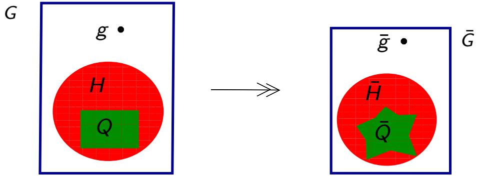
The Virtual Haken Conjecture and Relatively Hyperbolic Groups



Eduardo Martínez-Pedroza
Memorial University of Newfoundland.

August 14, 2012

Plan of the Talk.

Borel conjecture
1953

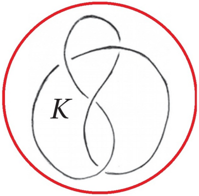
Virtual Haken Conjecture
1968

Geometrization conjecture
1980

Plan of the Talk.

- Borel 1953
- Kneser 1929, Milnor 1962
- Haken 1962
- Waldhausen 1968
- Thurston 1980
- Perelman 2002
- Kahn-Markovic 2009
- Haglund-Wise 2010
- Agol 2012

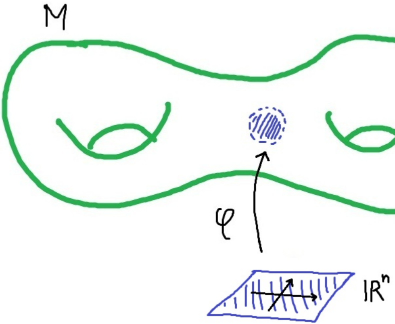
Borel conjecture 1953



Virtual Haken Conjecture 1968

Geometrization Conjecture 1980

n-dimensional Manifolds

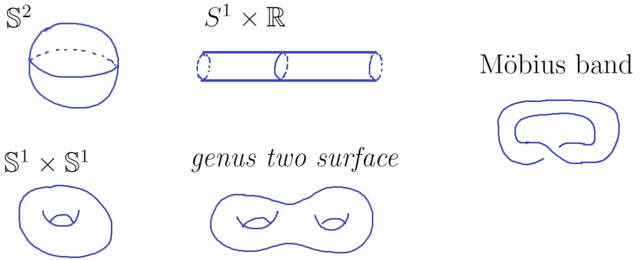


Definition. An n-manifold is a topological space that is locally homeomorphic to $\mathbb{R}^n$ ... and it is also Hausdorff and second countable.

Manifolds

Examples of 2-manifolds

- $\mathbb{R}^2$, the 2-sphere $\mathbb{S}^2$, the 2-torus $\mathbb{S}^1 \times \mathbb{S}^1$.
- The infinite cylinder $S^1 \times \mathbb{R}$.
- The genus two surface.



Closed = no boundary and compact .
Orientable = two sided = has no embedded Möbius bands.

Orientable and Closed 2-Manifolds

Classification and Geometrization

Closed = no boundary and compact .
Orientable = two sided = has no embedded Möbius bands.

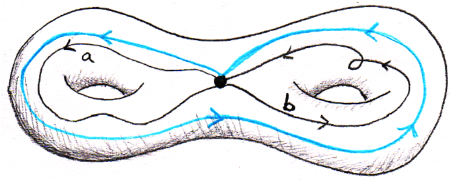
Dehn-Heegaard, 1907. [Classification of orientable closed surfaces]

Klein, 1870's. [Geometrization]
Any closed, orientable 2-manifold can be represented as a quotient of $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$ by a discrete group of isometries.

$$Isom(\mathbb{S}^2) = O(3), \quad Isom(\mathbb{E}^2) = \mathbb{R}^2 \rtimes O(1), \quad Isom(\mathbb{H}^2) = O(2,1)_+$$

The fundamental group

Recall that the fundamental group $\pi_1 M$ of a manifold M is the group of loops, up to homotopy:



- $\pi_1 \mathbb{S}^1 = \mathbb{Z}$ since $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.
- $\pi_1 \mathbb{S}^2 = 1$
- $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$, since the torus $\mathbb{S}^1 \times \mathbb{S}^1$ equals $\mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$.
- π_1 of the above surface is $\langle a, b, c, d \colon [a, b][c, d] = 1 \rangle$.

Figure by N. Dunfield; "Surfaces in finite covers of 3-manifolds"

Orientable and Closed 2-Manifolds

Classification and Geometrization

Dehn-Heegaard, 1907. [Classification of orientable closed surfaces]

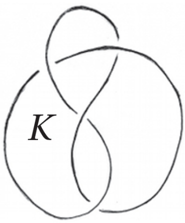
Corollary. [2-Dimensional Borel Conjecture] A pair of closed 2-manifolds are isomorphic iff they have isomorphic fundamental groups.

Klein, 1870's. [Geometrization] Any closed, orientable 2-manifold is modeled in one of the three geometries: $Isom(\mathbb{S}^2)$, $Isom(\mathbb{E}^2)$, $Isom(\mathbb{H}^2)$.

3-Manifolds

Examples

- $\mathbb{R}^3$, the 3-sphere $\mathbb{S}^3$, products as the 3-torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, or $\mathbb{S}^1 \times \mathbb{S}^2$.
- The 3-manifold $\mathbb{R}^3 \setminus \{point\}$ is isomorphic to the thick 2-sphere $\mathbb{S}^2 \times \mathbb{R}$.
- $\mathbb{R}^3 \setminus \{z - axis\}$ is isomorphic to the solid torus $\mathbb{S}^1 \times \mathbb{R}^2$.
- A knot complement: $\mathbb{S}^3 \setminus K$

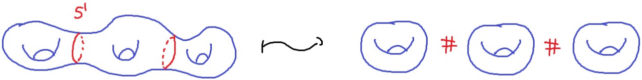


Splitting 3-manifolds into simpler pieces

Cutting 3-manifolds along spheres.

Prime 3-manifold. A 3-manifold is *prime* if it cannot be expressed as a non-trivial connected sum of two 3-manifolds. Non-trivial means that neither of the two is an 3-sphere.

A 2-dimensional illustration:



Kneser 1929, Milnor 1962.[Cutting along spheres] Every closed orientable 3-manifold M factors as a connected sum of prime 3-manifolds $M = P_1 \# \cdots \# P_n$.

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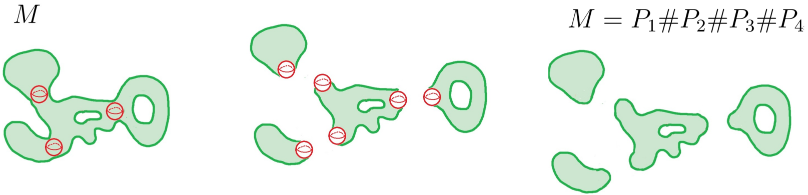

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The Borel Conjecture

Manifolds determined by their fundamental group

$$M \longrightarrow N$$

$$\pi_1 M \longrightarrow \pi_1 N$$

The conjectured appeared in a letter of May 2^{nd}, 1953 from Armand Borel to Jean Paul Serre. The letter discusses a paper of Mostow.

Borel Conjecture for 3-manifolds. Suppose that M and N are *irreducible* with infinite fundamental group. If $\pi_1 M$ and $\pi_1 N$ are isomorphic groups, then M and N are isomorphic 3-manifolds.

The Borel Conjecture

Manifolds determined by their fundamental group

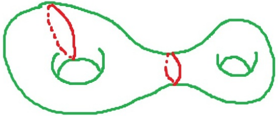
A letter of May 2^{nd}, 1953 from Armand Borel to Jean Paul Serre.

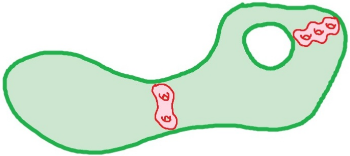
"...Nevertheless you have probably seen the abstract of Mostow announcing that if G_1 and G_2 are solvable Lie groups, and if G_1/H_1 and G_2/H_2 are compact and have isomorphic fundamental groups, they are homeomorphic. I read his paper, ... and noticed a basic point of the following kind: Let B_1 and B_2 two compact manifolds, classifying spaces for a group G (say, discrete) and in any dimension. Are they homeomorphic? and if so, are they homeomorphic by the projection of a homomorphism of universal spaces? Mostow, by clever choices of subgroups and inductions, essentially reduces to the case where B_1 and B_2 are tori, and the answer to both questions is then obviously yes. Overall, the paper is very interesting ..."

Borel Conjecture for 3-manifolds. Suppose that M and N are *irreducible* with infinite fundamental group. If $\pi_1 M$ and $\pi_1 N$ are isomorphic groups, then M and N are isomorphic 3-manifolds.

Haken, beyond the four-color theorem

Cutting 3-manifolds along incompressible surfaces





Haken manifold. A 3-manifold M is *Haken* if it is prime and contains an embedded 2-sided surface S such that $\pi_1 S \to \pi_1 M$ is injective and S is not a sphere.

Haken manifolds are called *sufficiently large*.

Haken, 1962 Algorithm to cut a Haken 3-manifold M into simpler 3-manifolds. Start cutting along an *incompressible surface*, the algorithm ends in a finite number of steps with information of how to reconstruct M starting from a collection of 3-balls.

Waldhausen: The modern study of 3-manifolds

A step forward towards the Borel Conjecture

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Waldhausen, 1968 The Borel conjecture holds in the class of Haken aspherical 3-manifolds.

Waldhausen: The modern study of 3-manifolds

A step forward towards the Borel Conjecture

On irreducible 3-manifolds which are sufficiently large

By FRIEDHELM WALDHAUSEN

We are mainly concerned with the questions whether any homotopy equivalence between compact orientable PL 3-manifolds can be induced by a homeomorphism, and whether homotopic homeomorphisms are also isotopic.

COROLLARY 6.5. *Let M and N be manifolds which are irreducible and boundary-irreducible. Suppose M is sufficiently large. Let $\psi\colon \pi_1(N) \to \pi_1(M)$ be an isomorphism which respects the peripheral structure. Then there exists a homeomorphism $f\colon N \to M$, which induces ψ.*

Remark. Of those irreducible manifolds, known to me, which have infinite fundamental group and are not sufficiently large [19], some (and possibly all) have a finite cover which is sufficiently large.

The Annals of Mathematics, Second Series, Vol. 87, No. 1 (Jan., 1968)

Waldhausen: The modern study of 3-manifolds

The Virtual Haken Conjecture

Borel Conjecture for 3-manifolds, 1953. Suppose that M and N are *prime* with infinite fundamental group. If $\pi_1 M$ and $\pi_1 N$ are isomorphic groups, then M and N are isomorphic 3-manifolds.

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The Virtual Haken Conjecture. Every 3-manifold has a finite cover that is Haken.

Remark. VHC implies BC for 3-manifolds

Thurston: A revolution in 3-manifold topology

Geometrization Conjecture

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 3, May 1982

THREE DIMENSIONAL MANIFOLDS, KLEINIAN GROUPS AND HYPERBOLIC GEOMETRY

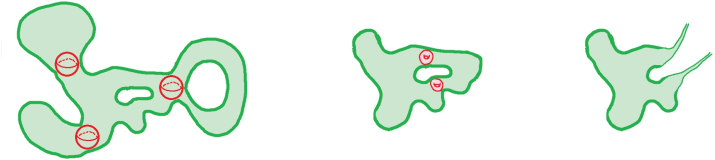
BY WILLIAM P. THURSTON

Three-manifolds are greatly more complicated than surfaces, and I think it is fair to say that until recently there was little reason to expect any analogous theory for manifolds of dimension 3 (or more)—except perhaps for the fact that so many 3-manifolds are beautiful. The situation has changed, so that I feel fairly confident in proposing the

1.1. CONJECTURE. *The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures.*

Thurston: A revolution in 3-manifold topology

The Geometrization Conjecture



A manifold M split into a connected sum of prime manifolds $M = P_1 \# \ldots \# P_m$. Each prime factor can be splitted along tori. Each component of the splitting admits a geometric structure modeled in one of the eight geometries.

$$\mathbb{S}^3,\ \mathbb{N}il,\ \mathbb{P}SL,\ \mathbb{S}ol,\ \mathbb{H}^3,\ \mathbb{S}^3 \times \mathbb{E}^1,\ \mathbb{E}^3,\ \mathbb{H}^2 \times \mathbb{E}^1.$$

Thurston: A revolution in 3-manifold topology

Strong Evidence for Geometrization of 3-manifolds

THREE DIMENSIONAL MANIFOLDS, KLEINIAN GROUPS AND HYPERBOLIC GEOMETRY

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1.1. CONJECTURE. *The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures.*

A 3-manifold M^3 is called a *Haken* manifold if it is prime and it contains a 2-sided incompressible surface (whose boundary, if any, is on ∂M) which is not a 2-sphere.

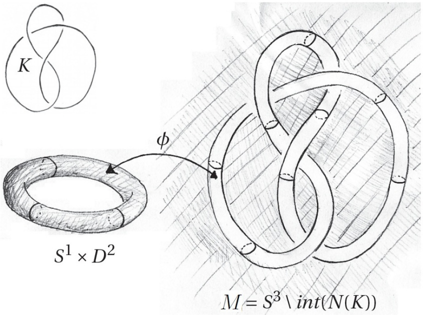
THEOREM. *Conjecture* 1.1 *is true for Haken manifolds.*

Thurston: A revolution in 3-manifold topology

A Change on the landscape of 3-manifolds

The Geometry and Topology of Three-Manifolds

William P. Thurston

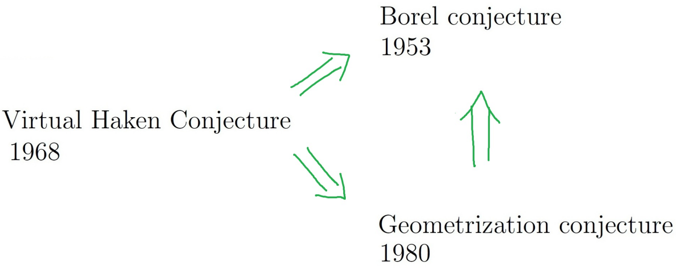


THEOREM. *The Dehn surgery manifold* $M_{(m,l)}$ *is* irreducible, *and it is a Haken-manifold if and only if* $(m,l) = (0, \pm 1)$ *or* $(\pm 4, \pm 1)$.

From the electronic edition of the 1980 Thurston's notes distributed by Princeton University.

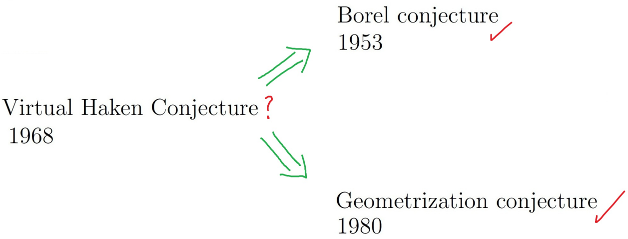
Figure from "Surfaces in finite convers of 3-manifolds" by N. Dunfield

After Thurston and before 2002...



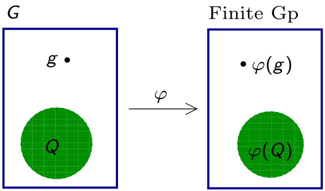
The solution of geometrization by Perelman in 2002

The virtual Haken the remaining major problem



After geometrization, it is enough to prove the virtual Haken conjecture for hyperbolic 3-manifolds. This is an algebraic problem on lattices of $PSL(2, \mathbb{C})$ solved by Ian Agol in 2012 relying on work of Kahn-Marcovic and Hanglund-Wise.

2. Subgroup Separability.



The Virtual Haken Conjecture

Solution

Strategy: Let M be a closed hyperbolic manifold. If $\pi_1 M$ is subgroup separable and contains a closed surface subgroup, then M is virtually Haken.

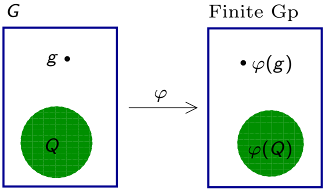
Khan-Markovic, 2009: If M is a hyperbolic closed manifold, then $\pi_1 M$ contains a surface subgroup.

Agol, Hanglund-Wise, 2012: If M is a hyperbolic closed manifold, then $\pi_1 M$ is subgroup separable.

Subgroup Separability

Separating f.g subgroups from elements in finite quotients

Definition. A subgroup Q of G is separable if for each $g \notin Q$, there is a homomorphism φ onto a finite group such that $\varphi g \notin \varphi(Q)$.



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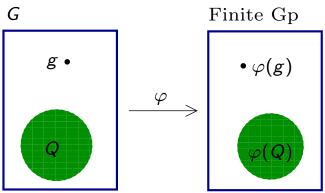

A group G is subgroup separable if every finitely generated subgroup is separable.

Subgroup Separability

Examples

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Examples of Subgroup Separable Groups

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Hall: Free groups.

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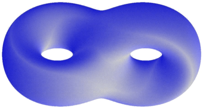
Scott: Surface groups.



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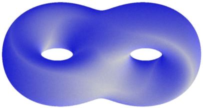

Agol-Long-Reid: Bianchi groups (arithmetic lattices in $PSL(2; \mathbb{C})$).

Subgroup Separability

Non-examples

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Mihailova: $F_2 \times F_2$ is not subgroup separable.

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Proof for $n \geq 4$: $SL(2, \mathbb{Z})$ contains a copy of F_2, and $SL(n, \mathbb{Z})$ contains a copy of $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ if $n \geq 4$.

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Burns-Karrass-Solitar: There are 3-manifold groups that are not subgroup separable.

Subgroup Separability

3-manifold connection

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A group G is subgroup separable if every finitely generated subgroup is separable.

Thurston's question: Are lattices of $PSL(2, \mathbb{C})$ subgroup separable?

Motivation: Let M be a hyperbolic manifold. If $\pi_1 M$ is subgroup separable and contains a closed surface subgroup, then M is virtually Haken.

Subgroup Separability

Separating in Non-uniform lattices

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A group G is subgroup separable if every finitely generated subgroup is separable.

Thurston's question 1980: Are discrete subgroups of $PSL(2,\mathbb{C})$ subgroup separable? Agol. Yes.

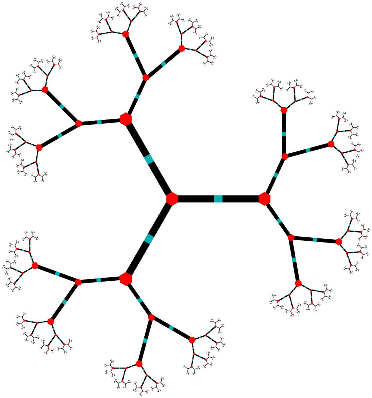
Proof.

- Morgan, 1984: any discrete subgroup of $PSL(2,\mathbb{C})$ embeds in a lattice of $PSL(2,\mathbb{C})$.
- Joint with J. Manning 2010: In $PSL(2,\mathbb{C})$, subgroup separability of uniform lattices implies subgroup separability of non-uniform lattices.
- Agol 2012: Uniform lattices in $PSL(2,\mathbb{C})$ are subgroup separable.

Toral Relatively Hyperbolic Groups

Main example

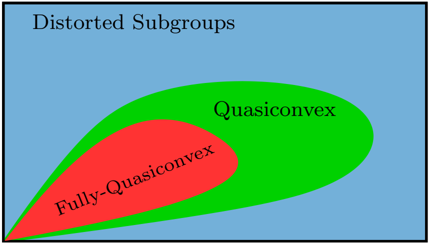
Farb. Any lattice in $PSL_2(\mathbb{C})$ is a hyperbolic group relative to its maximal parabolic subgroups. The maximal parabolic subgroups are virtually free abelian groups.



Subgroups of Toral Relatively Hyperbolic Groups

Geometric and Non-geometric subgroups

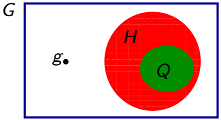
Subgroups of a (relatively) hyperbolic group:



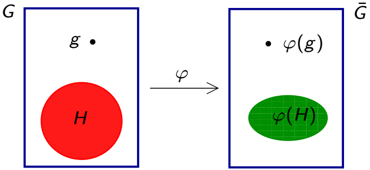
Separation of Quasiconvex Subgroups

MP-Manning: Let G be a (toral) relatively hyperbolic group.

Thm 1: If Q is quasiconvex and $g \notin Q$, then there is a fully quasiconvex $H < G$ such that $Q < H$ and $g \notin H$.



Thm 2: If H is fully-quasiconvex and $g \notin H$, then H and g are separated in a hyperbolic quotient $\bar{G}$. The image of H is quasiconvex in $\bar{G}$.



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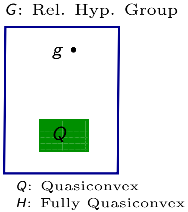

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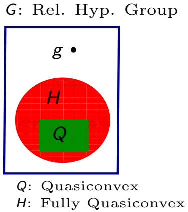

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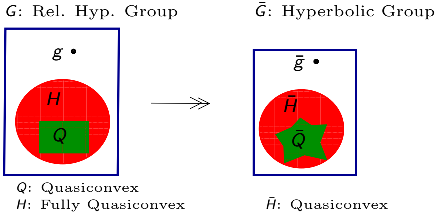

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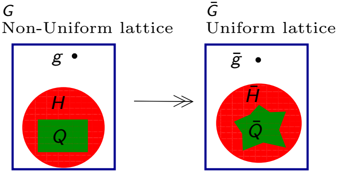
MP-Manning: Let G be a (toral) relatively hyperbolic group.

Thm 1: If Q is quasiconvex and $g \notin Q$, then there is a fully quasiconvex $H < G$ such that $Q < H$ and $g \notin H$.

Thm 2: If H is fully-quasiconvex and $g \notin H$, then H and g are separated in a hyperbolic quotient $\bar{G}$. The image of H is quasiconvex in $\bar{G}$.

Corollary In $PSL(2,\mathbb{C})$, subgroup separability of uniform lattices implies subgroup separability of non-uniform lattices.



Thurston: 2π-filling theorem

Agol, Calegari-Gabai: Geometrically infinite subgroups of lattices in $PSL(2,\mathbb{C})$ are separable.