Primitive Central Idempotents of Rational Group Algebras

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Outline

Introduction

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Semisimple group algebras

K field, G group

$KG = \bigoplus_{g \in G} Kg = \{\sum_{g \in G} \alpha_g g : \alpha_g \in K\}$ group algebra with

$$(\sum \alpha_g g)(\sum \beta_g g) = \sum \alpha_g \beta_h gh$$

KG semisimple $\Leftrightarrow$ $KG = \bigoplus_i A_i$, A_i simple algebras

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Wedderburn-Artin Theorem

KG is semisimple $\Leftrightarrow KG = \bigoplus_i M_{n_i}(D_i)$, D_i division algebras.

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Maschke Theorem

$$KG \text{ is semisimple} \Leftrightarrow \begin{cases} G \text{ finite} \\ char(K) \nmid |G| \end{cases}$$

Wedderburn components of KG

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- $\{\mathrm{Irr}_K(G)\} \leftrightarrow \{$Wedderburn components of $KG\} \leftrightarrow \{$P.c.i's $KG\}$
- $\{\mathrm{Irr}_{\mathbb{C}}(G)\} \leftrightarrow \{$Wedderburn components of $\mathbb{C}G\} \leftrightarrow \{$Conjugacy classes $G\}$

Primitive central idempotents of KG

$KG = \bigoplus_i A_i$

Every $A_i = KGe_i$, e_i primitive central idempotent of KG.

Every $e_i = e_K(\chi)$, for χ irreducible character of G over $\overline{K}$.

Computation of $e_K(\chi)$: classical method

- For $\mathbb{C}G$: $e_{\mathbb{C}}(\chi) = e(\chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$
- For KG: $e_K(\chi) = \sum_{\sigma \in G_\chi} e(\chi^\sigma) = \sum_{\sigma \in G_\chi} \sigma(e(\chi))$

where $G_\chi = Gal(K(\chi)/K)$ and $K(\chi) = F(\chi(g) | g \in G)$

I. Primitive central idempotents

1. Strongly monomial characters and strong Shoda pairs
2. Primitive central idempotents
3. Primitive idempotents for finite nilpotent groups

I. Primitive central idempotents

1. Strongly monomial characters and strong Shoda pairs

Strongly monomial groups

E. Jespers, G.Leal, A. Paques 2003

- Method for computing the p.c.i's of $\mathbb{Q}G$ for G nilpotent

A. Olivieri, A. del Rio, J.J. Simon 2004

- Method for computing the p.c.i's of $\mathbb{Q}G$ for G monomial
- Description of Wedderburn components for G strongly monomial (in particular, abelian-by-supersolvable)

Idempotents in $\mathbb{Q}G$

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- $H \leq G$ then $\widehat{H} = \frac{1}{|H|}\sum_{h\in H} h$ idempotent in $\mathbb{Q}G$, $\varepsilon(H,H) = \widehat{H}$
- $H \triangleleft G$ iff $\widehat{H}$ central idempotent in $\mathbb{Q}G$
- $K \triangleleft H \leq G$, $\mathcal{M}(H/K) = \{\text{minimal normal subgroups of } H/K\}$

$$\varepsilon(H,K) = \prod_{M/K\in\mathcal{M}(H/K)} (\widehat{K} - \widehat{M}) \text{ idempotent in } \mathbb{Q}G$$

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- G abelian then $\varepsilon(G,H)$ are P.C.I of $\mathbb{Q}G$ if G/H is cyclic
- Monomial group: all irreducible characters are monomial, i.e. induced by linear characters of subgroups of the group.

Shoda pairs

- Shoda pair (H,K) of G:
  $K \triangleleft H \leq G, \quad 1 \to K \to H \to^{\psi} \mathbb{C}$, with ψ^G irreducible
- $K \triangleleft H \leq G$, $\mathcal{M}(H/K) = \{$minimal normal subgroups of $H/K\}$

$$\varepsilon(H,K) = \prod_{M/K \in \mathcal{M}(H/K)} (\widehat{K} - \widehat{M}) \text{ idempotent in } \mathbb{Q}G$$

- T right transversal of $\mathrm{Cen}_G(\varepsilon(H,K))$ in G then

$$e(G,H,K) = \sum_{t \in T} \varepsilon(H,K)^t \text{ central in } \mathbb{Q}G$$

- G-conjugates of $\varepsilon(H,K)$ are orthogonal then $e(G,H,K)$ central idempotent

Shoda pairs

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- (H,K) Shoda pair of G, $\psi : H \to \mathbb{C}$ linear and $K = ker(\psi)$ then
$$e_{\mathbb{Q}}(\psi^G) = \frac{[Cen_G(\varepsilon(H,K)) : H]}{[\mathbb{Q}(\psi) : \mathbb{Q}(\psi^G)]} e(G,H,K)$$

Strong Shoda pairs

Strong Shoda pairs

- Strong Shoda pair (H, K) of G:
  - (H, K) Shoda pair
  - $H \triangleleft N_G(K)$
  - $\forall g \in G \setminus N_G(K)$, $\varepsilon(H, K)\varepsilon(H, K)^g = 0$

Strong Shoda pairs

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  - $\forall g \in G \setminus N_G(K)$, $\varepsilon(H, K)\varepsilon(H, K)^g = 0$
- (H, K) strong Shoda pair then $e(G, H, K)$ primitive central idempotent of $\mathbb{Q}G$
- G abelian-by-supersolvable, then $e \in \mathbb{Q}G$ primitive central idempotent iff $e = e(G, H, K)$ for (H, K) strong Shoda pair of G

I. Primitive central idempotents

2. Primitive central idempotents in $\mathbb{Q}G$, G finite group

Primitive central idempotents of $\mathbb{Q}G$, arbitrary finite G

Theorem

G arbitrary finite group, $|G| = n$, χ irreducible character of G then

$$e_{\mathbb{Q}}(\chi) = \frac{\chi(1)}{\mathbb{Q}(\zeta_n) : \mathbb{Q}(\chi)} \sum_i \frac{a_i}{[G : C_i]} [\mathbb{Q}(\zeta_n : \mathbb{Q}(\psi_i)] e(G, H_i, K_i),$$

- $a_i \in \mathbb{Z}$
- (H_i, K_i) strong Shoda pairs of subgroups M_i of G
- ψ_i linear characters of H_i with kernel K_i
- $C_i = \mathrm{Cen}_G(\varepsilon(H_i, K_i))$.

Primitive central idempotents of $\mathbb{Q}G$, arbitrary finite G

Theorem, G.J. 2012

G finite group, χ irreducible complex character of G, $C_i = \langle c_i \rangle$. Then

$$e_{\mathbb{Q}}(\chi) = \sum_{i=1}^{r} \frac{b_{C_i}\chi(1)}{[G : Cen_G(\varepsilon(C_i, C_i)]} e(G, C_i, C_i)$$

where the sum runs through a set $\{C_1, \ldots, C_r\}$ of representatives of conjugacy classes of cyclic subgroups of G and

$$b_{C_i} = \frac{[G : Cen_G(C_i)]}{[G : C_i]} \sum_{C_i^* \geq C_i} \mu([C_i^* : C_i])(\sum_{\sigma \in G_\chi} \sigma(\chi))(z^*)$$

where the sum rungs through all the cyclic subgroups C_i^* of G which contain C_i and z^* is a generator of C_i^*.

I. Primitive central idempotents

3. Primitive idempotents of $\mathbb{Q}Ge$, G finite nilpotent group

Primitive idempotents of $\mathbb{Q}Ge$, G finite nilpotent group

Theorem(Jepsers, Olteanu, Del Río, 2011)

Let G be a finite nilpotent group, e a p.c.i. of $\mathbb{Q}G$.

A complete set of orthogonal p.i of the simple compoent $\mathbb{Q}Ge$ consists of the conjugates of an element β_e by the elements of a set T_e, where β_e and T_e are defined algorithmatically in 4 cases.

Primitive idempotents of $\mathbb{Q}Ge$, G finite nilpotent group

Theorem

G finite nilpotent group, (H,K) strong shoda pair of G, $e=e(G,H,K)$, $\varepsilon=\varepsilon(H,K)$, $H/K=\langle\overline{a}\rangle$, $H/K=\langle\overline{a}\rangle$, $H_2/K=\langle\overline{a_2}\rangle$, $H_{2'}/K=\langle\overline{a_{2'}}\rangle$, $N=N_G(K)$. Then $\langle\overline{a_{2'}}\rangle$ has a cyclic complement $\langle\overline{b_{2'}}\rangle$ in $N_{2'}/K$.

1. H_2/K has a complement M_2/K in N_2/K
   - (i) $\overline{a_2}^{2^{n-2}}$ is central in N_2/K and M_2/K is cyclic
   - (ii) otherwise
2. H_2/K has no complement in N_2/K
   - (i) either $H_{2'}=K$ or the order of 2 modulo $m=\frac{[H_{2'}:K]}{[N_{2'}:H_{2'}]}$ is odd and $n-k\leq 2$
   - (ii) $H_{2'}\neq K$ and either the order of 2 modulo m is even or $n-k>2$

Primitive idempotents of $\mathbb{Q}Ge$, G finite nilpotent group

Theorem

Moreover, a complete set of primitive idempotents of $\mathbb{Q}Ge$ is given by the conjugates of $\beta_e = \widehat{b_{2'}}\beta_{2\epsilon}$ by the elements of $T_e = T_{2'} T_2 T_{G/N}$, where:

$$\beta_2 = \begin{cases} \widehat{M_2}, & \text{in case (1)} \\ \widehat{b_2}, & \text{in case (2.i)} \\ \widehat{b_2}\frac{1+xa_2^{2^{n-2}}+ya_2^{2^{n-2}}c_2}{2}, & \text{in case (2.ii)} \end{cases}$$

$T_{2'} = \{1, a_{2'}, \dots, a_{2'}^{[N_{2'}:H_{2'}]-1}\}$

$T_2 =$

$$\begin{cases} \{1, a_2, \dots, a_2^{2^k-1}\}, & \text{in (1.i), (2.i)} \\ \{1, a_2, \dots, a_2^{2^{k-1}-1}, a_2^{2^{n-2}}, a_2^{2^{n-2}+1}, \dots, a_2^{2^{n-2}+2^{k-1}-1}\}, & \text{in (1.ii)} \\ \{1, a_2, \dots, a_2^{2^{k-1}-1}, c_2, a_2c_2, \dots, a_2^{2^{k-1}-1}c_2\}, & \text{in (2.ii)} \end{cases}$$

$T_{G/N} =$ left transversal of N in G

finite group rings

Theorem (Olteanu, Van Gelder 2011)

Let G be a finite nilpotent group and $\mathbb{F}$ a finite field such that $\mathbb{F}G$ is semisimple and let e be a p.c.i of $\mathbb{F}G$.
A complete set of orthogonal p.i. of $\mathbb{F}Ge$ consists of the conjugates of β_e by the elements of T_e, where β_e and T_e are defined algorithmically in 3 cases.

Thank you