Nichols algebras of finite Gelfand-Kirillov dimension

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Plan of the talk: I. Preliminaries.

II. Nichols algebras of blocks.

III. Nichols algebras of diagonal type.

IV. Nichols algebras of one block + one point.

V. Nichols algebras of one pale block + one point.

VI. Nichols algebras of one block + several points.

VII. Nichols algebras of two blocks.

VIII. The general picture.

I. Preliminaries.

A finitely generated $\mathbb{C}$-algebra. If V is a finite-dimensional generating subspace of A and $A_{V,n} = \sum_{0\le j\le n} V^n$, then

$$\operatorname{GKdim} A := \overline{\lim}_{n\to\infty} \log_n \dim A_{V,n};$$

it does not depend on the choice of V. if A is not fin. gen., then

$$\operatorname{GKdim} A := \sup\{\operatorname{GKdim} B | B \subseteq A,\ B \text{ finitely generated}\}.$$

Example: A commutative $\Rightarrow$ $\operatorname{GKdim} A \in \mathbb{N}_0 \cup \infty$; if A fin. gen.,

$$\operatorname{GKdim} A = \operatorname{Krull\,dim} A = \dim \operatorname{Spec} A.$$

(Krull dim = sup of the lengths of all chains of prime ideals).

Problem. Classify Hopf algebras H with $\operatorname{GKdim} H < \infty$.

(V, c) braided vector space:
$c \in GL(V \otimes V)$ satisfies the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$

Γ **abelian group**

${}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ = category of Yetter-Drinfeld modules over $\mathbb{C}\Gamma$:

- $V = \oplus_{g \in \Gamma} V_g$ is a Γ-graded vector space;

- V is a left Γ-module such that $g \cdot V_h = V_h$ (compatibility).

$V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD} \implies V$ braided vector space:

$$c(v \otimes w) = g \cdot w \otimes v, \qquad\qquad v \in V_g, w \in V.$$

Given $V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$, the **Nichols algebra** of V is the graded Hopf algebra $\mathcal{B}(V) = \oplus_{n\geq 0}\mathcal{B}^n(V)$ in ${}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ such that

- $\mathcal{B}^0(V) = \mathbb{C}$,
- $\mathcal{B}^1(V) = P(V) := \{x : \Delta(x) = x \otimes 1 + 1 \otimes x\} \simeq V$,
- $\mathcal{B}(V) = \mathbb{C}\langle \mathcal{B}^1(V)\rangle$.

Remark: $\mathcal{B}(V)$ depends (e.g. as graded algebra) only on the underlying braided vector space.

Remark (Graña): $V_1, V_2 \in {}^{H}_{H}\mathcal{YD}$, $V = V_1 \oplus V_2$.

If $c_{V_2,V_1}c_{V_1,V_2} = c^2_{V_1,V_2} = \mathrm{id}_{V_1\otimes V_2} \implies \mathcal{B}(V) \simeq \mathcal{B}(V_1)\underline{\otimes}\mathcal{B}(V_2)$.

Problem. Classify Nichols algebras $\mathcal{B}(V)$ with $\operatorname{GKdim}\mathcal{B}(V) < \infty$.

Remark: R Hopf algebra in ${}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD} \implies R\#\mathbb{C}\Gamma$ Hopf algebra.

Therefore Γ finitely generated, $\dim V < \infty$, $\operatorname{GKdim}\mathcal{B}(V) < \infty \implies$
$H = \mathcal{B}(V)\#\mathbb{C}\Gamma$ Hopf algebra, $\operatorname{GKdim} H < \infty$.

II. Nichols algebras of blocks.

Γ abelian group.

Simple objects in ${}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$:
$\mathbb{C}_g^\chi \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ of dimension 1, homogeneous of degree g, where Γ acts by $\chi \in \widehat{\Gamma} \rightsquigarrow$ point of label $\chi(g)$.

Point of label $q \in \mathbb{C}^\times$: braided v. sp. (V, c) of dim. 1, $c = q\,\mathrm{id}$.

Fact: $\mathcal{B}(V) = \begin{cases} \mathbb{C}[X] & q = 1 \text{ or } \notin \mathbb{G}_\infty, \\ \mathbb{C}[X]/\langle X^N \rangle & q \in \mathbb{G}'_N, \end{cases}$

Notation: $\mathbb{G}_N$ group of N-th roots of unity, $\mathbb{G}'_N$ the subset of primitive roots; $\mathbb{G}_\infty = \bigcup_{N \in \mathbb{N}} \mathbb{G}_N$.

$V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD} \hookrightarrow V = \oplus_{g\in\Gamma} V_g$, each $V_g \in \Gamma-$Mod

Assume $\dim V < \infty \implies V_g$ is a direct sum of indecomposables.

$\mathbb{Z} = \langle \mathbf{g} \rangle$. $\mathcal{V}(\epsilon, \ell) \in {}^{\mathbb{C}\mathbb{Z}}_{\mathbb{C}\mathbb{Z}}\mathcal{YD}$ homogeneous of degree $\mathbf{g}$, dimension $\ell > 1$, the action of $\mathbf{g}$ given by a Jordan block of size ℓ and eigenvalue ϵ. These braided vector spaces are called *blocks*. A block has a basis $(x_i)_{i\in\mathbb{I}_\ell}$,

$$c(x_i \otimes x_j) = \begin{cases} \epsilon x_1 \otimes x_i, & j = 1 \\ (\epsilon x_j + x_{j-1}) \otimes x_i, & j \geq 2, \end{cases} \qquad i \in \mathbb{I}_\ell.$$

Theorem. $\operatorname{GKdim} \mathcal{B}(\mathcal{V}(\epsilon, \ell)) < \infty \iff \ell = 2$ and $\epsilon \in \{\pm 1\}$.

Proposition.
$\mathcal{B}(\mathcal{V}(1,2)) = \mathbb{C}\langle x_1, x_2 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2\rangle$ (Jordan plane).

$\{x_1^a x_2^b : a, b \in \mathbb{N}_0\}$ is a basis of $\mathcal{B}(\mathcal{V}(1,2))$.

GKdim $\mathcal{B}(\mathcal{V}(1,2)) = 2$; $\mathcal{B}(\mathcal{V}(1,2))$ is a domain.

We pass to $\mathcal{V}(-1,2)$. Let $x_{21} = \operatorname{ad}_c x_2\, x_1 = x_2x_1 + x_1x_2$.

Prop. $\mathcal{B}(\mathcal{V}(-1,2)) = \mathbb{C}\langle x_1, x_2 | x_1^2, x_2x_{21} - x_{21}x_2 - x_1x_{21}\rangle$
(super Jordan plane).

$\{x_1^a x_{21}^b x_2^c : a \in \{0,1\}, b, c \in \mathbb{N}_0\}$ is a basis of $\mathcal{B}(\mathcal{V}(-1,2))$.
GKdim $\mathcal{B}(\mathcal{V}(-1,2)) = 2$.

III. Nichols algebras of diagonal type.

Semisimple objects in ${}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$:

$$V = \oplus_{i\in\mathbb{I}_\theta}\mathbb{C}^{\chi_i}_{g_i} \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}, \qquad g_i \in \Gamma, \qquad \chi_i \in \widehat{\Gamma}.$$

As a braided vector space, (V, c) is of diagonal type: There exist a basis $(x_i)_{i\in\mathbb{I}_\theta}$ of V and a matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$, such that

$$c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i, \qquad i, j \in \mathbb{I}_\theta.$$

By technical reasons, we may assume that $q_{ii} \neq 1$ for all i.

Set $\widetilde{q}_{ij} = q_{ij}q_{ji}$. Generalized Dynkin diagram: $\cdots \overset{q_i}{\circ}\overset{\widetilde{q}_{ij}}{\rule[3pt]{30pt}{0.4pt}}\overset{q_j}{\circ}\cdots$. By technical reasons, we may assume that the diagram is connected.

Conjecture. $\operatorname{GKdim} \mathcal{B}(V) < \infty \Rightarrow V$ has a finite Generalized Root System (it has a PBW basis with finite set of generators) $\rightsquigarrow$ Heckenberger's classification.

Theorem. [AAH]. The conjecture is true when $\dim V = 2$.

$\mathfrak{q}$ of **Cartan type**: $\exists$ a generalized Cartan matrix $\mathfrak{a} = (a_{ij})_{i,j \in \mathbb{I}_\theta}$ such that $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$, $\forall i \neq j \in \mathbb{I}_\theta$.

Theorem. [A.–Angiono,Rosso] Assume $\mathfrak{q}$ generic (every q_{ij} is 1 or $\notin \mathbb{G}_\infty$). Then $\operatorname{GKdim} \mathcal{B}(V) < \infty \Leftrightarrow \mathfrak{q}$ of finite Cartan type.

Conjecture. $\mathfrak{q}$ of Cartan type, $\operatorname{GKdim} \mathcal{B}(V) < \infty \Rightarrow$ finite type.

Theorem. [AAH]. If $\mathfrak{a}$ is of affine type, then $\operatorname{GKdim} \mathcal{B}(V) = \infty$.

IV. Nichols algebras of one block + one point.

Let $g_1 \in \Gamma$, $\chi_1 \in \widehat{\Gamma}$ and $\eta : \Gamma \to \mathbb{C}$ a (χ_1, χ_1)-derivation.
Let $V_1 = \mathcal{V}_{g_1}(\chi_1, \eta) \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ homogeneous of degree g_1 (indecomposable) with basis $(x_i)_{i \in \mathbb{I}_2}$ s. t.

$$\rho(g) = \begin{pmatrix} \chi_1(g) & \eta(g) \\ 0 & \chi_1(g) \end{pmatrix}.$$

We assume that $\eta(g_1) = 0$ (otherwise V_1 is of diagonal type, discussed later); up to normalization, $\eta(g_1) = 1$ and $V_1 \simeq \mathcal{V}(\epsilon, 2)$ as braided vector space, where $\epsilon = \chi_1(g_1)$.

Let $g_2 \in \Gamma$, $\chi_2 \in \widehat{\Gamma}$ and $V_2 = \mathbb{C}_{g_2}^{\chi_2} \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ with base (x_3).

Consider $V = V_1 \oplus V_2$. **Block plus point.**
Question. Determine all V s. t. $\operatorname{GKdim} \mathcal{B}(V) < \infty$.

Let $q_{ij} = \chi_j(g_i)$, $i,j \in \mathbb{I}_2$; $\epsilon = q_{11}$; $a = q_{21}^{-1}\eta(g_2)$.

Then the braiding in the basis $(x_i)_{i\in\mathbb{I}_3}$ is $(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_3} =$

$$= \begin{pmatrix} \epsilon x_1 \otimes x_1 & (\epsilon x_2 + x_1) \otimes x_1 & q_{21}x_3 \otimes x_1 \\ \epsilon x_1 \otimes x_2 & (\epsilon x_2 + x_1) \otimes x_2 & q_{21}x_3 \otimes x_2 \\ q_{21}x_1 \otimes x_3 & q_{21}(x_2 + a x_1) \otimes x_3 & q_{22}x_3 \otimes x_3 \end{pmatrix}. \tag{1}$$

$c^2_{|V_1\otimes V_2} = \mathrm{id} \iff q_{21}q_{21} = 1$ and $a = 0$.

$\widetilde{q}_{12} = q_{21}q_{21} =$ *interaction* between the block V_1 and the point V_2; we say that it is

weak if $\widetilde{q}_{12} = 1$, **mild** if $\widetilde{q}_{12} = -1$, **strong** if $\widetilde{q}_{12} \notin \{\pm 1\}$.

We may assume $\epsilon^2 = 1$.

We introduce a normalized version of a, called the *ghost*:

$$\mathscr{G} = \begin{cases} -2a, & \epsilon = 1, \\ a, & \epsilon = -1. \end{cases} \qquad \mathscr{G} \in \mathbb{N} \underset{\text{def.}}{\Longleftrightarrow} \text{ the ghost is } \textit{discrete}.$$

Lemma. $\diamond$ If the interaction is **strong**, then $\operatorname{GKdim} \mathcal{B}(V) = \infty$.
$\diamond$ If the interaction is mild and $\epsilon = 1$, then $\operatorname{GKdim} \mathcal{B}(V) = \infty$.

Pictorial description. Weak interaction:

$\epsilon = 1$: $\boxplus \overset{\mathscr{G}}{\rule[0.5ex]{3em}{0.4pt}} \overset{q_{22}}{\bullet}$, $\quad \boxplus \qquad\qquad \overset{q_{22}}{\bullet}$ when $\mathscr{G} = 0$;

$\epsilon = -1$: $\boxminus \overset{\mathscr{G}}{\rule[0.5ex]{3em}{0.4pt}} \overset{q_{22}}{\bullet}$, $\quad \boxminus \qquad\qquad \overset{q_{22}}{\bullet}$ when $\mathscr{G} = 0$;

Mild interaction: $\epsilon = -1$: $\boxminus \overset{(-,\mathscr{G})}{\rule[0.5ex]{3em}{0.4pt}} \overset{q_{22}}{\bullet}$

Below

$$z_n := (\mathrm{ad}_c\, x_2)^n x_3; \qquad f_n = \mathrm{ad}_c\, x_1(z_n), \qquad n \in \mathbb{N}_0.$$

Theorem. $V = V_1 \oplus V_2$, braiding (1); assume that the interaction is **weak**, $\mathscr{G} \neq 0$ and $\operatorname{GKdim} \mathcal{B}(V) < \infty$. Then either of the following holds:

$\boxplus \overset{\mathscr{G}}{\rule[0.5ex]{6em}{0.4pt}} \overset{1}{\bullet}$, $\mathscr{G}$ discrete. $V = \mathfrak{L}(1, \mathscr{G})$ (*Laistrygonian*).

$$\begin{aligned}\mathcal{B}(\mathfrak{L}(1,\mathscr{G})) \simeq \mathbb{C}\langle x_1, x_2, x_3 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2,\\ x_1x_3 - q_{21}\, x_3x_1,\\ z_{1+\mathscr{G}}, z_t z_{t+1} - q_{21}q_{22}\, z_{t+1}z_t, 0 \leq t < \mathscr{G}\rangle\end{aligned}$$

$\operatorname{GKdim} \mathcal{B}(\mathfrak{L}(1,\mathscr{G})) = \mathscr{G} + 3$; $\mathfrak{L}(1,\mathscr{G})$ is a domain.

$\boxplus \overset{\mathscr{G}}{\rule{3cm}{0.4pt}} \overset{-1}{\bullet}$, $\mathscr{G}$ discrete. $V = \mathfrak{L}(-1, \mathscr{G})$

$$\begin{aligned}\mathcal{B}(\mathfrak{L}(-1,\mathscr{G})) \simeq \mathbb{C}\langle x_1, x_2, x_3 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2,\\ x_1x_3 - q_{21}\, x_3x_1,\\ z_{1+\mathscr{G}},\\ z_t^2,\, 0 \leq t \leq \mathscr{G}\rangle\rangle\end{aligned}$$

$\operatorname{GKdim} \mathcal{B}(\mathfrak{L}(-1,\mathscr{G})) = 2.$

$\boxplus \overset{1}{\text{———}} \overset{\omega}{\bullet}$, $\omega \in \mathbb{G}'_3$. $V = \mathfrak{L}(\omega, 1)$

$$\begin{aligned}\mathcal{B}(\mathfrak{L}(\omega,1)) \simeq \mathbb{C}\langle x_1, x_2, x_3 | \, & x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, \\ & x_1x_3 - q_{21}\, x_3x_1, \\ & z_2, \quad x_3^3, \\ & z_1^3, \quad z_{1,0}^3 \rangle\end{aligned}$$

$\operatorname{GKdim} \mathcal{B}(\mathfrak{L}(\omega, 1)) = 2.$

$\boxminus \overset{\mathscr{G}}{\rule{2cm}{0.4pt}} \overset{1}{\bullet}$, $\mathscr{G}$ discrete. $V = \mathfrak{L}_-(1, \mathscr{G})$.

$$\begin{aligned}\mathcal{B}(\mathfrak{L}_-(1,\mathscr{G})) \simeq \mathbb{C}\langle x_1, x_2, x_3 | x_1^2,\ & x_2x_{21} - x_{21}x_2 - x_1x_{21},\\ & x_1x_3 - q_{21}\, x_3x_1,\\ & x_{21}x_3 - q_{21}^2 x_3x_{21},\\ & z_{1+2\mathscr{G}},\\ & z_{2k+1}^2,\\ & z_{2k}z_{2k+1} - q_{21}q_{22}\, z_{2k+1}z_{2k}, 0 \leq k < \mathscr{G}\rangle\end{aligned}$$

$\operatorname{GKdim} \mathcal{B}(\mathfrak{L}_-(1,\mathscr{G})) = \mathscr{G} + 3$.

$\boxminus \overset{\mathscr{G}}{\rule{2cm}{0.4pt}} \overset{-1}{\bullet}$, $\mathscr{G}$ discrete. $V = \mathfrak{L}_-(-1, \mathscr{G})$

$$\begin{aligned}\mathcal{B}(\mathfrak{L}_-(-1,\mathscr{G})) \simeq \mathbb{C}\langle x_1, x_2, x_3 | & x_1^2, x_2x_{21} - x_{21}x_2 - x_1x_{21}, \\ & x_3^2, \\ & x_1x_3 - q_{21}\, x_3x_1, \\ & x_{21}x_3 - q_{21}^2 x_3x_{21}, \\ & z_{1+2\mathscr{G}}, \\ & z_{2k}^2, \\ & z_{2k-1}z_{2k} - q_{21}q_{22}\, z_{2k}z_{2k-1}, 0 < k \le \mathscr{G}\rangle\end{aligned}$$

$\operatorname{GKdim} \mathcal{B}(\mathfrak{L}_-(-1,\mathscr{G})) = \mathscr{G} + 2.$

Theorem. $V = V_1 \oplus V_2$, braiding (1); assume that the interaction is **mild** and $\operatorname{GKdim} \mathcal{B}(V) < \infty$.
Suppose that the Conjecture on diagonal type is true. Then $V = \mathfrak{C}_1$ (*Cyclop*) has a diagram $\boxminus \overset{(-,1)}{\rule{2cm}{0.4pt}} \overset{-1}{\bullet}$.

$$\mathcal{B}(\mathfrak{C}_1) \simeq \mathbb{C}\langle x_1, x_2, x_3 | x_1^2, x_2x_{21} - x_{21}x_2 - x_1x_{21}, \\ x_3^2, \\ f_1^2, \\ z_1^2, \\ x_{21}x_3 - q_{21}^2 x_3 x_{21}, \\ x_2 z_1 + q_{21} z_1 x_2 - q_{21}\, f_0 x_2 - \frac{1}{2} f_1 \rangle$$

$\operatorname{GKdim} \mathcal{B}(\mathfrak{C}_1) = 2$.

V. Nichols algebras of one pale block + one point.

Let $g_1 \in \Gamma$, $\chi_1 \in \widehat{\Gamma}$ and $\eta : \Gamma \to \mathbb{C}$ a (χ_1, χ_1)-derivation.
Let $V_1 = \mathcal{V}_{g_1}(\chi_1, \eta) \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ homogeneous of degree g_1 with basis $(x_i)_{i \in \mathbb{I}_2}$ s. t.

$$\rho(g) = \begin{pmatrix} \chi_1(g) & \eta(g) \\ 0 & \chi_1(g) \end{pmatrix}.$$

We assume that $\eta(g_1) = 0 \implies V_1$ is of diagonal type.

Let $g_2 \in \Gamma$, $\chi_2 \in \widehat{\Gamma}$ and $V_2 = \mathbb{C}^{\chi_2}_{g_2} \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ with base (x_3).

$$V = V_1 \oplus V_2.$$

We assume $\eta(g_2) \neq 0$ so that V is not of diagonal type.

Question. Determine all V s. t. $\operatorname{GKdim} \mathcal{B}(V) < \infty$.

Let $q_{ij} = \chi_j(g_i)$, $i, j \in \mathbb{I}_2$; $\epsilon = q_{11}$; we may assume $q_{21}^{-1}\eta(g_2) = 1$.

Then the braiding is given in the basis $(x_i)_{i\in\mathbb{I}_3}$ by

$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_3} = \begin{pmatrix} \epsilon x_1 \otimes x_1 & \epsilon x_2 \otimes x_1 & q_{12}x_3 \otimes x_1 \\ \epsilon x_1 \otimes x_2 & \epsilon x_2 \otimes x_2 & q_{12}x_3 \otimes x_2 \\ q_{21}x_1 \otimes x_3 & q_{21}(x_2 + x_1) \otimes x_3 & q_{22}x_3 \otimes x_3 \end{pmatrix}.$$

We call this a **pale block plus point.**

Theorem. $\operatorname{GKdim} \mathcal{B}(V) < \infty \iff \epsilon = -1$ and either

- $\widetilde{q}_{12} = 1$ and $q_{22} = \pm 1$; in this case $\operatorname{GKdim} \mathcal{B}(V) = 1$; or else
- $q_{22} = -1 = \widetilde{q}_{12}$; in this case $\operatorname{GKdim} \mathcal{B}(V) = 2$.

VI. Nichols algebras of one block + several points.

We now consider:

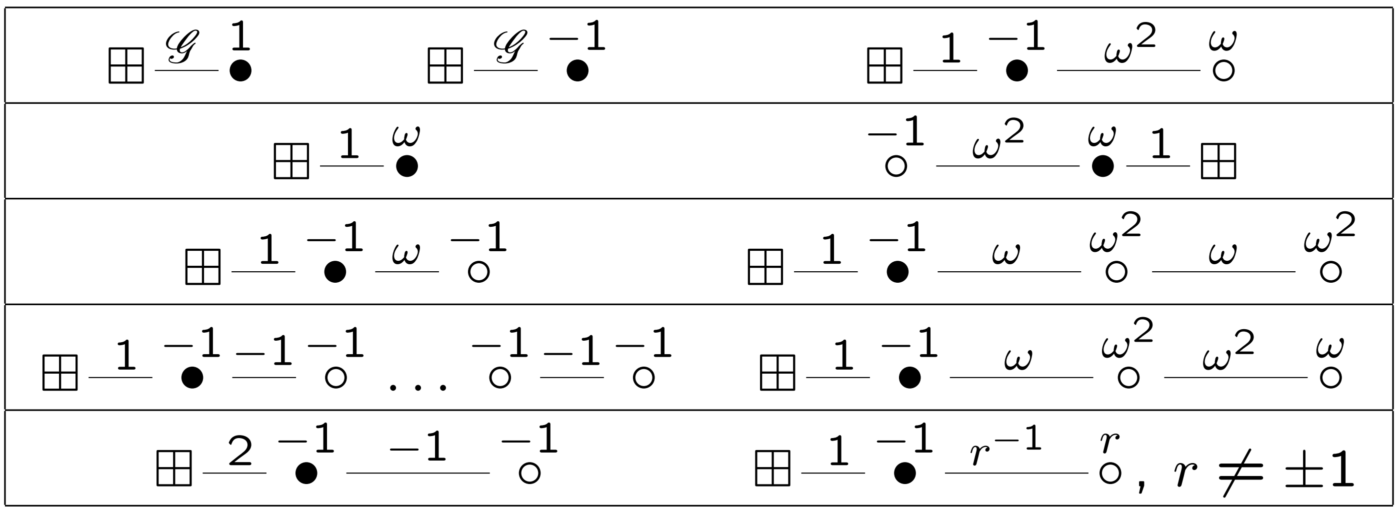
$V = V_1 \oplus V_2 \oplus \cdots \oplus V_\theta,$ $\theta \geq 3;$
where V_1 is a ϵ-block, $\epsilon = 1,\ h \in \mathbb{I}_t;$
V_i is a q_{ii}-point, with $q_{ii} \in \mathbb{C}^\times,$ $i \in \mathbb{I}_{2,\theta};$
$c(V_i \otimes V_j) = V_j \otimes V_i,\ i, j \in \mathbb{I}_\theta,$ $c_{ij} = c_{|V_i \otimes V_j},\ i \neq j \in \mathbb{I}_\theta.$
$V_{\text{diag}} = V_2 \oplus \cdots \oplus V_\theta,$ $\mathcal{X} = \{\text{connected components of } V_{\text{diag}}\}.$

Suppose that the Conjecture on diagonal type is true.

Theorem. ($\epsilon = 1$). $\operatorname{GKdim} \mathcal{B}(V) < \infty \iff$ for every $J \in \mathcal{X}$, either $\mathscr{G}_J = 0$, or else V_J is as in Table 1. In this case,

$$\operatorname{GKdim} \mathcal{B}(V) = 2 + \sum_{J \in \mathcal{X}} \operatorname{GKdim} \mathcal{B}(K_J).$$

Connecting components with 1-block; $\mathscr{G} \in \mathbb{N}$, $\omega \in \mathbb{G}_3'$.



Notice the interaction is always weak.

Theorem. ($\epsilon = -1$). $\operatorname{GKdim} \mathcal{B}(V) < \infty \iff$ for every $J \in \mathcal{X}$, either of the following holds:

(a) The interaction of J is weak and $\mathscr{G}_J = 0$.

(b) $J = \{i\}$, $\boxminus \overset{\mathscr{G}}{\rule{6em}{0.4pt}} \overset{\pm 1}{\bullet}$, $\mathscr{G}$ discrete.

(c) $J = \{i\}$, $\boxminus \overset{(-,1)}{\rule{6em}{0.4pt}} \overset{-1}{\bullet}$.

(d) $J = \{i,j\}$, $\boxminus \overset{(-1,1)}{\rule{6em}{0.4pt}} \overset{-1}{\bullet} \overset{-1}{\rule{6em}{0.4pt}} \overset{-1}{\circ}$ $V = \mathfrak{C}_2$ (*Cyclop*).

Furthermore, if there is one $J \in \mathcal{X}$ with mild interaction, i.e. of type (c) or (d), and no components disconnected from the block, i.e. of type (a), then the diagram is connected, i.e. $J = \mathbb{I}_{2,\theta}$. In this case, $\operatorname{GKdim} \mathcal{B}(V) = 2 + \sum_{J \in \mathcal{X}} \operatorname{GKdim} \mathcal{B}(K_J)$.

VII. Nichols algebras of two blocks.

$\mathbb{I}^\ddagger = \mathbb{I}_1^\ddagger \cup \mathbb{I}_2^\ddagger = \{1, \frac{3}{2}, 2, \frac{5}{2}\}$.

$g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ and $\eta_i : \Gamma \to \mathbb{C}$ a (χ_i, χ_i)-derivation, $i \in \mathbb{I}_2$.

$\mathcal{V}_{g_i}(\chi_i, \eta_i) \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ indecomposable with basis $(x_h)_{h \in \mathbb{I}_i^\ddagger}$.

$V = V_1 \oplus V_2$, $V_i = \mathcal{V}_{g_i}(\chi_i, \eta_i)$, $i \in \mathbb{I}_2$; $(x_h)_{h \in \mathbb{I}^\ddagger}$ is a basis of V.

$q_{ij} = \chi_j(g_i)$, $a_{ij} = q_{ij}^{-1}\eta_j(g_i)$, $i, j \in \mathbb{I}_2$.

We suppose that V is neither of diagonal type nor of the form *1 block & 2 points*, hence $a_{ii} \neq 0$, $i \in \mathbb{I}_2$; we may assume that $\eta_i(g_i) = 1$ by normalizing x_i.

We seek to know when $\operatorname{GKdim} \mathcal{B}(V) < \infty$. Since $\mathcal{B}(V_i \oplus \mathbb{C}_{g_j}^{\chi_j}) \hookrightarrow \mathcal{B}(V)$ for $i \neq j \in \mathbb{I}_2$, we may assume that

$$q_{ii}^2 = 1 \text{ hence } a_{ii} = q_{ii} =: \epsilon_i, \qquad q_{12}q_{21} \in \{\pm 1\}, \qquad \mathscr{G} \in \mathbb{N}_0^2.$$

With the previous conventions, the braiding is given in the basis $(x_i)_{i \in \mathbb{I}_\ddagger}$ by $(c(x_i \otimes x_j))_{i,j \in \mathbb{I}_\ddagger} =$

$$\begin{pmatrix} \epsilon_1 x_1 \otimes x_1 & (\epsilon_1 x_{\frac{3}{2}} + x_1) \otimes x_1 & q_{12} x_2 \otimes x_1 & q_{12}(x_{\frac{5}{2}} + a_{12} x_2) \otimes x_1 \\ \epsilon_1 x_1 \otimes x_{\frac{3}{2}} & (\epsilon_1 x_{\frac{3}{2}} + x_1) \otimes x_{\frac{3}{2}} & q_{12} x_2 \otimes x_{\frac{3}{2}} & q_{12}(x_{\frac{5}{2}} + a_{12} x_2) \otimes x_{\frac{3}{2}} \\ q_{21} x_1 \otimes x_2 & q_{21}(x_{\frac{3}{2}} + a_{21} x_1) \otimes x_2 & \epsilon_2 x_2 \otimes x_2 & (\epsilon_2 x_{\frac{5}{2}} + x_2) \otimes x_2 \\ q_{21} x_1 \otimes x_{\frac{5}{2}} & q_{21}(x_{\frac{3}{2}} + a_{21} x_1) \otimes x_{\frac{5}{2}} & \epsilon_2 x_2 \otimes x_{\frac{5}{2}} & (\epsilon_2 x_{\frac{5}{2}} + x_2) \otimes x_{\frac{5}{2}} \end{pmatrix}.$$

Theorem. $\operatorname{GKdim} \mathcal{B}(V) < \infty \iff c^2_{|V_1 \otimes V_2} = \mathrm{id}$.

VIII. The general picture.

Here is the class of braided vector spaces (V, c) we consider:

$$V = V_1 \oplus \cdots \oplus V_t \oplus V_{t+1} \oplus \cdots \oplus V_\theta,$$

$$c(V_i \otimes V_j) = V_j \otimes V_i,\ i, j \in \mathbb{I}_\theta, \qquad c_{ij} = c_{|V_i \otimes V_j},\ i \neq j \in \mathbb{I}_\theta.$$

where V_h is a ϵ_h-block, $\epsilon_h^2 = 1,\ h \in \mathbb{I}_t$;

V_i is a q_{ii}-point, with $q_{ii} \in \mathbb{C}^\times$, $i \in \mathbb{I}_{t+1,\theta}$.

Set $i \sim j$ when $c_{ij}c_{ji} \neq \mathrm{id}_{V_j \otimes V_i}$, $i \neq j \in \mathbb{I}_\theta$. Assume V is connected wrt $\sim$.

We attach a graph to V as already discussed:

• If $h \in \mathbb{I}_t$, then corresponding vertex is depicted as $\boxplus$ when $\epsilon_h = 1$, respectively $\boxminus$ when $\epsilon_h = -1$.

• If $j \in \mathbb{I}_{t+1,\theta}$, then the corresponding vertex is depicted as $\overset{q_{jj}}{\circ}$.

• There is an edge between i and $j \in \mathbb{I}_\theta$ iff $i \sim j$.

• If $i \in \mathbb{I}_t$, $j \in \mathbb{I}_{t+1,\theta}$, $\mathscr{I}_{ij}$ is weak and $\mathscr{G}_{ij} \neq 0$, respectively $\mathscr{I}_{ij}$ is mild, then the edge between i and j is labelled by $\mathscr{G}_{ij}$, respectively by $(-, \mathscr{G}_{ij})$.

• If $i \neq j \in \mathbb{I}_{t+1,\theta}$ and $q_{ij}q_{ji} \neq 1$, then the corresponding edge is decorated by $\widetilde{q}_{ij} = q_{ij}q_{ji}$.

Theorem. $\operatorname{GKdim} \mathcal{B}(V) < \infty \iff$

(a) There are no edges between blocks.

(b) The only possible connections between a connected component $J \in \mathcal{X}$ and one block are as described in the section *one block + several points*.

(c) Let $J \in \mathcal{X}$ (a connected component of $\mathcal{D}_{\text{diag}}$). Then there is a unique $j \in J$ connected to a block.

(d) If $J \in \mathcal{X}$ has $|J| > 1$, then it is connected to a unique block i.

(e) If $J = \{j\} \in \mathcal{X}$ and $q_{jj} \in \mathbb{G}'_3$, then it is connected to a unique block i.

(f) If a block h is connected to a point j by an edge labelled $(-, \mathscr{G}_{hj})$ for some $\mathscr{G}_{hj} \in \mathbb{C}^\times$, then there is no other edge connecting h with a point and there is no other edge connecting j with a block.

As you set out for Ithaka
hope the voyage is a long one,
full of adventure, full of discovery.
Laistrygonians and Cyclops,
angry Poseidon–dont be afraid of them:
youll never find things like that on your way
as long as you keep your thoughts raised high,
as long as a rare excitement
stirs your spirit and your body.
Laistrygonians and Cyclops,
wild Poseidon–you wont encounter them
unless you bring them along inside your soul,
unless your soul sets them up in front of you.
C. P. Cavafy