Invariant decompositions of H-(co)module algebras and their applications to polynomial identities

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Outline

1. Invariant decompositions
   - H-(co)module algebras
   - Invariant Wedderburn — Mal'cev and Levi theorems
   - Stability of radicals
   - Other invariant decompositions
2. Polynomial identities
   - Definitions, examples, conjecture
   - H-identities of associative algebras
   - H-identities of Lie algebras
   - Criteria for H-simplicity

Original Wedderburn — Mal'cev and Levi theorems

Theorem (J.H.M. Wedderburn, A.I. Mal'cev)

Let A be a finite dimensional associative algebra over a field F of characteristic 0. Then there exists a maximal semisimple subalgebra $B \subseteq A$ such that $A = B \oplus J(A)$ (direct sum of subspaces) where $J(A)$ is the Jacobson radical of A. Moreover, if A is unitary and $A = B' \oplus J(A)$ for another subalgebra B', then $B' = (1-j)^{-1}B(1-j)$ for some $j \in J(A)$.

Theorem (E. Levi)

Let L be a finite dimensional Lie algebra over a field F of characteristic 0. Then there exists a maximal semisimple subalgebra $B \subseteq L$ such that $L = B \oplus R$ (direct sum of subspaces) where R is the solvable radical of L.

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Examples of H-(co)module algebras

- *graded algebras.* Let G be a group. We say that an algebra $A = \bigoplus_{g \in G} A^{(g)}$ is *graded* if $A^{(g)} A^{(h)} \subseteq A^{(gh)}$ for all $g, h \in G$.
- *algebras with an action of a group by automorphisms.* Let G be a group. We say that an algebra A is a *G-algebra* if A is endowed with a homomorphism $G \to \mathrm{Aut}(A)$. In particular, $(ab)^g = a^g b^g$, $a, b \in A$.
- *algebras with an action of a Lie algebra by derivations.* We say that a Lie algebra $\mathfrak{g}$ *acts on an algebra A by derivations* if A is endowed with a homomorphism $\mathfrak{g} \to \mathrm{Der}(A)$. In particular, $u(ab) = (ua)b + a(ub)$ for all $a, b \in A$, $u \in \mathfrak{g}$.

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Definition

Let A be an algebra over a field F. Suppose A is a left H-module where H is a Hopf algebra. We say that A is an *H-module algebra* if $h(ab) = (h_{(1)}a)(h_{(2)}b)$ for all $h \in H$ and $a, b \in A$. Here we use Sweedler's notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Example

Let A be an algebra. Suppose a group G acts on A by automorphisms. Denote by FG the group algebra of G. Then FG is a Hopf algebra with the comultiplication $\Delta(g) = g \otimes g$, the counit $\varepsilon(g) = 1$, and the antipode $S(g) = g^{-1}$, $g \in G$. Moreover A is an FG-module algebra.

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Let A be an algebra and $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra where $\Delta(v) = v \otimes 1 + 1 \otimes v$, $\varepsilon(v) = 0$, $S(v) = -v$ for all $v \in \mathfrak{g}$. Suppose $\mathfrak{g}$ is acting on A by derivations. Consider the corresponding homomorphism $U(\mathfrak{g}) \to \mathrm{End}_F(A)$ of associative algebras. Then A becomes an $U(\mathfrak{g})$-module algebra.

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Let $A = \bigoplus_{g\in G} A^{(g)}$ be an algebra over a field F graded by a group G. Then A is an FG-comodule algebra where $\rho(a^{(g)}) = a^{(g)} \otimes g$ for all $g \in G$ and $a^{(g)} \in A^{(g)}$.

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Affine algebraic groups

Let G be an affine algebraic group over a field F.
Denote by $\mathcal{O}(G)$ the coordinate algebra of G.
Then $\mathcal{O}(G)$ is a Hopf algebra where the comultiplication $\Delta \colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ is dual to the multiplication $G \times G \to G$, the counit $\varepsilon \colon \mathcal{O}(G) \to F$ is defined by $\varepsilon(f) = f(1_G)$, and the antipode $S \colon \mathcal{O}(G) \to \mathcal{O}(G)$ is dual to the map $g \to g^{-1}$, $g \in G$.

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H-comodule algebras

Let A be an algebra over a field F and let G be an affine algebraic group.

Suppose A is endowed with a *rational action* of G by automorphisms, i.e. there is a fixed homomorphism $G \to \mathrm{Aut}(A) \subseteq GL(A)$ such that for some basis $e_1, \ldots, e_m$ of A we have $ge_j = \sum_{i=1}^m \omega_{ij}(g)e_i$ where ω_{ij} are polynomials in the coordinates of $g \in G$.

Then A is an $\mathcal{O}(G)$-comodule algebra where $\rho(e_j) = \sum_{i=1}^m e_i \otimes \omega_{ij}$, $1 \leqslant j \leqslant m$, and $ga = a_{(1)}(g)a_{(0)}$, $g \in G$.

Furthermore, each $\mathcal{O}(G)$-subcomodule of A is a G-invariant subspace and vice versa.

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H-(co)module algebras

If $\dim H < +\infty$, then we can define a Hopf algebra structure on the space $H^* := \mathrm{Hom}_F(H, F)$ using the dual operators.
In this case, every H-comodule algebra A is an H^*-module algebra and vice versa.
The correspondence between the H-coaction and the H^*-action is given by the formula $h^*a = h^*(a_{(1)})a_{(0)}$ where $h^* \in H^*$, $a \in A$.

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Integrals on Hopf algebras

Let H be a Hopf algebra.

Recall that $t \in H^*$ is a *left integral on* H if $t(a_{(2)})a_{(1)} = t(a)1$ for all $a \in H$.

We say that a left integral is ad-*invariant* if $t(a_{(1)}\, b\, S(a_{(2)})) = \varepsilon(a)t(b)$ for all $a, b \in H$.

In the main results we assume that there exists an ad-invariant left integral $t \in H^*$ such that $t(1) = 1$.

Now we list three main examples of such Hopf algebras H.

First, we notice that the existence of an integral $t \in H^*$, such that $t(1) = 1$, is equivalent to the cosemisimplicity of H.

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Integrals on Hopf algebras

Let H be a Hopf algebra.
Recall that $t \in H^*$ is a *left integral on H* if $t(a_{(2)})a_{(1)} = t(a)1$ for all $a \in H$.
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Examples of Hopf algebras with ad-invariant integrals

Example

Let G be any group. Consider $t \in (FG)^*$,

$$t(g) = \begin{cases} 0 & \text{if} \quad g \neq 1, \\ 1 & \text{if} \quad g = 1. \end{cases}$$

Then t is an ad-invariant left integral on FG. Note that $t(1) = 1$.

Example

Let G be an affine algebraic group over a field F. If F is algebraically closed of characteristic 0 and G is reductive, then there exists an ad-invariant left integral $t \in \mathcal{O}(G)^*$ such that $t(1) = 1$.

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We conclude the subsection with an example of a Hopf algebra that does not have nonzero integrals.

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Let L be a Lie algebra over a field F. If $L \neq 0$, F is of characteristic 0, and $t \in U(L)^*$ is a left integral of $U(L)$, then $t = 0$.

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In 1957 E.J. Taft proved the G-invariant Levi and Wedderburn — Mal'cev theorems for G-algebras with an action of a finite group G by automorphisms and anti-automorphisms.

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The study of Wedderburn decompositions for H-module algebras started by A. V. Sidorov in 1986.

Theorem (D. Ştefan, F. Van Oystaeyen, 1999)

Let A be a finite dimensional associative H-comodule algebra over a field F of characteristic 0 where H is a Hopf algebra with an ad-invariant left integral $t \in H^*$, $t(1) = 1$. Suppose $J(A)$ is an H-subcomodule. Then there exists an maximal semisimple subalgebra $B \subseteq A$ such that $A = B \oplus J(A)$ (direct sum of H-subcomodules).

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The graded Levi theorem for finite dimensional Lie algebras over an algebraically closed field of characteristic 0, graded by a finite group, was proved by D. Pagon, D. Repovš, and M.V. Zaicev in 2011.

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Let L be a finite dimensional H-comodule Lie algebra over a field F of characteristic 0 where H is a Hopf algebra. Suppose R is an H-subcomodule and there exists an ad-invariant left integral $t \in H^*$ such that $t(1) = 1$. Then there exists a maximal semisimple subalgebra B in L such that $L = B \oplus R$ (direct sum of H-subcomodules).

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Example of an algebra without an H-invariant Levi decomposition

Let $L = \left\{ \left(\begin{array}{cc} C & D \\ 0 & 0 \end{array} \right) \,\middle|\, C \in \mathfrak{sl}_m(F), D \in M_m(F) \right\} \subseteq \mathfrak{sl}_{2m}(F)$, $m \geqslant 2$. Then

$$R = \left\{ \left(\begin{array}{cc} 0 & D \\ 0 & 0 \end{array} \right) \,\middle|\, D \in M_m(F) \right\}$$

is the solvable (and nilpotent) radical of L.

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Then L is a G-algebra and an FG-module algebra where $G = \langle \varphi \rangle \cong \mathbb{Z}$. However there is no FG-invariant semisimple subalgebra B such that $L = B \oplus R$ (direct sum of FG-submodules).

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on itself.

Then L is a $U(L)$-module algebra.

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Cohomologies of Lie algebras

Let $\psi \colon L \to \mathfrak{gl}(V)$ be a representation of a Lie algebra L on some vector space V over a field F.

Denote by $C^k(L; V) \subseteq \mathrm{Hom}_F(L^{\otimes k}; V)$, $k \in \mathbb{N}$, the subspace of all alternating multilinear maps, $C^0(L; V) := V$.

Recall that the elements of $C^k(L; V)$ are called *k-cochains* with coefficients in V.

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The *coboundary operators* $\partial \colon C^k(L; V) \to C^{k+1}(L; V)$ are defined on these spaces in such a way that $\partial^2 = 0$.

$(\partial v)(x) = \psi(x)v$ if $v \in C^0(L; V)$,

$$(\partial\omega)(x_1, \ldots, x_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \psi(x_i)\omega(x_1, \ldots, \hat{x}_i, \ldots, x_{k_1}) +$$

$$\sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1})$$

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(H, L)-modules

Suppose L is an H-comodule algebra for some Hopf algebra H.
Consider a representation $\psi\colon L \to \mathfrak{gl}(V)$.
We say that (V, ψ) is an *(H, L)-module* if V is an H-comodule and

$$\rho_V(\psi(a)v) = \psi(a_{(0)})v_{(0)} \otimes a_{(1)}v_{(1)} \text{ for all } a \in L,\ v \in V$$

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H-colinear cohomologies of Lie algebras

Denote by $\tilde{C}^k(L; V)$ the subspace of *H-colinear cochains*, i.e. such maps $\omega \in C^k(L; V)$ that

$$\rho_V(\omega(a_1, a_2, \ldots, a_k)) = \omega(a_{1(0)}, a_{2(0)}, \ldots, a_{k(0)}) \otimes a_{1(1)} a_{2(1)} \ldots a_{k(1)}$$

for all $a_i \in L$.

If (V, ψ) is an (H, L)-module and H is commutative, then, clearly, the coboundary of an H-colinear cochain is again an H-colinear cochain.

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However, for 1-cochains and a symmetric (H, L)-module (V, ψ) this is true even if H is not commutative.

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If L is an H-comodule Lie algebra, then the adjoint representation $\mathrm{ad}\colon L \to \mathfrak{gl}(L)$ defines on L the structure of a symmetric (H, L)-module.

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If (V, ψ) is a symmetric (H, L)-module, then $\partial(\tilde{C}^1(L; V)) \subseteq \tilde{C}^2(L; V)$.

Let $\tilde{Z}^2(L; \psi) := Z^2(L; \psi) \cap \tilde{C}^2(L; V)$ and $\tilde{B}^2(L; \psi) := \partial(\tilde{C}^1(L; V))$.

The lemma above enables us to define *the second H-colinear cohomology group* $\tilde{H}^2(L; \psi) := \tilde{Z}^2(L; \psi)/\tilde{B}^2(L; \psi)$.

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Maschke's trick for cosemisimple Hopf algebras

Lemma

Let $r\colon V \to W$ be a linear map where V and W are H-comodules for a Hopf algebra H. Let $t \in H^*$ be a left integral on H. Then $\tilde r\colon V \to W$ where

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is an H-colinear map. If, in addition, $\pi \circ r = \mathrm{id}_V$ for some H-colinear map $\pi\colon W \to V$ and $t(1) = 1$, then $\pi \circ \tilde r = \mathrm{id}_V$ too.

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Remark

If G is an arbitrary group and $H = FG$, then $V = \bigoplus_{g\in G} V^{(g)}$ and $W = \bigoplus_{g\in G} W^{(g)}$ are graded spaces.

Suppose $t(g) = \begin{cases} 0 & \text{if} \quad g \neq 1, \\ 1 & \text{if} \quad g = 1. \end{cases}$.

Then $\tilde r(x) = \sum_{g\in G} p_{W,g}\, r(p_{V,g}x)$ for $x \in V$ and this is a graded map.

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Lemma

Let (V, ψ) *be a finite dimensional symmetric* (H, L)*-module where* L *is a finite dimensional* H*-comodule semisimple Lie algebra over a field* F *of characteristic* 0 *and* H *is a Hopf algebra with an* ad*-invariant left integral* $t \in H^*$, $t(1) = 1$*. Then* $\tilde{H}^2(L; \psi) = 0$.

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An important condition in the invariant Levi and Wedderburn — Mal'cev theorems is the stability of the radicals.

In the case of G-algebras the stability is clear since the radicals are invariant under automorphisms and anti-automorphisms.

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Stability of radicals

Theorem

Let L be a finite dimensional H-module Lie algebra over a field F of characteristic 0 and H be a finite dimensional (co)semisimple Hopf algebra. Then the solvable and the nilpotent radicals R and N of L are H-invariant.

If A is a finite dimensional associative algebra over a field F of characteristic 0, then $J(A)$ is invariant under all derivations of A (see e.g. J. Dixmier's book).

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Example of an algebra with a non-stable radical

Let $H = \langle 1, g, x, gx \rangle_F$ be the 4-dimensional Sweedler's Hopf algebra over a field F of characteristic 0. Here $g^2 = 1$, $x^2 = 0$, $xg = -gx$, $\Delta(g) = g \otimes g$, $\Delta(x) = g \otimes x + x \otimes 1$, $\varepsilon(g) = 1$, $\varepsilon(x) = 0$, $S(g) = g$, $S(x) = -gx$. Note that $J(H) = \langle x, gx \rangle_F \neq 0$, i.e. H is not semisimple. Let V be a three-dimensional vector space. Fix some linear isomorphism $\varphi \colon \mathfrak{sl}_2(F) \to V$. Consider the Lie algebra $L = \mathfrak{sl}_2(F) \oplus V$ with the Lie commutator

$$[a+\varphi(b), c+\varphi(u)] = [a, c]+\varphi([a, u]+[b, c]) \text{ where } a, b, c, u \in \mathfrak{sl}_2(F),$$

i.e. V is an Abelian ideal of L that coincides with the solvable and the nilpotent radicals of L. Define $g(a + \varphi(b)) = a - \varphi(b)$ and $x(a + \varphi(b)) = b$ where $a, b \in \mathfrak{sl}_2(F)$. Then L is an H-module Lie algebra, however V is not H-invariant.

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H-(co)invariant decompositions of semisimple algebras

Theorem

Let B be a finite dimensional semisimple H-module Lie algebra over a field of characteristic 0 where H is an arbitrary Hopf algebra. Then $B = B_1 \oplus B_2 \oplus \ldots \oplus B_s$ (direct sum of ideals and H-submodules) for some H-simple subalgebras B_i.

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H-coinvariant analog of the Weyl theorem

We say that an (H, L)-module (V, ψ) is *irreducible* if it has no nontrivial L-submodules that are H-subcomodules at the same time.

Theorem

Let L be an H-comodule Lie algebra over a field of characteristic 0, let H be a Hopf algebra with an ad-invariant integral $t \in H^*$, $t(1) = 1$, and let (V, ψ) be a finite dimensional (H, L)-module completely reducible as an L-module disregarding the H-coaction. Then $V = V_1 \oplus V_2 \oplus \ldots \oplus V_s$ for some irreducible (H, L)-submodules V_i.

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Graded analog of the Weyl theorem

Again, we have the variants of the theorem for $H = FG$ and for finite dimensional H.

Let G be a group, let $L = \bigoplus_{g \in G} L^{(g)}$ be a graded Lie algebra, and let $V = \bigoplus_{g \in G} V^{(g)}$ be a G-graded vector space. We say that (V, ψ), where $\psi \colon L \to \mathfrak{gl}(V)$, is a *graded L-module* if $\psi(a^{(g)})v^{(h)} \in V^{(gh)}$ for all $g, h \in G$, $a^{(g)} \in L^{(g)}$, $v^{(h)} \in V^{(h)}$. We say that an graded L-module (V, ψ) is *irreducible* if it has no nontrivial graded L-submodules.

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Let L be a Lie algebra over a field of characteristic 0 graded by any group, and let (V, ψ) be a finite dimensional graded L-module completely reducible as an L-module disregarding the grading. Then

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Let L be a finite dimensional Lie algebra over an algebraically closed field F of characteristic 0 and let G be a reductive affine algebraic group over F. Suppose L is endowed with a rational action of G by automorphisms. Let (V, ψ) be a finite dimensional (G, L)-module with a rational G-action, completely reducible as an L-module disregarding the G-action. Then

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Polynomial identities

$$\begin{tikzcd}
\boxed{\text{algebras}} \arrow[r] & \boxed{\text{polynomial identities}} \arrow[d] \\
 & \boxed{\text{their codimensions}} \arrow[ul, dotted]
\end{tikzcd}$$

Ordinary polynomial identities

- Let F be a field of characteristic 0.
- Let $F\langle X\rangle$ be the free associative algebra on the countable set $X = \{x_1, x_2, \dots\}$, i.e. the algebra of polynomials without constant term in the noncommuting variables X.
- Let A be an associative F-algebra and $f = f(x_1, \dots, x_n) \in F\langle X\rangle$.
- We say that f is a polynomial identity of A if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$.
- The set $\mathrm{Id}(A)$ of polynomial identities of A is a T-ideal of $F\langle X\rangle$, i.e. $\psi(\mathrm{Id}(A)) \subseteq \mathrm{Id}(A)$ for all $\psi \in \mathrm{End}(F\langle X\rangle)$.
- E.g., $[x, y] = xy - yx \equiv 0$ is a polynomial identity for all commutative algebras.

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Ordinary codimensions

- Let P_n be the space of multilinear polynomials in the noncommuting variables $x_1, x_2, \ldots, x_n$.
- The non-negative integer $c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}$ is called the nth *codimension* of the algebra A.
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Polynomial G-identities

Analogously for graded, G- and H-identities.

Example

Let $M_2(F)$ be the algebra of 2×2 matrices. Consider $\psi \in \mathrm{Aut}(M_2(F))$ defined by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\psi} := \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Then $[x + x^{\psi}, y + y^{\psi}] \in \mathrm{Id}^G(M_2(F))$ where $G = \langle \psi \rangle \cong \mathbb{Z}_2$. Here $[x, y] := xy - yx$.

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Consider $L = \left\{ \begin{pmatrix} \mathfrak{gl}_2(F) & 0 \\ 0 & \mathfrak{gl}_2(F) \end{pmatrix} \right\} \subseteq \mathfrak{gl}_4(F)$ with the following grading by the third symmetric group S_3:

$$L^{(e)} = \left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \right\}, \quad L^{((12))} = \left\{ \begin{pmatrix} 0 & \beta & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

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S.A. Amitsur's conjecture

Conjecture

There exists $\lim\limits_{n\to\infty} \sqrt[n]{c_n(A)} \in \mathbb{Z}_+$.

Example (A. Regev)

$\lim\limits_{n\to\infty} \sqrt[n]{c_n(M_k(F))} = k^2$ where $M_k(F)$ is the algebra of $k \times k$ matrices.

S.A. Amitsur's conjecture

Conjecture

There exists $\lim\limits_{n\to\infty} \sqrt[n]{c_n(A)} \in \mathbb{Z}_+$.

Example (A. Regev)

$\lim\limits_{n\to\infty} \sqrt[n]{c_n(M_k(F))} = k^2$ where $M_k(F)$ is the algebra of $k \times k$ matrices.

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- I.B. Volichenko gave an example of an infinite dimensional Lie algebra L with a nontrivial polynomial identity for which the growth of codimensions $c_n(L)$ of ordinary polynomial identities is overexponential.
- M.V. Zaicev and S.P. Mishchenko gave an example of an infinite dimensional Lie algebra L with a nontrivial polynomial identity such that there exists fractional $\mathrm{PIexp}(L) := \lim\limits_{n\to\infty} \sqrt[n]{c_n(L)}$.

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- as a consequence, they proved the analog of the conjecture for G-codimensions for any associative PI-algebra with an action of a finite Abelian group G by automorphisms;
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Theorems for associative algebras

Theorem

Let A be a finite dimensional non-nilpotent associative algebra over a field F of characteristic 0. Suppose a finite not necessarily Abelian group G acts on A by automorphisms and anti-automorphisms. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^G(A) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.

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Let A be a finite dimensional non-nilpotent H-module associative algebra over a field F of characteristic 0, where H is a finite dimensional semisimple Hopf algebra. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^H(A) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.

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Generalized Hopf action

Let H be an associative algebra with 1. We say that an associative algebra A is an algebra with a *generalized H-action* if A is endowed with a homomorphism $H \to \mathrm{End}_F(A)$ and for every $h \in H$ there exist $h'_i, h''_i, h'''_i, h''''_i \in H$ such that

$$h(ab) = \sum_i \left((h'_i a)(h''_i b) + (h'''_i b)(h''''_i a)\right) \text{ for all } a, b \in A.$$

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Let $A = B_1 \oplus B_2 \oplus \ldots \oplus B_q$ be an algebra with a generalized H-action, where B_i are finite dimensional H-simple semisimple algebras and H is a finite dimensional associative algebra with 1. Let $d := \max_{1 \leqslant k \leqslant q} \dim B_k$. Then there exist $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

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Example

Let $G = S_3$ and $A = M_2(F) \oplus M_2(F)$. Consider the following G-grading on A:

$$A^{(e)} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix} \right\},$$

$$A^{((12))} = \left\{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right\} \oplus 0, \qquad A^{((23))} = 0 \oplus \left\{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right\},$$

the other components are zero. Then there exist $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} 4^n \leqslant c_n^{\mathrm{gr}}(A) \leqslant C_2 n^{r_2} 4^n \text{ for all } n \in \mathbb{N}.$$

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Let $A = FG$ where G is a finite group. Consider the natural G-grading $A = \bigoplus_{g\in G} A^{(g)}$ where $A^{(g)} = Fg$. Then there exist $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} |G|^n \leqslant c_n^{\mathrm{gr}}(A) \leqslant C_2 n^{r_2} |G|^n \text{ for all } n \in \mathbb{N}.$$

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Let $A = Fe_1 \oplus \ldots \oplus Fe_m$ (direct sum of ideals) where $e_i^2 = e_i$, $m \in \mathbb{N}$. Suppose $G \subseteq S_m$ acts on A by the formula $\sigma e_i := e_{\sigma(i)}$, $\sigma \in G$. Let $\{1, 2, \ldots, m\} = \coprod_{i=1}^{q} O_i$ where O_i are orbits of the G-action on $\{1, 2, \ldots, m\}$. Let $d := \max_{1 \leqslant i \leqslant q} |O_i|$. Then there exist $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that $C_1 n^{r_1} d^n \leqslant c_n^G(A) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.

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Let $A = A_1 \oplus \ldots \oplus A_m$ (direct sum of ideals), $A_i \cong M_k(F)$, $1 \leqslant i \leqslant m$, and $k, m \in \mathbb{N}$. The group $\mathrm{Aut}^*(M_k(F)) \times S_m$ acts on A in the following way: if $(\varphi, \sigma) \in \mathrm{Aut}^*(M_k(F)) \times S_m$ and $(a_1, \ldots, a_m) \in A$, then

$$(\varphi, \sigma) \cdot (a_1, \ldots, a_m) := (a^{\varphi}_{\sigma^{-1}(1)}, \ldots, a^{\varphi}_{\sigma^{-1}(m)}).$$

Suppose $G \subseteq \mathrm{Aut}^*(M_k(F)) \times S_m$ is a subgroup. Denote by $\pi : \mathrm{Aut}^*(M_k(F)) \times S_m \to S_m$ the natural projection on the second component. Let $\{1, 2, \ldots, m\} = \coprod_{i=1}^{q} O_i$ where O_i are orbits of the $\pi(G)$-action on $\{1, 2, \ldots, m\}$. Let $d := k^2 \max_{1 \leqslant i \leqslant q} |O_i|$. Then there exist $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that $C_1 n^{r_1} d^n \leqslant c_n^G(A) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.

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Let $A = A_1 \oplus \ldots \oplus A_m$ (direct sum of ideals) where $A_i \cong \mathrm{UT}_k(F)$, $1 \leqslant i \leqslant m$; $k, m \in \mathbb{N}$; and $\mathrm{UT}_k(F)$ is the associative algebra of $k \times k$ upper-triangular matrices. Suppose $G \subseteq S_m$ acts on A in the following way: if $\sigma \in G$ and $(a_1, \ldots, a_m) \in A$, then

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Let H be the 4-dimensional Sweedler's Hopf algebra. Then H is a left H^*-module with the action defined by $gh = g(h_{(2)})h_{(1)}$, $h \in H$, $g \in H^*$. We prove

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Let F be a field of characteristic 0. There exist $C > 0$ and $r \in \mathbb{R}$ such that $Cn^r 4^n \leqslant c_n^{H^*}(H) \leqslant 4^{n+1}$ for all $n \in \mathbb{N}$.

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Let $L = L_1 \oplus \ldots \oplus L_s$ (direct sum of H-invariant ideals) be an H-module Lie algebra over an algebraically closed field F of characteristic 0 where H is a Hopf algebra. Suppose L_i are H-nice algebras. Then there exists $\mathrm{PIexp}^H(L) = \max_{1 \leqslant i \leqslant s} \mathrm{PIexp}^H(L_i)$.

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Let $L = L_1 \oplus \ldots \oplus L_s$ (direct sum of H-invariant ideals) be an H-module Lie algebra over an algebraically closed field F of characteristic 0 where H is a Hopf algebra. Suppose L_i are H-nice algebras. Then there exists $\mathrm{PIexp}^H(L) = \max_{1 \leqslant i \leqslant s} \mathrm{PIexp}^H(L_i)$.

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Formula for the Hopf PI-exponent
of Lie algebras

Suppose L is an H-nice Lie algebra.
Consider H-invariant ideals $I_1, I_2, \ldots, I_r, J_1, J_2, \ldots, J_r$, $r \in \mathbb{Z}_+$, of the algebra L such that $J_k \subseteq I_k$, satisfying the conditions

1. I_k/J_k is an irreducible (H, L)-module;
2. for any H-invariant B-submodules T_k such that $I_k = J_k \oplus T_k$, there exist numbers $q_i \geqslant 0$ such that

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Let M be an L-module. Denote by $\operatorname{Ann} M$ its annihilator in L. Let

$$d(L) := \max \left(\dim \frac{L}{\operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_r/J_r)} \right)$$

where the maximum is found among all $r \in \mathbb{Z}_+$ and all $I_1, \dots, I_r$, $J_1, \dots, J_r$ satisfying Conditions 1–2. Then that $\operatorname{PIexp}^H(L) = d(L)$.

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Let $G = S_3$ and $L = \mathfrak{gl}_2(F) \oplus \mathfrak{gl}_2(F)$. Consider the following G-grading on L:

$$L^{(e)} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix} \right\},$$

$$L^{((12))} = \left\{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right\} \oplus 0, \qquad L^{((23))} = 0 \oplus \left\{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right\},$$

the other components are zero. Then there exist $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

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