Hopf Algebras in Combinatorics

Mitja Mastnak

Saint Mary's University

History

- 1629 – Girard, Newton, Cauchy, Jacobi, . . . Symmetric Functions
- 1941 Hopf Cohomology of connected Lie groups
- 1960's Milnor-Moore Structure theory
- 1979 Joni-Rota Disassembly of comb. structures
- 1983 Gessel Quasi-symmetric functions
- 1980's Schmitt Hopf algebras built on posets, graphs, matroids, . . .
- 1990's BOOM Hopf zoo

  95 : Malvenuto-Reutenauer Hopf algebra of permutations

  Gelfand-Krob-Lascoux-Leclerc-Retakh-Thibon NCSym

  98 : Connes-Kreimer Hopf algebra of rooted trees

  Loday-Ronco Hopf algebra of binary trees

  ⋮

- 2006 – Aguiar-Mahajan

  Species

Preliminaries

$\mathbb{C}$ = ground field

$U \otimes V = U \otimes_{\mathbb{C}} V$

$L(U,V) = \{f : U \to V \mid f \text{ linear}\}$

Graded vector space : $U = \bigoplus_{n=0}^{\infty} U_n = U_0 \oplus U_1 \oplus \dots \oplus U_n \oplus \dots$

homogeneous elements of degree n

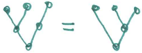
we assume $\dim U_n < \infty$

Dual = graded dual : $U^* = \bigoplus_{n=0}^{\infty} U_n^*$

gcb = graded connected bialgebra

Graded connected bialgebra $(B, m, u, \Delta, \varepsilon)$

Associative multiplication $\quad m : B \otimes B \to B$, $\quad a \otimes b \mapsto ab \qquad a(bc) = (ab)c$



Unit $\quad u : \mathbb{C} \to B$, $\quad \lambda \mapsto \lambda 1_B \qquad x \cdot 1_B = 1_B x = x$

Coassociative comultiplication $\quad \Delta : B \to B \otimes B \qquad (id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$

Counit $\quad \varepsilon : B \to \mathbb{C} \qquad (\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta$

Compatibility $\quad \underline{(\Delta x)(\Delta y) = \Delta(xy)}$

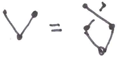
$\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$

Graded $\quad B = \bigoplus_{n=0}^{\infty} B_n = B_0 \oplus B_1 \oplus \dots \oplus B_n \oplus \dots$

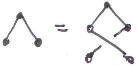
$B_0 = \mathbb{C}$ connected

B_n: homogeneous component of degree n

Commutative $\quad ab = ba$



Cocommutative



$\Delta c = 2u \otimes V + w \otimes w + 2V \otimes w$ ✓

$\Delta c = u \otimes V + 2V \otimes u$ ✗

Bialgebras in combinatorics

Main idea :

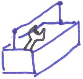
Joni, Rota

$\Delta \rightsquigarrow$ disassembly $\qquad m \rightsquigarrow$ assembly

$\varepsilon \rightsquigarrow$ char. function of the trivial object $\qquad u \rightsquigarrow$ trivial object

deg $\rightsquigarrow$ size

Core purpose : organize information

Toolbox 

algebraic structure theory

universal properties : free, cofree, ...

ubiquity of symmetric and quasi-symmetric functions

duality (interplay between B and B^*)

antipode (if B is a gcb, then $\exists ! S : B \to B \ni m(id \otimes S)\Delta = \varepsilon = m(S \otimes id)\Delta$)

Calculus in the convolution algebra

$\vdots$

Common questions

1. B has interesting bases/generating sets

   (a) find explicit formulas for conversion

   (b) find explicit formulas for the antipode

   (c) find explicit formulas for primitives

efficient?

interpretation? (comb., algebraic, geometric, ...)

2. Is B related to other cha's?

   - isomorphic / subobj. / quotient
   - some kind of extension
   - some kind of deformation
   - new kind of equivalence relation

3. Is B free/cofree (s.t.)?

   over $\mathbb{C}, \mathbb{Q}, \mathbb{Z}, \ldots$

Example

$$B = \mathbb{C}[x] = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}x^2 \oplus \dots \oplus \mathbb{C}x^n \oplus \dots$$

$$\Delta x^n = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i} \qquad \underbrace{\Delta 1 = 1 \otimes 1}_{\text{group like}}, \quad \underbrace{\Delta x = x \otimes 1 + 1 \otimes x}_{\text{primitive}}, \quad \Delta x^2 = x^2 \otimes 1 + 2x \otimes x + 1 \otimes x^2, \dots$$

Compatibility: $\Delta(x^n) = (\Delta x)^n = (x \otimes 1 + 1 \otimes x)^n$, $\Delta(p(x)) = p(x \otimes 1 + 1 \otimes x)$

Counit: $\varepsilon(x^n) = \delta_{n,0} = \begin{cases} 1; & n = 0 \\ 0; & \text{otherwise} \end{cases}$, $\varepsilon(p(x)) = p(0)$

Remark. $\mathbb{C}[x] \otimes \mathbb{C}[x] = \mathbb{C}[x] \otimes \mathbb{C}[y] = \mathbb{C}[x,y]$.

$m : \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x]$, $\underline{m(p(x,y)) = p(x,x)}$

$\Delta : \mathbb{C}[x] \longrightarrow \mathbb{C}[x,y]$, $\underline{\Delta p(x) = p(x+y)}$

Rota's Hopf algebra of ranked posets $\mathcal{R} = \mathrm{span}\{\text{ranked posets}\}$

$$I \cdot J = I \times J, \quad 1_{\mathcal{R}} = \{*\}$$

$$\Delta I = \sum_{x \in I} [0_I, x] \otimes [x, 1_I], \quad \varepsilon(I) = \delta_{0_I, 1_I}$$

Malvenuto-Reutenauer Hopf algebra of permutations $MR = \bigoplus_{n=0}^{\infty} \mathbb{C} S_n$

product = shuffle product

$$M_{132} \cdot M_{\substack{21\\54}} = M_{13254} + M_{13524} + M_{13524} + M_{15324} + M_{15342} + M_{15432} + M_{51324} + M_{51342} + M_{51432} + M_{54132}$$

coproduct = deconcatenation

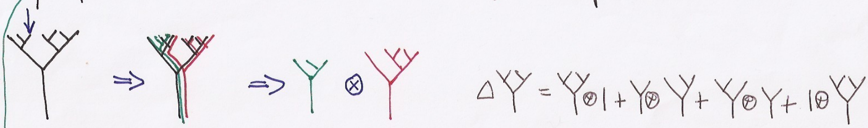
$$\Delta M_{3124} = M_{\substack{3124\\3124}} \otimes 1 + M_{\substack{312\\312}} \otimes M_{\substack{\cancel{4}\\1}} + M_{\substack{\cancel{31}\\21}} \otimes M_{\substack{\cancel{24}\\12}} + M_{\substack{\cancel{3}\\1}} \otimes M_{\substack{\cancel{124}\\123}} + 1 \otimes M_{\substack{3124\\3124}}$$

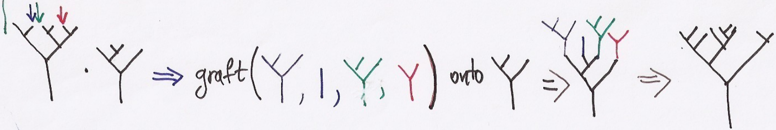
Loday-Ronco Hopf algebras of trees LR

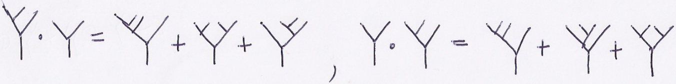
basis = planary binary trees :

coproduct = splitting along the trunk (in all possible ways)

product = split the first tree and graft the pieces onto the second tree.



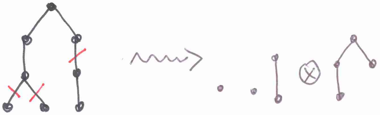


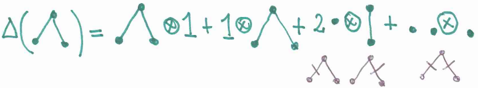


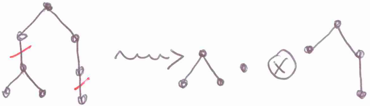
Connes-Kreimer Hopf algebra CK = span{rooted forests}

multiplication = disjoint union

comultiplication = admissible cuts
(no cut below a cut)







Renormalization

$$x(c) = \int_0^\infty \cdots \Rightarrow t \in CK \Rightarrow S(t) \Rightarrow \int_0^\infty \cdots \Rightarrow x^R(c) = x(c) - c.t.$$

divergent integral(s) counter term(s) convergent integral(s)

Examples

$$x(c) = \int_0^\infty \frac{1}{y+c}\,dy, \quad x^R(c) = \text{“}x(c) - x(1)\text{”} = \int_0^\infty \left[\frac{1}{y+c} - \frac{1}{y+1}\right] dy = \text{finite}.$$

$$x_2(c) = \int_0^\infty \int_0^{y_1} \frac{1}{y_1+c}\,\frac{1}{y_2+y_1}\,dy_2\,dy_1 \Rightarrow x_2^R(c) = \text{“}x_2(c) - x(c)x(1) - x_2(1) + x(1)^2\text{”} \ldots$$

$QSym \subseteq \mathbb{C}[[x_1, x_2, \ldots]]_f$ formal power series of finite total degree

f is quasi symmetric if $\forall a_1, \ldots, a_r;\ i_1 < i_2 < \ldots < i_r$ $\mathrm{coef}_f(x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}) = \mathrm{coef}_f(x_1^{a_1} \cdots x_r^{a_r})$

Example $x_1^2x_2 + x_1^2x_3 + \ldots + x_2^2x_3 + x_2^2x_4 + \ldots \in QSym$
(but not $x_1x_2^2$)

Coproduct $f(x_1, x_2, \ldots; y_1, y_2, \ldots) = \sum_i f_i'(x_1, x_2, \ldots) f_i''(y_1, y_2, \ldots)$ $\Delta f = \sum_i f_i' \otimes f_i''$

Monomial basis $\underset{\alpha = (a_1, \ldots, a_r)}{M_\alpha} = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$

$$\Delta M_\alpha = \sum_{\beta\gamma = \alpha} M_\beta \otimes M_\gamma$$

$$\Delta M_{(2,1,3)} = M_{(2,1,3)} \otimes 1 + M_{(2,1)} \otimes M_{(3)} + M_{(2)} \otimes M_{(1,3)} + 1 \otimes M_{(2,1,3)}$$

Zeta function $\zeta : QSym \to \mathbb{C}$

$$\zeta(f) = f(1, 0, 0, \ldots) \quad ; \quad \zeta(M_\alpha) = \begin{cases} 1; & \alpha = \emptyset \text{ or } \alpha = (n) \\ 0; & \text{otherwise} \end{cases}$$

$Sym \subseteq QSym$ symmetric functions

$$Sym = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[p_1, p_2, \dots]$$

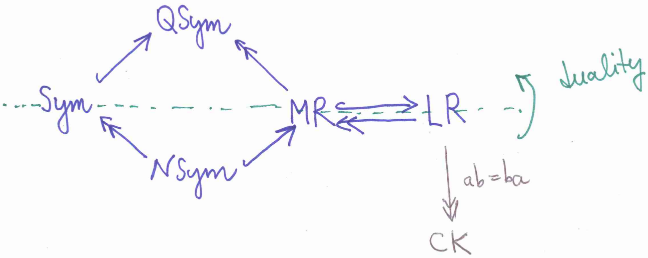
$$\Delta h_n = \sum_{i+j=n} h_i \otimes h_j \qquad \Delta e_n = \sum_{i+j=n} e_i \otimes e_j \qquad \Delta p_n = p_n \otimes 1 + 1 \otimes p_n$$

$NSym = \mathbb{C}\langle H_1, H_2, \dots \rangle$ noncommutative symmetric functions

$$\Delta H_n = \sum_{i+j=n} H_i \otimes H_j$$

$NCSym \subseteq \mathbb{C}\langle\langle x_1, x_2, \dots \rangle\rangle$ symmetric functions in noncommuting variables

Big Picture



Universality of quasi symmetric functions

Theorem (Aguiar–Bergeron–Sottile)

If B is a gcb and $\xi : B \to \mathbb{C}$ is a character, then there is a unique morphism $\Phi : B \to QSym$ such that

$$\begin{tikzcd} B \arrow[rr, "\Phi"] \arrow[dr, "\xi"'] & & QSym \arrow[dl, "\zeta"] \\ & \mathbb{C} & \end{tikzcd}$$

Proof (sketch):

$\xi \in B^0$, $\Delta\xi = \xi \otimes \xi$, $\xi = \sum_n \xi_n$, $\xi_n \in B_n^* \Rightarrow \Delta \xi_n = \sum_{i+j=n} \xi_i \otimes \xi_j$.

Let $\Psi : NSym \to B^*$ be s.t. $\Psi(H_n) = \xi_n$.

Now let $\Phi = \Psi^*$:

$$\begin{tikzcd} B \simeq B^{**} \arrow[r, "\Psi^*"] \arrow[rr, bend right, "\Phi"'] & NSym^* = QSym & {} \end{tikzcd}$$

$\blacksquare$

Convolution algebra $L(B,A) = \{ f : B \to A \mid f \text{ linear} \}$

$\Delta c = \sum_{i=1}^{n} c_i' \otimes c_i''$

$(f*g)(c) = \sum_{i=1}^{n} f(c_i') g(c_i'')$

convolution product $f*g : B \xrightarrow{\Delta} B\otimes B \xrightarrow{f\otimes g} A \otimes A \xrightarrow{m} A$

unit $1_{L(B,A)} = u_A \varepsilon_C : B \xrightarrow{\varepsilon} \mathbb{C} \xrightarrow{u} A$

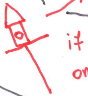
antipode : convolution inverse of identity in $L(B,B)$ $\quad S = id^{-1} : B \to B$

(automatically exists in a gcb)

Calculus $f : B \to A$, $f(1) = 0 \Rightarrow f$ is locally nilpotent

$n > m$, $\underbrace{f*\dots*f}_{f^n}(B_m) = 0$:

$$f^n : B_m \xrightarrow{\Delta_n} \bigoplus_{p_1+\dots+p_n=m} B_{p_1} \otimes \dots \otimes B_{p_n} \xrightarrow{f\otimes\dots\otimes f} A\otimes\dots\otimes A \xrightarrow{\text{prod.}} A$$



if $n>m$, then one of p_j's is 0

$F = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}[[t]] \Rightarrow F(f) = \sum_{i=0}^{\infty} a_i f^i$ locally finite sum

Characters $\mathcal{X}(B) = \{ f : B \to \mathbb{C} \mid f \text{ algebra map} \}$

$f(1) = 1$

$f(ab) = f(a) f(b)$

= group under $*$

'First year' calculus

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \dots\right) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Convolution algebra calculus

$$S = id^{-1} = (\varepsilon - (\varepsilon - id))^{-1} = \sum_{n=0}^{\infty} (\varepsilon - id)^n$$ Takeuchi's formula

$$\exp(f) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n$$ ← makes sense if $f(1) = 0$

$$\log(g) = \log(\varepsilon - (\varepsilon - g)) = -\sum_{n=1}^{\infty} \frac{1}{n} (\varepsilon - g)^n$$ ← makes sense whenever $g(1) = 1$

! Identities like $\log(fg) = \log(f) + \log(g)$, $\exp(f+g) = \exp(f) * \exp(g)$ still hold!

(provided that $f * g = g * f$).

If $f, g \in \mathbb{X}(B)$ such that $f * g = g * f$, then $\log(f * g) = \log(f) + \log(g)$.

(one can also define $f^\lambda = \exp(\lambda \log(f))$.

Free probability

$VN(F_n) \cong VN(F_m) \overset{?}{\Rightarrow} m = n$

Invented by D.V. Voiculescu ~1986 to study the free group factor isomorphism problem

Many parallels with classical probability (central limit thm., convolution, infinite divisibility...)

There is a free analogue of the information-theoretic notion of entropy

in this talk

Important applications to Von Neumann algebras and to Random Matrix Theory and...

certain collections of $N \times N$ matrices are asymptotically free as $N \to \infty$

$\Longleftarrow$ computations of the eigenvalue distributions

sometimes free probability methods apply also for small N

Noncommutative probability space (ncps)

$(\mathcal{A},\varphi)$ $\quad \mathcal{A}$ = unital algebra

$\varphi : \mathcal{A} \to \mathbb{C}$ state (linear functional s.t. $\varphi(1_{\mathcal{A}}) = 1$)

random variable $\quad a \in \mathcal{A}$

distribution of a : $\underbrace{(\varphi(a), \varphi(a^2), \varphi(a^3), \dots)}_{\text{moments}} \leftrightsquigarrow \mu : \mathbb{C}[X] \to \mathbb{C}, \; \mu(x^i) = \varphi(a^i)$

joint distribution of $a_1, \dots, a_k$: $\mu \langle X_1, \dots, X_k \rangle \to \mathbb{C}, \quad \mu(X_{i_1} \dots X_{i_n}) = \varphi(a_{i_1} \dots a_{i_n})$

$$\mathcal{D}_{alg}(k) = \{ \mu : \mathbb{C}\langle X_1, \dots, X_k \rangle \to \mathbb{C} \mid \mu \text{ a distribution of some } a_1, \dots, a_k \text{ in some } \mathcal{A} \}$$

$$= \{ \mu : \mathbb{C}\langle X_1, \dots, X_k \rangle \to \mathbb{C} \mid \mu \text{ linear}, \; \mu(1) = 1 \}.$$

$$\mathcal{G}_k = \{ \mu \in \mathcal{D}_{alg}(k) \mid \mu(X_1) = \mu(X_2) = \dots = \mu(X_k) = 1 \}$$

Examples

e.g. $\Omega = [0,1]$, $Q =$ Lebesgues measurable sets, $P =$ Lebesgues measure

1. (Ω, Q, P) probability space

   $\mathcal{A} = L^\infty(\Omega, P) =$ bounded measurable functions $f : \Omega \to \mathbb{R}$

   $\varphi(a) = \int_\Omega a(\omega)\, dP(\omega)$ $\left[= \int_0^1 f(t)\, dt\right]$

Remark: Distribution of f = compactly sup. measure μ on $\mathbb{R}$ $\mu(X) = P(f(t) \in X)$ $= P(f^{-1}(X))$.

$\rightsquigarrow \int_{\mathbb{R}} t^n \, d\mu(t) = \int_\Omega a^n(\omega)\, dP(\omega) = \varphi(a^n)$

Stone-Weierstrass $\rightsquigarrow$ $(\varphi(a^n))_{n=0}^\infty$ determines μ.

2. $(M_d(\mathbb{C})$, normalized trace$)$

3. $(M_d(\mathcal{A}), \varphi)$, $\varphi(a) = \int_\Omega tr(a(\omega))\, dP(\omega)$

random matrices

4. $\mathcal{H} =$ Hilbert space, $\xi_0 \in \mathcal{H}$, $\|\xi_0\| = 1$

   $\mathcal{A} = B(\mathcal{H})$, $\varphi(a) = \langle a\xi_0, \xi_0 \rangle$

representations of ncps's.

Free independence

Unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ of $\mathcal{A}$ are **freely independent** if $\varphi(a_1 a_2 \dots a_m) = 0$ whenever

- $a_j \in \mathcal{A}_{i(j)}$, $j = 1, \dots, m$ and $i(1) \neq i(2) \neq i(3) \neq \dots \neq i(m-1) \neq i(m)$
- $\varphi(a_j) = 0$, $j = 1, \dots, m$.

$\{a_1, \dots, a_k\}$ is freely independent from $\{b_1, \dots, b_k\}$ if the unital subalgebras generated by these k-tuples are freely independent

Recipe for computing φ:

Example $\mathcal{A}, \mathcal{B}$ freely independent; $a_1 a_2, \dots \in \mathcal{A}$, $b_1, b_2, \dots \in \mathcal{B}$

$\varphi(a_1 b_1) = ?$ $\qquad \varphi(a_1 - \varphi(a_1)) = 0 = \varphi(b_1 - \varphi(b_1))$

$$0 = \varphi((a_1 - \varphi(a_1))(b_1 - \varphi(b_1))) = \varphi(a_1 b_1) - \varphi(a_1 \varphi(b_1)) - \varphi(\varphi(a_1) b_1) + \varphi(\varphi(a_1)\varphi(b_1))$$
$$= \varphi(a_1 b_1) - \varphi(a_1)\varphi(b_1) - \varphi(a_1)\varphi(b_1) + \varphi(a_1)\varphi(b_1)$$

$\Rightarrow \varphi(a_1 b_1) = \varphi(a_1)\varphi(b_1)$

Similarly $\quad \varphi(a_1 b_1 a_2) = \varphi(a_1 a_2)\varphi(b_1)$

$$\varphi(a_1 b_1 a_2 b_2) = \varphi(a_1 a_2)\varphi(b_1)\varphi(b_2) + \varphi(a_1)\varphi(a_2)\varphi(b_1 b_2) - \varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2)$$

$\vdots$

Free multiplicative convolution

$\mathcal{D}_{alg}(k) = \{\mu : \mathbb{C}\langle x_1,\dots,x_k\rangle \to \mathbb{C} \mid \mu \text{ a distribution}\}$

Q. How do we multiply (convolve) $\mu, \nu \in \mathcal{D}_{alg}(k)$?

A. Find $(\mathcal{A}, \varphi)$ ncps and $a_1,\dots,a_k, b_1,\dots,b_k \in \mathcal{A}$ s.t.

- μ = distribution of $a_1,\dots,a_k$, ν = distribution of $b_1,\dots,b_k$
- $\{a_1,\dots,a_k\}$ is free from $b_1,\dots,b_k$

Then $\mu \boxtimes \nu$ = distribution of $a_1b_1, a_2b_2, \dots, a_kb_k$.

$\mathcal{G}_k = \{\mu \in \mathcal{D}_{alg}(k) \mid \mu(x_i)=1, i=1,\dots,k\}$ is a group; unit $= \mu_0 : \mathbb{C}\langle X_1,\dots,X_k\rangle \to \mathbb{C}$, $\mu_0(X_{i_1}\dots X_{i_n}) = 1$

Remark. $\mu \boxplus \nu$ = distribution of $a_1+b_1,\dots,a_k+b_k$

$(\mathcal{D}_{alg}(k), \boxplus)$ is a vector space

Transforms

One variable case:

$$\mathcal{D}_{alg}(k) \ni \mu \longmapsto \begin{array}{l} M_\mu = \sum \mu(x^i) z^i \\ R_\mu \\ S_\mu \end{array} \in \mathbb{C}[[z]]$$

$$S_\mu(z) = \frac{1+z}{z} M_\mu^{\langle -1 \rangle}(z) = \frac{1}{z} R_\mu^{\langle -1 \rangle}(z)$$

composition inverse

Theorem (Voiculescu)

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu$$

$$S_{\mu \boxtimes \nu} = S_\mu \cdot S_\nu$$

Multivariable case: Speicher, Nica-Speicher: R generalizes to multivariable context

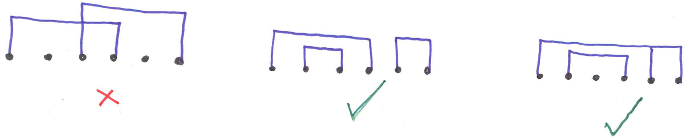
$$\mathcal{D}_{alg}(k) \ni \mu \longmapsto R_\mu \in \mathbb{C}\langle\langle z_1, \dots, z_k \rangle\rangle$$

$$\mu : \mathbb{C}\langle X_1, \dots, X_k \rangle \to \mathbb{C} \qquad \mathrm{coef}_{z_{i_1} \dots z_{i_n}}(R_\mu) = \text{'free cumulant'}(x_{i_1} \dots x_{i_n})$$

related to moments via summation over noncrossing partitions

Q What about multivariable S?

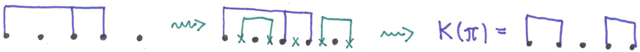
NC(n) = noncrossing partitions of $[n] = \{1, 2, \dots, n\}$

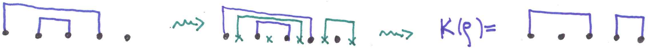


$$|NC(n)| = \frac{1}{n+1}\binom{2n}{n} = \text{Catalan number}$$

NC(n) is a lattice

Kreweras complement

$\pi =$ 

$\rho =$ 

◁ □ ▷ ◁ ⊡ ▷ ◁ ≣ ▷ ◁ ≣ ▷ ≣ ↺ 🔍 ↻

Bialgebra $\mathcal{Y}^{(k)}$

$$\mathcal{Y}^{(k)} = \mathbb{C}\left[Y_w \mid w \in [k]^*, |w| \geq 2\right]$$

Convention: $Y_{(i)} = 1$, $\pi = \{A_1, \dots, A_n\}$ partition of $\{1, 2, \dots, |w|\}$

$$Y_{w;\pi} = Y_{w|A_1} Y_{w|A_2} \cdots Y_{w|A_n}, \text{ e.g.,}$$

$$Y_{i_1 i_2 i_3 i_4;} = Y_{i_1 i_4} Y_{i_2 i_3}, \quad Y_{i_1 i_2 i_3 i_4;} = Y_{i_1 i_2 i_4} Y_{i_3} = Y_{i_1 i_2 i_4}$$

Grading: $\deg(Y_w) = |w| - 1$

Coproduct: $$\Delta Y_w = \sum_{\pi \in NC(w)} Y_{w;\pi} \otimes Y_{w;K(\pi)}$$

$$\Delta Y_{i_1 i_2 i_3} = Y_{i_1 i_2 i_3} \otimes 1 + Y_{i_1 i_2} \otimes Y_{i_2 i_3} + Y_{i_1 i_3} \otimes Y_{i_1 i_2} + Y_{i_2 i_3} \otimes Y_{i_1 i_3} + 1 \otimes Y_{i_1 i_2 i_3}$$

not cocommutative unless $k = 1$

Theorem (M., Nica)

$\mathcal{Y}^{(k)}$ is a graded, connected bialgebra and $\mathcal{G}_k \cong \mathbb{X}(\mathcal{Y}^{(k)})$

Sketch.

$$\mu \in \mathcal{G}_k \rightsquigarrow R_\mu : \mathbb{C}\langle\langle z_1, \dots, z_k \rangle\rangle \to \mathbb{C} \rightsquigarrow \chi_\mu : \mathcal{Y}^{(k)} \to \mathbb{C}$$

$\chi_\mu(Y_{i_1 \dots i_n})$ = coef. of R_μ in front of $z_{i_1} \dots z_{i_n}$. ∎

LS-transform

$$\mathcal{G}_k \ni \mu \longmapsto LS_\mu \in \mathbb{C}\langle\langle z_1, \dots, z_k \rangle\rangle$$

$$LS_\mu(z_1, \dots, z_k) = \sum_{\substack{w \in [k]^* \\ |w| \geq 2}} \left[(\log \chi_\mu)(Y_w) \right] z_w$$

$z_{i_1} z_{i_2} \dots z_{i_n}$

Theorem

$$\mu \boxtimes \nu = \nu \boxtimes \mu \Rightarrow LS_{\mu \boxtimes \nu} = LS_\mu + LS_\nu$$

Proof $\mu \leftrightarrow \nu \Rightarrow \chi_\mu \leftrightarrow \chi_\mu \Rightarrow \log \chi_{\mu \boxtimes \nu} = \log \chi_\mu * \chi_\nu = \log \chi_\mu + \log \chi_\nu$ ∎

Remark

$$\mu \boxtimes \nu = \nu \boxtimes \mu \not\Rightarrow LS_\mu \cdot LS_\nu = LS_\nu \cdot LS_\mu .$$

Combinatorial description of LS_μ

$\Gamma = 0 = \pi_0 < \pi_1 < \ldots < \pi_\ell = 1_n$ chain in $NC(n)$

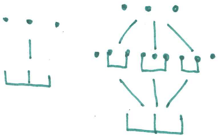
$f \in \mathbb{C}\langle\langle z_1, \ldots, z_k \rangle\rangle$, $\quad Cf_w^{(\Gamma)}(f) = \prod_{j=1}^{\ell} Cf_{w; K_{\pi_j}(\pi_{j-1})}(f)$

<u>Theorem</u>

$$Cf_w(LS_\mu) = (\log \mathcal{X}_\mu)(Y_w) = \sum_{\substack{\Gamma \text{ chain} \\ \text{in } NC(n)}} \frac{(-1)^{1+|\Gamma|}}{|\Gamma|} \underbrace{Cf_w^{(\Gamma)}(R_\mu)}_{\mathcal{X}_\mu(Y_w^{(\Gamma)})}$$

<u>Example</u>

Chains in $NC(3)$



$$Cf_{(i_1 i_2 i_3)}(LS_\mu) = Cf_{i_1 i_2 i_3}(R_\mu) - \frac{1}{2} Cf_{i_1 i_2}(R_\mu) Cf_{i_1 i_3}(R_\mu) - \frac{1}{2} Cf_{i_1 i_3}(R_\mu) Cf_{i_2 i_3}(R_\mu) - \frac{1}{2} Cf_{i_2 i_3}(R_\mu) Cf_{i_1 i_3}(R_\mu).$$

<u>Challenge</u>: Prove the property of LS_μ from combinatorial description.

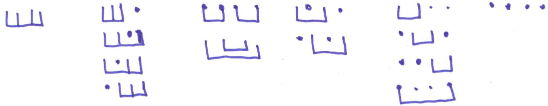
$\boxed{k=1}$

$\mathcal{Y}^{(1)} = \mathbb{C}[Y_n \mid n \geq 2] \qquad Y_n = Y_{\underbrace{11\dots1}_{n}}, \quad Y_1 = Y_0 = 1$

$$\Delta Y_n = \sum_{\substack{\pi=\{A_1,\dots,A_r\}\in NC(n)\\ K(\pi)=\{B_1,\dots,B_s\}}} \underbrace{Y_{|A_1|}\cdots Y_{|A_r|}}_{Y_\pi} \otimes \underbrace{Y_{|B_1|}\cdots Y_{|B_s|}}_{Y_{K(\pi)}}$$

$$\Delta Y_3 = Y_3 \otimes 1 + 3\, Y_2 \otimes Y_2 + 1 \otimes Y_3$$

$$\Delta Y_4 = Y_4 \otimes 1 + 4\, Y_3 \otimes Y_2 + 2\, Y_2^2 \otimes Y_2 + 2\, Y_2 \otimes Y_2^2 + 4\, Y_2 \otimes Y_3 + 1 \otimes Y_4$$



Theorem

$$L S_\mu(z) = -z \log S_\mu(z).$$

Sym = symmetric functions $\subseteq \mathbb{C}[[x_1, x_2, \dots]]$

power symmetric functions $p_n = x_1^n + x_2^n + \dots$

complete monomial symmetric functions $h_n = \sum_{\substack{\alpha_1+\dots+\alpha_r = n \\ i_1 < \dots < i_r}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r}$

elementary symmetric functions

$$e_n = \sum_{i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$\mathrm{Sym} = \mathbb{C}[p_1, p_2, \dots] = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[e_1, e_2, \dots]$$

$$\Delta p_n = p_n \otimes 1 + 1 \otimes p_n \qquad \Delta h_n = \sum_{i+j=n} h_i \otimes h_j \qquad \Delta e_n = \sum_{i+j=n} e_i \otimes e_j$$

Theorem

$$\mathcal{Y}^{(1)} \xrightarrow{\sim} \mathrm{Sym}$$

$$Y_n \longmapsto y_n, \quad y_n = \sum_{\pi \in \{A_1, \dots, A_q\} \in NC(n-1)} e_{|A_1|} e_{|A_2|} \cdots e_{|A_q|}$$

Dictionary

$$\mathcal{G}_1 \xrightarrow{\sim} \mathcal{X}(\mathcal{Y}^{(1)}) \xrightarrow{\sim} \mathcal{X}(\mathrm{Sym})$$

$$\mu \longmapsto \theta_\mu$$

$$\theta_\mu(y_n) = \mathrm{coef}_n(R_\mu)$$

$$\theta_\mu(h_n) = (-1)^n \mathrm{coef}_n(S_\mu)$$

$$\theta_\mu(e_n) = \mathrm{coef}_n\left(\frac{1}{S_\mu}\right)$$

$$\theta_\mu(p_n) = (-1)^{n+1} n\, \mathrm{coef}_n \log(S_\mu) = (-1)^{n-1} n\, \mathrm{coef}_n(LS_\mu).$$

Exercise

$$y_n = \sum_{\pi = \{A_1, \dots, A_q\} \in NC(n-1)} \underbrace{e_{|A_1|} \cdot e_{|A_2|} \cdots e_{|A_q|}}_{e_\pi}$$

$y_1 = 1$
$y_2 = e_1$
$y_3 = e_2 + e_1^2$
$y_4 = e_3 + 3e_1e_2 + e_1^3$
$y_5 = e_4 + 4e_1e_3 + 2e_2^2 + 6e_1^2e_2 + e_1^4$
$\vdots$

$$\Delta e_n = \sum_{i=0}^{n} e_i \otimes e_{n-i}$$

$\Delta e_1 = e_1 \otimes 1 + 1 \otimes e_1, \quad \Delta e_2 = e_2 \otimes 1 + e_1 \otimes e_1 + 1 \otimes e_2, \ \dots$

Find a $\underline{\text{nice}}$ direct combinatorial proof that $\Delta y_n = \sum_{\pi \in NC(n)} y_\pi \otimes y_{K(\pi)}$

$y_{\{A_1, \dots, A_q\}} = y_{|A_1|} \cdots y_{|A_q|}$

Remarks

- y_n's appear in the works of M. Haiman and F. Bergeron on diagonal harmonics; they correspond to signed permutation representation of S_n
- $$y_n = \sum_{\substack{\Gamma = (\pi_0 < \dots < \pi_\ell) \\ \text{chain in } NC(n)}} M_{(|\pi_0| - |\pi_1|, \dots, |\pi_{\ell-1}| - |\pi_\ell|)}, \qquad M_{(a_1, \dots, a_\ell)} = \sum_{i_1 < \dots < i_\ell} x_{i_1}^{a_1} \cdots x_{i_\ell}^{a_\ell}.$$

'Big Picture'

$$
\begin{tikzcd}
 & \mathcal{R} \arrow[dr] & \\
\mathcal{Y}^{(1)} \arrow[ur, "Y_n \mapsto {[NC(n)]}"] \arrow[rr] \arrow[dr, "Y_n \to y_n"'] & & QSym \\
 & Sym \arrow[ur] &
\end{tikzcd}
$$

Rota's bialgebra of ranked posets

Ehrenborg's morphism

Bialgebra of quasi-symmetric functions (Gessel)

Aguiar-Bergeron-Sottile universal morphism (relative to $\zeta_{\mathcal{Y}^{(1)}}$ given by $\zeta_{\mathcal{Y}^{(1)}}(Y_n) = 1$).

Thank You!