_i(\mathfrak{q})_{jk} = \mathbf{q}(s_i^{\mathfrak{q}}(\alpha_j), s_i^{\mathfrak{q}}(\alpha_k)), \qquad j, k \in \mathbb{I}.$$

$$\rho_i(V) = \text{ braided v. sp. of diagonal type with matrix } \quad \rho_i(\mathfrak{q}).$$

$$\mathcal{X} = \{\rho_{i_1} \dots \rho_{i_N}(V) : N \in \mathbb{N}_0, i_1, \dots, i_N \in \mathbb{I}\}.$$

$$R^W = \{\text{degrees of the generators of a PBW-basis of } \mathcal{B}(W)\}, \quad W \in \mathcal{X}.$$

(existence of PBW-basis by Kharchenko's theorem).

From now on $\mathfrak{q}$ is assumed to be symmetric.

Let $\mathfrak{u}(V) = \mathcal{B}(V) \otimes \mathbb{C}\mathit{\Gamma} \otimes \mathcal{B}(V)$ = Hopf algebra with triangular decomposition associated to V, $\mathit{\Gamma}$.

Theorem. (Heckenberger). If $i \in \mathbb{I}$, then there is an algebra isomorphism $T_i : \mathfrak{u}(V) \to \mathfrak{u}(\rho_i(V))$ (Lusztig-like isomorphism).

This gives a representation of the Weyl groupoid of V.

Modular contragredient Lie superalgebras

$\Bbbk$ field char $\Bbbk = \ell \neq 2$, $\quad A = (a_{ij}) \in \mathbb{C}^{\theta\times\theta}$,

$\mathbf{p} = (p_i) \in (\mathbb{Z}/2)^{\theta}$, *parity vector.* $\quad \mathfrak{h}$ $\Bbbk$-v. sp., $\dim \mathfrak{h} = 2\theta - \operatorname{rk} A$,

$\xi_j \in \mathfrak{h}^*$, $h_i \in \mathfrak{h}$ L. I., such that $\xi_j(h_i) = a_{ij}$ for all $i, j \in \mathbb{I}$.

∘ $\widetilde{\mathfrak{g}} := \widetilde{\mathfrak{g}}(A, \mathbf{p}) =$ Lie superalgebra generated by e_i, f_i, $i \in \mathbb{I}$, and $\mathfrak{h}$, with parity and relations

$$
\begin{aligned}
|e_i| = |f_i| &= p_i,\ i \in \mathbb{I}, & |h| &= 0,\ h \in \mathfrak{h},\\
[h, e_j] &= \xi_j(h) e_j, & [h, h'] &= 0,\\
[h, f_j] &= -\xi_j(h) f_j, & [e_i, f_j] &= \delta_{ij} h_i.
\end{aligned}
$$

∘ $\mathfrak{r} :=$ maximal ideal among those that intersect $\mathfrak{h}$ trivially;
∘ $\mathfrak{g}(A, \mathbf{p}) := \widetilde{\mathfrak{g}}(A, \mathbf{p})/\mathfrak{r}$ *contragredient Lie superalgebra*.

Theorem. (N. A.–I. Angiono).

(a) If $\dim \mathfrak{g}(A, \mathbf{p}) < \infty$, then it has an associated finite Generalized Root System.

(b) Each of these GRS also appears as a GRS of a Nichols algebra of diagonal type.

Remark. There are still 11 unidentified braided vector spaces of diagonal type in the list.

III. Pre-Nichols and Post-Nichols algebras.

$R = \oplus_{n\geq 0} R^n \in {}^H_H\mathcal{YD}$ graded connected Hopf algebra, $V = R^1$

R is a Nichols algebra when

♣ **R is coradically graded $\iff$ $P(R) = V$.**

♦ **R is generated by V.**

R is a **pre-Nichols** algebra when ♦ holds (Masuoka).

R is a **post-Nichols** algebra when ♣ holds.

$$
\begin{tikzcd}
T(V) \arrow[rr] \arrow[dr] \arrow[rrrr, bend left, "\Omega=\oplus_{n\geq 2}\Omega_n"] & & \mathcal{B}(V) \arrow[rr] \arrow[dr] & & T^c(V) \\
& \mathcal{B} \arrow[ur] & & \mathcal{E} \arrow[ur] &
\end{tikzcd}
$$

$T(V)$ free algebra $\quad T^c(V)$ free coalgebra $\quad \mathcal{B}(V)$ Nichols alg.

Ω_n quantum symm. $\quad \mathcal{B}$ pre-Nichols $\quad \mathcal{E}$ post-Nichols

$\mathfrak{Pre}(V)$: poset of pre-Nichols, $\leq$ is $\twoheadrightarrow$; min. $T(V)$, max. $\mathcal{B}(V)$.

$\mathfrak{Post}(V)$: poset of post-Nichols, $\leq$ is $\subseteq$; min. $\mathcal{B}(V)$, max. $T^c(V)$.

($\dim V < \infty$): $\Phi : \mathfrak{Pre}(V) \to \mathfrak{Post}(V^*)$, $\Phi(R) = R^d$, anti-isom.

Post-Nichols algebras with finite GKdim are relevant for the classification of Hopf algebras with finite GKdim.

Let K be a Hopf algebra, $V \in {}^K_K\mathcal{YD}$ finite-dimensional.

Lemma. $\mathcal{B}$ a pre-Nichols algebra of a V, $\mathcal{E} = \mathcal{B}^d$. Then $\operatorname{GKdim}\mathcal{E} \leq \operatorname{GKdim}\mathcal{B}$. If $\mathcal{E}$ is fin. gen., then $\operatorname{GKdim}\mathcal{E} = \operatorname{GKdim}\mathcal{B}$.

V is *pre-bounded* if every chain

$$\cdots < \mathcal{B}[3] < \mathcal{B}[2] < \mathcal{B}[1] < \mathcal{B}[0] = \mathcal{B}(V), \tag{1}$$

of pre-Nichols algebras over V with finite GKdim, is finite.

Lemma. $\mathcal{E} \in {}^K_K\mathcal{YD}$ post-Nichols algebra of V, $\operatorname{GKdim}\mathcal{E} < \infty$. If V^* is pre-bounded, then $\mathcal{E}$ is fin. gen. and $\operatorname{GKdim}\mathcal{E} = \operatorname{GKdim}\mathcal{E}^d$.

In particular, if the only pre-Nichols algebra of V^* with finite GKdim is $\mathcal{B}(V^*)$, then $\mathcal{E} = \mathcal{B}(V)$.

Examples: the only pre-Nichols or post-Nichols algebra of V with finite GKdim is $\mathcal{B}(V)$ when:

- ◦ V is of finite generic Cartan type,

- ◦ V is of Jordan or super Jordan type.

- • (V, c) $\theta := \dim V < \infty$; $c =$ the usual flip. Then

- • $\mathcal{B}(V) = S(V)$, $T(V) \simeq U(L(V))$ (free Lie algebra on V).
- • $\mathcal{B}$ a pre-Nichols algebra of $V \Rightarrow U(P(\mathcal{B}))$; $P(\mathcal{B})$ graded Lie algebra generated by V.
- • L $\mathbb{N}$-graded Lie algebra generated by $L_1 \simeq V \Rightarrow U(L)$ is a pre-Nichols algebra of V. L finite-dimensional $\Rightarrow$ GKdim $U(L) < \infty$. Such Lie algebras are nilpotent, there are infinitely many.

Distinguished pre-Nichols algebras. $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta} \rightsquigarrow V$.

Assume $\dim \mathcal{B}(V) < \infty$**.** Notation: $\mathcal{B}_{\mathfrak{q}} = \mathcal{B}(V) = T(V)/\mathcal{J}_{\mathfrak{q}}$.

$$c_{ij}^{\mathfrak{q}} := -\min\left\{n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij} q_{ji}) = 0\right\}, \qquad i \neq j.$$

∘ $i \in \mathbb{I}$ is a *Cartan vertex* of $\mathfrak{q}$ if $q_{ij}q_{ji} \overset{(\star)}{=} q_{ii}^{c_{ij}^{\mathfrak{q}}}$, for all $j \neq i$.

∘ $\alpha = s_{i_1}^{\mathfrak{q}} s_{i_2} \dots s_{i_k}(\alpha_i) \in \Delta_+^{\mathfrak{q}}$ Cartan root of $\mathfrak{q}$ if $i \in \mathbb{I}$ Cartan vertex of $\rho_{i_k} \dots \rho_{i_2}\rho_{i_1}(\mathfrak{q})$.

$\mathfrak{S}$ = set of generators of $\mathcal{J}_{\mathfrak{q}}$ (Angiono). Let

$$\mathcal{J}_{\mathfrak{q}} \supset \mathcal{I}_{\mathfrak{q}} := \langle \mathfrak{S} \cup \mathfrak{S}_2 - \mathfrak{S}_1 \rangle$$

$\mathfrak{S}_1$ = {powers root vectors $E_\alpha^{N_\alpha}$, α Cartan root};

$\mathfrak{S}_2$ = {quantum Serre rel.$(\mathrm{ad}_c\, E_i)^{1-c_{ij}^{\mathfrak{q}}} E_j, i \neq j$ s. t. $(\star)$.}

Definition (Angiono). The *distinguished pre-Nichols algebra* of V is $\widetilde{\mathcal{B}}_{\mathfrak{q}} = T(V)/\mathcal{I}_{\mathfrak{q}}$. Let $\widetilde{\mathfrak{u}}(V) = \widetilde{\mathcal{B}}(V) \otimes \mathbb{C}\Gamma \otimes \widetilde{\mathcal{B}}(V)$.

Theorem. (Angiono) (a) GKdim $\widetilde{\mathcal{B}}_{\mathfrak{q}} = |\{\text{Cartan roots}\}|$.

(b) The Lusztig automorphisms can be lifted to $\widetilde{\mathfrak{u}}(V)$.

The *Lusztig algebra* $\mathcal{L}_\mathfrak{q}$ of V is the graded dual of $\widetilde{\mathcal{B}}_\mathfrak{q}$.

$$\begin{tikzcd}
T(V) \arrow[rr] \arrow[dr] & & \mathcal{B}_\mathfrak{q} \arrow[rr] \arrow[dr, hook] & & T^c(V) \\
& \widetilde{\mathcal{B}}_\mathfrak{q} \arrow[ur] & & \mathcal{L}_\mathfrak{q} \arrow[ur] &
\end{tikzcd}$$

Theorem. [AAR]. $\operatorname{GKdim}_\mathfrak{q} \mathcal{L}_\mathfrak{q} = |\{\text{Cartan roots}\}|$.

Example: $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ of Cartan type, i. e. $q_{ij}q_{ji} = q_{ii}^{c_{ij}^\mathfrak{q}}$, for all $j \neq i$. Assume $(\operatorname{ord} q_{ij}q_{ji}, 210) = 1$ for all i, j. Then

$\mathcal{B}_\mathfrak{q} = \mathbb{C}\langle x_1, \dots, x_\theta |$ quantum Serre rel., power root vectors $\rangle$;

$\widetilde{\mathcal{B}}_\mathfrak{q} = \mathbb{C}\langle x_1, \dots, x_\theta |$ quantum Serre rel. $\rangle$

$\operatorname{GKdim} \widetilde{\mathcal{B}}_\mathfrak{q} = \operatorname{GKdim}_\mathfrak{q} \mathcal{L}_\mathfrak{q} = |\Delta^+|$.

IV. An exact sequence and Lie algebras.

Let $\pi : \widetilde{\mathcal{B}}_{\mathfrak{q}} \to \mathcal{B}_{\mathfrak{q}}$ be the projection. Let $Z_{\mathfrak{q}}$ be the subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ generated by $x_{\beta}^{N_\beta}$, β Cartan root.

We need that if α, β are Cartan roots, then $q_{\alpha\beta}^{N_\beta} = 1$.

Theorem. (Angiono). $Z_{\mathfrak{q}} = \widetilde{\mathcal{B}}_{\mathfrak{q}}^{\mathrm{co}\,\pi}$ is a commutative normal Hopf subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ and the following sequence is exact:

$$\mathbb{C} \to Z_{\mathfrak{q}} \xrightarrow{\iota} \widetilde{\mathcal{B}}_{\mathfrak{q}} \xrightarrow{\pi} \mathcal{B}_{\mathfrak{q}} \to \mathbb{C}.$$

Let $\mathfrak{Z}_\mathfrak{q}$ be the graded dual of $Z_\mathfrak{q}$.

Theorem. [AAR] $\mathfrak{Z}_\mathfrak{q}$ is a cocommutative Hopf algebra and the following sequence is exact:

$$\mathbb{C} \to \mathcal{B}_\mathfrak{q} \xrightarrow{\pi^*} \mathcal{L}_\mathfrak{q} \xrightarrow{\iota^*} \mathfrak{Z}_\mathfrak{q} \to \mathbb{C}.$$

$\mathfrak{Z}_\mathfrak{q}$ is the enveloping algebra of the Lie algebra $\mathfrak{n}_\mathfrak{q} = \mathcal{P}(\mathfrak{Z}_\mathfrak{q})$.

This Lie algebra can be explicitly described and computed.

Diagram	Parameter	Type	$\mathfrak{n}_{\mathfrak{q}} = \mathfrak{g}^{+}$
$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$	$q \neq 1$	Cartan A	A_2
$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}$	$q \neq \pm 1$	Super A	A_1
$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$	$q \neq \pm 1$	Cartan B	B_2
$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{-1}{\circ}$	$q \notin \mathbb{G}_4$	Super B	$A_1 \times A_1$
$\overset{\zeta}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$	$\zeta \in \mathbb{G}_3 \not\ni q$	$\mathfrak{br}(2, a)$	$A_1 \times A_1$
$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}_3'$	Standard B	0
$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-\zeta^{2}}{\circ}$	$\zeta \in \mathbb{G}_{12}'$	$\mathfrak{ufo}(7)$	0
$\overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{———}}\overset{-\zeta^{2}}{\circ}$	$\zeta \in \mathbb{G}_{12}'$	$\mathfrak{ufo}(8)$	A_1

Diagram	Parameter	Type	$\mathfrak{n}_{\mathfrak{q}}$
$\overset{-\zeta}{\circ}\overset{\zeta^{-2}}{\rule[0.5ex]{3em}{0.4pt}}\overset{\zeta^{3}}{\circ}$	$\zeta \in \mathbb{G}'_9$	$\mathfrak{brj}(2;3)$	$A_1 \times A_1$
$\overset{q}{\circ}\overset{q^{-3}}{\rule[0.5ex]{3em}{0.4pt}}\overset{q^{3}}{\circ}$	$q \notin \mathbb{G}_2 \cup \mathbb{G}_3$	Cartan G_2	G_2
$\overset{\zeta^{2}}{\circ}\overset{\zeta}{\rule[0.5ex]{3em}{0.4pt}}\overset{\zeta^{-1}}{\circ}$	$\zeta \in \mathbb{G}'_8$	Standard G_2	$A_1 \times A_1$
$\overset{\zeta^{6}}{\circ}\overset{\zeta}{\rule[0.5ex]{3em}{0.4pt}}\overset{\zeta^{-1}}{\circ}$	$\zeta \in \mathbb{G}'_{24}$	$\mathfrak{ufo}(9)$	$A_1 \times A_1$
$\overset{\zeta}{\circ}\overset{\zeta^{2}}{\rule[0.5ex]{3em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_5$	$\mathfrak{brj}(2;5)$	B_2
$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\rule[0.5ex]{3em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_{20}$	$\mathfrak{ufo}(10)$	$A_1 \times A_1$
$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\rule[0.5ex]{3em}{0.4pt}}\overset{\zeta^{5}}{\circ}$	$\zeta \in \mathbb{G}'_{15}$	$\mathfrak{ufo}(11)$	$A_1 \times A_1$
$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\rule[0.5ex]{3em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_7$	$\mathfrak{ufo}(12)$	G_2

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