On torsion units in $\mathbb{Z}\mathrm{Sz}(q)$

Andreas Bächle

Universität Stuttgart
Fachbereich Mathematik

St. John's, June 2009

Notations

G	finite group
$\mathbb{Z}G$	integral group ring of G
$\mathrm{U}(\mathbb{Z}G)$	group of units of $\mathbb{Z}G$
$\mathrm{V}(\mathbb{Z}G)$	group of normalized units of $\mathbb{Z}G$, i.e.

$$\mathrm{V}(\mathbb{Z}G) = \left\{ \sum_{g \in G} u_g g \in \mathrm{U}(\mathbb{Z}G) : \sum_{g \in G} u_g = 1 \right\}$$

As $\mathrm{U}(\mathbb{Z}G) = \pm\mathrm{V}(\mathbb{Z}G)$ we don't lose much, if we restrict our interest to $\mathrm{V}(\mathbb{Z}G)$.

First Zassenhaus conjecture

Question

How far does the structure of the group G determine the structure of the torsion units in $\mathbb{Z}G$?

G abelian $\Longrightarrow$ all torsion elements of $\mathrm{V}(\mathbb{Z}G)$ are G (Higman, 1939)

Hans Zassenhaus conjectured, that for an arbitrary group the normalized torsion units of $\mathbb{Z}G$ are not too "far away" from the group:

Conjecture (First Zassenhaus conjecture, 1970s)

(ZC1) *Every normalized torsion unit of the integral group ring $\mathbb{Z}G$ is rationally conjugate to an element of the group.*

[u and v rationally conjugate: $\exists\, x \in \mathrm{U}(\mathbb{Q}G)$ with $x^{-1}ux = v$]

(ZC1) has been verified for the following finite groups

- ✓ A_5 (Luthar, Passi, 1989)
- ✓ finite nilpotent groups (Weiss, 1991)
- ✓ groups of order at most 71 (Höfert, 2004)
- ✓ $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,13)$ (Hertweck, 2004)
- ✓ A_6 (Hertweck, 2008)
- ✓ finite metacyclic groups (Hertweck, 2008)

But especially for simple groups it seems to be hard to attack this conjecture and it seems plausible to study a weakened version first:

Prime graph question

The prime graph (or Gruenberg-Kegel graph) of a group H is the undirected loop-free graph $\Pi(H)$ with

- Vertices: primes p, for which there exists an element of order p in H
- Edges: primes p and q joined iff there is an element of order pq in H

Example: $\Pi(\mathrm{A}_7)$ $|\mathrm{A}_7| = 2^3 \cdot 3^2 \cdot 5 \cdot 7$

Question (Prime graph question, Kimmerle, 2005)

(PQ) *Is it true, that* $\Pi(\mathrm{V}(\mathbb{Z}G)) = \Pi(G)$*?*

Clearly: **(ZC1)** $\Longrightarrow$ **(PQ)**.

Furthermore there is an affirmative answer to **(PQ)** for finite

- ✓ Frobenius groups (Kimmerle, 2006)
- ✓ soluble groups (Höfert, Kimmerle, 2006)
- ✓ 13 sporadic simple groups (Bovdi, Konovalov, et. al., 2005 - 2008)

A criterion for rational conjugacy

Let C be a conjugacy class of G, $u = \sum_{g \in G} u_g g \in \mathbb{Z}G$ then

$$\varepsilon_C(u) = \sum_{g \in C} u_g$$

is the partial augmentation of u with respect to C.

Theorem (Marciniak, Ritter, Sehgal, Weiss, 1987; Luthar, Passi, 1989)

Let $u \in \mathrm{V}(\mathbb{Z}G)$ be a torsion unit of order k. u is rationally conjugate to an element of G $\iff \forall\, d \,|\, k$ all partial augmentations of u^d but one vanish.

For confirming **(ZC1)**: Fix a possible order k of torsion units in $\mathrm{V}(\mathbb{Z}G)$ and

- if there is an element of this order in G, try to show that all units of this order in $\mathrm{V}(\mathbb{Z}G)$ satisfy the premise of the above theorem
- if there is no element of this order in G, try to show there is no element of this order in $\mathrm{V}(\mathbb{Z}G)$

Which orders of torsion units may appear?

Theorem (Cohn, Livingstone, 1965)

Let $u \in \mathbb{Z}G$ be a normalized torsion unit of order k. Then k divides the exponent of G.

Which conjugacy classes have to be taken into account?

Theorem (Berman, 1955; Higman, 1939)

Let $u = \sum_{g \in G} u_g g \in \mathbb{Z}G$ a normalized torsion unit, $u \neq 1$. Then $u_1 = 0$.

Theorem (Hertweck, 2004)

Let $u \in \mathbb{Z}G$ be a normalized torsion unit and C a conjugacy class of G. If the order of the elements of C does not divide the order of u, then $\varepsilon_C(u) = 0$.

In many cases this is not sufficient. A method developed by Luthar and Passi, and improved by Hertweck, may provide help.

Theorem (Luthar, Passi, 1989; Hertweck, 2004)

Let $u \in \mathbb{Z}G$ be a torsion unit of order k and let

- *$p = 0$ and χ an ordinary character of G or*
- *p a prime not dividing k and χ be a p-modular Brauer character of G (as function with values in $\mathbb{C}$).*

If $\zeta \in \mathbb{C}$ is a primitive k-th root of unity, then for every integer ℓ the number

$$\mu_\ell(u, \chi, p) = \frac{1}{k} \sum_{d|k} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\zeta^{-d\ell})$$

is a non-negativ integer.

$$\mu_\ell(u,\chi,p) = \frac{1}{k}\sum_{d|k} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\zeta^{-d\ell})$$
$$= \underbrace{\frac{1}{k}\sum_{\substack{d|k\\ d\neq 1}} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\zeta^{-d\ell})}_{=:a_{\chi,\ell}} + \frac{1}{k}\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(u)\zeta^{-\ell})$$

As $\chi(u) = \sum_C \varepsilon_C(u)\chi(C)$, where $\chi(C)$ is the value of χ on the class C, we obtain linear equations

$$t_{C_1}\varepsilon_{C_1}(u) + t_{C_2}\varepsilon_{C_2}(u) + \ldots + t_{C_h}\varepsilon_{C_h}(u) + a_{\chi,\ell} = \mu_\ell(u,\chi,p).$$

for the partial augmentations $\varepsilon_{C_i}(u)$, with "known" coefficients $t_{C_j}, a_{\chi,\ell}$ and $\mu_\ell(u,\chi,p)$.

Example: The smallest Suzuki group $\mathrm{Sz}(8)$ ...

- ... is a simple group
- ... of order $|\mathrm{Sz}(8)| = 2^6 \cdot 5 \cdot 7 \cdot 13 = 29\,120$
- ... has conjugacy classes

  `1a 2a 4a 4b 5a 7a 7b 7c 13a 13b 13c`

- ... and prime graph

Example: torsion units of order 13 in $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$

Let $u \in \mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$ be a torsion unit of order 13, then u has possibly non-trivial partial augmentation on the following conjugacy classes

1a 2a 4a 4b 5a 7a 7b 7c **13a 13b 13c**

We are not done yet. But using the Luthar-Passi method with φ_2, one of the irreducible degree 4 Brauer characters modulo 2 we get the inequalities

Example: torsion units of order 13 in $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$

$$
\begin{array}{rcrrrrl}
13\,\mu_1(u,\varphi_2,2) & = & 9\,\varepsilon_{\mathsf{13a}}(u) & -\ 4\,\varepsilon_{\mathsf{13b}}(u) & -\ 4\,\varepsilon_{\mathsf{13c}}(u) & +\ 4 & \geq 0 \\
13\,\mu_2(u,\varphi_2,2) & = & -\ 4\,\varepsilon_{\mathsf{13a}}(u) & +\ 9\,\varepsilon_{\mathsf{13b}}(u) & -\ 4\,\varepsilon_{\mathsf{13c}}(u) & +\ 4 & \geq 0 \\
 & & -\ 5\,\varepsilon_{\mathsf{13a}}(u) & -\ 5\,\varepsilon_{\mathsf{13b}}(u) & -\ 5\,\varepsilon_{\mathsf{13c}}(u) & +\ 5 & \geq 0 \\
\hline
 & & & & -\ 13\,\varepsilon_{\mathsf{13c}}(u) & +\ 13 & \geq 0
\end{array}
$$

We obtain the third inequality since u is normalizd. By adding the three upper inequalites, we obtain the fourth.
Hence $\varepsilon_{\mathsf{13c}}(u) \leq 1$. Similarly

$$0 \ \leq \ \varepsilon_{\mathsf{13a}}(u), \varepsilon_{\mathsf{13b}}(u), \varepsilon_{\mathsf{13c}}(u) \ \leq \ 1$$

and as $\varepsilon_{\mathsf{13a}}(u) + \varepsilon_{\mathsf{13b}}(u) + \varepsilon_{\mathsf{13c}}(u) = 1$ there must be exactly one non-vanishing partial augmentation.

Theorem (Marciniak, Ritter, Sehgal, Weiss, 1987; Luthar, Passi, 1989)

Let $u \in \mathrm{V}(\mathbb{Z}G)$ be a torsion unit of order k. u is rationally conjugate to an element of $G \iff \forall\, d \mid k$ all partial augmentations of u^d but one vanish.

Example: torsion units of order 2 in $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$

Let $u \in \mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$ be a torsion unit of order 2, then u has possibly non-trivial partial augmentation on the following conjugacy classes

1a **2a** 4a 4b 5a 7a 7b 7c 13a 13b 13c

Hence: u is rationally conjugate to an element of $\mathrm{Sz}(8)$.

More general:

With group theoretical arguments one can show, that for all possible $q = 2^{2m+1}$, positive $m \in \mathbb{Z}$, there is exactly one conjugacy class of elements of order 2 and 5 in $\mathrm{Sz}(q)$.

With the same argumentation as above we have

Proposition

All torsion units of $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(q))$ of order 2 or 5 are rationally conjugate to group elements.

Proposition

Let $u \in \mathbb{Z}\mathrm{Sz}(8)$ be a normalized torsion unit, then

1. *the order of u coincides with the order of an element of $\mathrm{Sz}(8)$,*
2. *if u is of order $2, 5$ or 13, then u is rationally conjugate to an element of $\mathrm{Sz}(q)$,*
3. *if u is of order 4, there are at most 12 possible tupels of partial augmentations for u,*
4. *if u is of order 7, there are at most 6 possible tupels of partial augmentations for u.*

This proposition gives a postive answer to **(PQ)** for $\mathrm{Sz}(8)$:

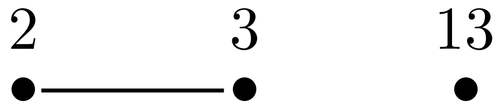
$$\Pi(\mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))) = \Pi(\mathrm{Sz}(8))$$

$$\overset{2}{\bullet} \qquad \overset{5}{\bullet} \qquad \overset{7}{\bullet} \qquad \overset{13}{\bullet}$$

We even obtain that every cyclic subgroup of $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$ is isomorphic to one of $\mathrm{Sz}(8)$.

Example: The group $\mathrm{PSL}(3,3)$ ...

- ... is a simple group of order $|\mathrm{PSL}(3,3)| = 2^4 \cdot 3^3 \cdot 13 = 5\,616$
- ... and has the prime graph



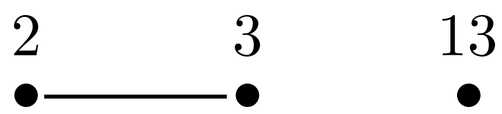
Proposition

Let $u \in \mathbb{Z}\mathrm{PSL}(3,3)$ be a normalized torsion unit, then

1. *if u is a 2- or a 13-element, then u is rationally conjugate to an element of $\mathrm{PSL}(3,3)$.*
2. *u can not have order 26 or 39.*

And we have again an affirmative answer to the prime graph question for $\mathrm{PSL}(3,3)$:

$$\Pi(\mathrm{V}(\mathbb{Z}\mathrm{PSL}(3,3))) = \Pi(\mathrm{PSL}(3,3))$$



Turning our attention to subgroups of the torsion units:

Question (Subgroup question)

$$H \le \mathrm{V}(\mathbb{Z}G) \text{ finite} \quad \Longrightarrow \quad H \text{ isomorphic to a subgroup of } G?$$

The answer is in general no. (Hertweck, 1998)

If we focus on elementary abelian subgroups, we can prove for the series of the Suzuki groups:

Proposition

For any prime p the elementary abelian p-subgroups of $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(q))$ are isomorphic to subgroups of $\mathrm{Sz}(q)$.

Corollary

If $p \in \{2, 5\}$ this isomorphism can be taken as conjugation with an unit of $\mathbb{Q}\mathrm{Sz}(q)$.

Proposition

For any prime p the elementary abelian p-subgroups of $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(q))$ are isomorphic to subgroups of $\mathrm{Sz}(q)$.

Proof. If $p = 2$: Let $H \leq \mathrm{V}(\mathbb{Z}\mathrm{Sz}(q))$, $H \cong \mathrm{C}_2^k$.
Let $q = 2^{2m+1}$, $r = 2^m$ and $t \in \mathrm{Sz}(q)$ be an involution:

	1	x^a	y^b	z^c	t	f	f^{-1}
δ_1	$r(q-1)$	$.$	1	-1	$-r$	ri	$-ri$

We have

$$\langle (\delta_1)_H, 1_H \rangle_H = \frac{1}{|H|} \sum_{h \in H} \delta_1(h) = \frac{1}{|H|} \Big(r(q-1) + (2^k - 1)(-r) \Big) \geq 0.$$

Hence

$$2^{2m+1} - 1 \geq 2^k - 1,$$

and $k \leq 2m+1$. On the other hand $\mathrm{Z}(F) \cong \mathrm{C}_2^{2m+1}$, where $F \in \mathrm{Syl}_2(\mathrm{Sz}(q))$.

A minimal simple group is isomorphic to a group of the series

- $\mathrm{PSL}(2, q)$,
- $\mathrm{Sz}(q)$,

or to

- $\mathrm{PSL}(3, 3)$.

Combinig the last result,

Theorem (Hertweck, Höfert, Kimmerle, 2008)

For any prime p the elementary abelian p-subgroups of $\mathrm{V}(\mathbb{Z}\mathrm{PSL}(2, q))$, q a prime power, are isomorphic to subgroups of $\mathrm{PSL}(2, q)$.

and checking $\mathrm{PSL}(3, 3)$ separately, we obtain

Proposition

Let G be a minimal simple group, p a prime number. Then any elementary abelian p-group $H \leq \mathrm{V}(\mathbb{Z}G)$ is isomorphic to a subgroup of G, except possibly the case that $G \cong \mathrm{PSL}(3, 3)$ and $H \cong \mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_3$.

Quantum subgroups of $GL_{\alpha,\beta}(n)$

Gastón Andrés García

Universidad Nacional de Córdoba, Argentina
CIEM-CONICET

Groups and Hopf Algebras

International Workshop & Special Session of Summer Meeting of CMS

June 2009, St. John's, Canada.

Plan

I Introduction

II Construction and characterization of quantum subgroups of $GL_{\alpha,\beta}(n)$.

III Application: new families of Hopf algebras.

$$\mathrm{Alg}(\mathcal{O}(G), \mathbb{C}) = G \longleftrightarrow \mathcal{O}(G) \text{ comm. Hopf alg.}$$

$$\Gamma \hookrightarrow G \longleftrightarrow \mathcal{O}(G) \twoheadrightarrow \mathcal{O}(\Gamma)$$

$$G_q \longleftrightarrow \mathcal{O}_q(G) \text{ non-comm. Hopf alg.}$$

$$\Gamma_q \hookrightarrow G_q \longleftrightarrow \mathcal{O}_q(G) \twoheadrightarrow A$$

Category of quantum groups : $\mathcal{H}opf^{\text{op}}$

Quantum subgroups of $G_{\mathbf{q}} \rightsquigarrow$ Hopf algebras quotients of $\mathcal{O}_{\mathbf{q}}(G)$

Problem

Determine all quantum subgroups of a given quantum group.

- First considered by Podlés for $\mathcal{O}_q(SU(2))$ y $\mathcal{O}_q(SO(3))$.
- E. Müller solved it for finite-dimensional quotients of $\mathcal{O}_q(SL(n))$.
- N. Andruskiewitsch and G.A.G for $\mathcal{O}_\epsilon(G)$, G connected, simply connected, simple complex Lie group, ϵ a root of 1.

Strategy:

(1) Give a general construction of the quotients.

(2) Prove that the construction is exhaustive.

Definitions

$k = \mathbb{C}$.

Definition (Takeuchi)

Let $\alpha,\ \beta \in k^{\times}$, $n \in \mathbb{N}$. $\mathcal{O}_{\alpha,\beta}(M_n)$ *is the* k*-algebra generated by* $\{x_{ij} :\ 1 \leq i,j \leq n\}$ *satisfying:*

$$\begin{aligned} x_{ik}x_{ij} &= \alpha x_{ij}x_{ik} && \text{if } j < k, \\ x_{jk}x_{ik} &= \beta x_{ik}x_{jk} && \text{if } i < j, \\ x_{jk}x_{il} &= \beta\alpha^{-1}x_{il}x_{jk} && \text{if} \\ x_{jl}x_{ik} - x_{ik}x_{jl} &= (\beta - \alpha^{-1})x_{il}x_{jk} && \text{if } i < j \text{ and } k < l. \end{aligned}$$

- $\mathcal{O}_{\alpha,\beta}(M_n)$ is the non-commutative polynomial algebra in x_{ij}, without zero divisors. It is a bialgebra with

$$\Delta(x_{ij}) = \sum_{s=1}^{n} x_{is} \otimes x_{sj} \quad \text{y} \quad \varepsilon(x_{ij}) = \delta_{ij}.$$

Quantum determinant:

$$g = \sum_{\sigma \in \mathbb{S}_n}^{n} (-\beta)^{-\ell(\sigma)} x_{\sigma(1),1} \cdots x_{\sigma(n),n} = \sum_{\sigma \in \mathbb{S}_n}^{n} (-\alpha)^{-\ell(\sigma)} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)},$$

where $\mathbb{S}_n$ is the symmetric group and $\ell(\sigma) =$ lenght of σ.

- g is group-like and satisfies

$$x_{ij} g = (\beta\alpha^{-1})^{i-j} g x_{ij} \quad \text{for all} \quad 1 \leq i, j \leq n.$$

- The powers of g satisfy the Ore conditions: the localization of

$\mathcal{O}_{\alpha,\beta}(M_n)$ at g^{-1} gives a Hopf algebra $\mathcal{O}_{\alpha,\beta}(GL_n) := \mathcal{O}_{\alpha,\beta}(M_n)[g^{-1}]$.
The antipode $\mathcal{S}$ is determined by

$$\mathcal{S}(x_{ij}) = (-\beta)^{j-i} g^{-1} |X_{ji}| = (-\alpha)^{j-i} |X_{ji}| g^{-1},$$

In particular, $\mathcal{S}^2(x_{ij}) = (\alpha\beta)^{j-i} x_{ij}$.

Remarks

(*i*) Taking $(\alpha, \beta) = (q, q)$ or $(1, q)$ we obtain the usual one-parameter deformations of $\mathcal{O}(GL_n)$ (Dipper-Donkin, Jimbo).

(*ii*) [Takeuchi] $\mathcal{O}_{\alpha,\beta}(GL_n)$ can not be obtained as a *twist-deformation* of $\mathcal{O}(GL_n)$.

(*iii*) Several authors (Reshetikin, Sudbery, *et al.*) defined multiparameter deformations of $\mathcal{O}(GL_n)$. They turn to be twist equivalent to $\mathcal{O}(GL_n)$ [Artin-Schelter-Tate].

Definition (Benkart-Witherspoon, Takeuchi)

Let $\alpha \neq \beta$. $U_{\alpha,\beta}(\mathfrak{gl}_n)$ *is the* k*-algebra generated by* $\{a_i, a_i^{-1}, b_i, b_i^{-1}, e_j, f_j : \ 1 \leq i \leq n, 1 \leq j < n\}$ *satisfying*

$$\begin{aligned}
a_i, b_k & \quad \text{commute, } a_i a_i^{-1} = a_i^{-1} a_i = b_i b_i^{-1} = b_i^{-1} b_i = 1, \\
a_i e_j &= \alpha^{\delta_{ij} - \delta_{i,j+1}} e_j a_i, \qquad b_i e_j = \beta^{\delta_{ij} - \delta_{i,j+1}} e_j b_i \\
a_i f_j &= \alpha^{-\delta_{ij} + \delta_{i,j+1}} f_j a_i, \qquad b_i f_j = \beta^{-\delta_{ij} + \delta_{i,j+1}} f_j b_i, \\
[e_j, f_l] &= \frac{\delta_{jl}}{\alpha - \beta}(a_j b_{j+1} - a_{j+1} b_j), \\
[e_j, e_l] &= [f_j, f_l] = 0 \qquad \text{if } |j - l| > 1, \\
0 &= e_j^2 e_{j+1} - (\alpha + \beta) e_j e_{j+1} e_j + \alpha\beta e_{j+1} e_j^2, \qquad (1) \\
0 &= e_j e_{j+1}^2 - (\alpha + \beta) e_{j+1} e_j e_{j+1} + \alpha\beta e_{j+1}^2 e_j, \\
0 &= f_j^2 f_{j+1} - (\alpha^{-1} + \beta^{-1}) f_j f_{j+1} f_j + \alpha^{-1}\beta^{-1} f_{j+1} f_j^2, \qquad (2) \\
0 &= f_j f_{j+1}^2 - (\alpha^{-1} + \beta^{-1}) f_{j+1} f_j f_{j+1} + \alpha^{-1}\beta^{-1} f_{j+1}^2 f_j,
\end{aligned}$$

Remarks

(i) Let $w_j = a_j b_{j+1}$ and $w'_j = a_{j+1} b_j$ for $1 \leq j < n$. $U_{\alpha,\beta}(\mathfrak{gl}_n)$ is a Hopf algebra determined by: $\Delta(a_i) = a_i \otimes a_i, \Delta(b_i) = b_i \otimes b_i$, $\Delta(e_j) = e_j \otimes 1 + w_j \otimes e_j$ and $\Delta(f_j) = 1 \otimes f_j + f_j \otimes w'_j$.

(ii) Taking different values of α, β, one obtain the known one-parameter deformations (Drinfeld-Jimbo) of $U(\mathfrak{gl}_n)$ as quotients.

(iii) [Chin-Musson] Multiparameter deformations of $U(\mathfrak{gl}_n)$ via twist.

(iv) [Benkart-Witherspoon] There exists a Hopf pairing between $\mathcal{O}_{\alpha,\beta}(GL_n)$ and $U_{\alpha,\beta}(\mathfrak{gl}_n)$:

$$\langle a_i, x_{st} \rangle = \delta_{st} \alpha^{\delta_{is}}, \qquad \langle b_i, x_{st} \rangle = \delta_{st} \beta^{\delta_{is}},$$

$$\langle e_j, x_{st} \rangle = \delta_{js} \delta_{j+1,t}, \qquad \langle f_j, x_{st} \rangle = \delta_{j+1,s} \delta_{jt}.$$

Finite quantum subgroups with $\alpha^{-1}\beta$ not a root of 1

Theorem

If $\alpha^{-1}\beta \notin \mathbb{G}_\infty$, then any finite-dimensional quotient of $\mathcal{O}_{\alpha,\beta}(GL_n)$ is isomorphic to k^Γ, with Γ a subgroup of the diagonal torus of $GL_n(k)$.

Proof.

Let $q : \mathcal{O}_{\alpha,\beta}(GL_n) \to A$ a Hopf algebra epimorphism, $\dim A < \infty$. Then, $\operatorname{ord} \mathcal{S}_A = 2t$ is finite. But

$$q(x_{ij}) = \mathcal{S}_A^{2t}(q(x_{ij})) = q(\mathcal{S}^{2t}(x_{ij})) = (\alpha^{-1}\beta)^{t(j-i)}q(x_{ij}).$$

Thus $q(x_{ij}) = 0$ for all $i \neq j$ and hence A is a quotient of $k[x_{11}, x_{22}, \dots, x_{nn}]$. □

Quantum subgroups with $\alpha^{-1}\beta$ a root of 1

- Assume $\alpha \neq \pm\beta, \beta^{-1}$, $\ell = \operatorname{ord} \alpha^{-1}\beta$ odd, and $\alpha^\ell = 1 = \beta^\ell$.
- [Du-Parshall-Wang] Generalization of the quantum Frobenius map

$$F^\# : \quad \mathcal{O}(M_n) \to \mathcal{O}_{\alpha,\beta}(M_n), \qquad X_{ij} \mapsto x_{ij}^\ell.$$

It is a bialgebra monomorphism with $F^\#(|X|) = g^\ell$. In particular, $F^\#$ gives a Hopf algebra monomorphism $F^\# : \mathcal{O}(GL_n) \to \mathcal{O}_{\alpha,\beta}(GL_n)$.

Proposition (Du-Parshall-Wang)

(*i*) *$\mathcal{O}(GL_n)$ is central in $\mathcal{O}_{\alpha,\beta}(GL_n)$, $\mathcal{O}_{\alpha,\beta}(GL_n)$ is faith. flat in $\mathcal{O}(GL_n)$.*

(*ii*) $\dim(\overline{H} := \mathcal{O}_{\alpha,\beta}(GL_n)/\mathcal{O}(GL_n)^+\mathcal{O}_{\alpha,\beta}(GL_n)) = \ell^{n^2}$.

(*iii*) $1 \to \mathcal{O}(GL_n) \to \mathcal{O}_{\alpha,\beta}(GL_n) \to \overline{H} \to 1$ *is exact.*

$U_{\alpha,\beta}(\mathfrak{gl}_n)$ contains a Hopf ideal I_ℓ generated by central elements.

[Benkart-Witherspoon] The quotient $u_{\alpha,\beta}(\mathfrak{gl}_n) = U_{\alpha,\beta}(\mathfrak{gl}_n)/I_\ell$ is the restricted quantum group of dimension ℓ^{n^2+n}. Dividing out by central group-likes one obtains $\hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)$.

Lemma

(a) $\hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)$ *is a pointed Hopf algebra* $\dim \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) = \ell^{n^2}$.

(b) *The Hopf pairing* $\langle -,- \rangle : U_{\alpha,\beta}(\mathfrak{gl}_n) \times \mathcal{O}_{\alpha,\beta}(GL_n) \to k$ *induces*

$$\langle -,- \rangle' : \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \times \overline{H} \to k.$$

(c) *The Hopf algebra map* $\psi : \overline{H} \to \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^*$ *given by* $\psi(\pi(x))(r(u)) = \langle r(h), \pi(x) \rangle'$ *for all* $x \in \mathcal{O}_{\alpha,\beta}(GL_n)$, $h \in U_{\alpha,\beta}(\mathfrak{gl}_n)$ *is injective. In particular,* $\overline{H} \simeq \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^*$.

Let $q : \mathcal{O}_{\alpha,\beta}(GL_n) \twoheadrightarrow A$ be a Hopf algebra epimorphism.
Then we have the commutative diagram of central exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(GL_n) & \xrightarrow{\iota} & \mathcal{O}_{\alpha,\beta}(GL_n) & \xrightarrow{\pi} & \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow q & & \downarrow & & \\
1 & \longrightarrow & B & \xrightarrow{\hat{\iota}} & A & \xrightarrow{\hat{\pi}} & H & \longrightarrow & 1,
\end{array}$$

where $B = q(\mathcal{O}(GL_n)) = \mathcal{O}(\Gamma)$ for $\Gamma \subseteq GL_n$ and $H = A/AB^+$.

Idea: Construct the quotients using extensions.

First step

Lemma

H^* is parameterized by (Σ, I_+, I_-), where $\Sigma \subseteq \mathbb{T} = G(\hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^*)$ $\simeq (\mathbb{Z}/\ell\mathbb{Z})^n$, $I_+ = \{i|\ e_i \in H^*,\ 1 \leq i < n\}$ and $I_- = \{i|\ f_i \in H^*,\ 1 \leq i < n\}$ such that $w_i \in \Sigma$ if $i \in I_+$ and $w'_j \in \Sigma$ if $j \in I_-$.

Let $\hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l}) \subseteq \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{gl}_n)$ be the Hopf subalgebra determined by $(\mathbb{T}, I_+, I_-)$. Define

$$\mathcal{I}_+ = \{(i,j)|\ i \leq k < j,\ k \notin I_+\}, \quad \mathcal{I}_- = \{(i,j)|\ j \leq k < i,\ k \notin I_-\},$$

and let $\mathcal{I} = (x_{i,j} :\ (i,j) \in \mathcal{I}_+ \cup \mathcal{I}_-) \subseteq \mathcal{O}_{\alpha,\beta}(GL_n)$.

Then $\mathcal{I}$ is a Hopf ideal and one has the central exact sequence

$$1 \longrightarrow \mathcal{O}(GL_n)/\mathcal{J} \hookrightarrow \mathcal{O}_{\alpha,\beta}(GL_n)/\mathcal{I} \twoheadrightarrow \overline{H}/\pi(\mathcal{I}) \longrightarrow 1,$$

where $\mathcal{J} = \mathcal{I} \cap \mathcal{O}(GL_n)$.

Since $\mathcal{O}(GL_n)/\mathcal{J}$ is commutative, there exists an algebraic subgroup L of GL_n such that $\mathcal{O}(GL_n)/\mathcal{J} \simeq \mathcal{O}(L)$. Let $\mathcal{O}_{\alpha,\beta}(L) = \mathcal{O}_{\alpha,\beta}(GL_n)/\mathcal{I}$.

Theorem

(a) *$\mathcal{O}(L)$ is central in $\mathcal{O}_{\alpha,\beta}(L)$.*

(b) *The following diagram commutes*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_{\alpha,\beta}(GL_n) & \xrightarrow{\pi} & \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{gl}_n)^* & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_{\alpha,\beta}(L) & \xrightarrow{\pi_L} & \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})^* & \longrightarrow & 1. \end{array}$$

(c) *π_L admits a coalgebra section ψ.*

Second Step: The Pushout construction

Let $\sigma : \Gamma \to GL_n$ such that $\sigma(\Gamma) \subseteq L$. Denote ${}^t\sigma : \mathcal{O}(GL_n) \to \mathcal{O}(\Gamma)$ the surjective Hopf algebra map, $\mathcal{J} = \operatorname{Ker}\ {}^t\sigma$ and $(\mathcal{J}) = \mathcal{O}_{\alpha,\beta}(GL_n)\mathcal{J}$.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(GL_n) & \xrightarrow{\iota} & \mathcal{O}_{\alpha,\beta}(GL_n) & \xrightarrow{\pi} & \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{gl}_n)^* & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_{\alpha,\beta}(L) & \xrightarrow{\pi_L} & \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & \downarrow {}^t\sigma & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\bar{\iota}} & A_{\alpha,\beta,\sigma,\mathfrak{l}} & \xrightarrow{\bar{\pi}} & \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})^* & \longrightarrow & 1,
\end{array}$$

where $A_{\alpha,\beta,\sigma,\mathfrak{l}} := \mathcal{O}_{\alpha,\beta}(L)/(\mathcal{J})$.

Third Step

Let $H^* \subseteq \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{gl}_n)$ be determined by (Σ, I_+, I_-). Since $\hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})$ is determined by $(\mathbb{T}, I_+, I_-)$ with $\Sigma \subseteq \mathbb{T}$, we have that $H^* \subseteq \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l}) \subseteq \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{gl}_n)$.
Let $\mathbb{T}_I = \langle w_i, w_j' : \; i \in I_+, \; j \in I_- \rangle \subseteq \mathbb{T}$ and denote $\rho : \widehat{\mathbb{T}} \to \widehat{\Sigma}$, $\rho_I : \widehat{\mathbb{T}} \to \widehat{\mathbb{T}_I}$ the group maps between the character groups induced by the inclusions. Set $N = \operatorname{Ker} \rho$ and $M_I = \operatorname{Ker} \rho_I$.

Lemma

(a) *Every χ of M_I defines an element $\bar{\chi}$ of $G(\hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})^*)$ which is central. In particular, since $N \subseteq M_I$, $\bar{\chi} \in G(\hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})^*) \cap \mathcal{Z}(\hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})^*)$ for all $\chi \in N$.*

(b) $H \simeq \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})^*/(\bar{\chi} - 1 | \; \chi \in N)$.

Using the coalgebra section ψ of π_L, we divide out by ideals generated by central elements related to Σ: for $\chi \in N$, $\psi(\bar{\chi}) \in \mathcal{Z}(A_{\alpha,\beta,\sigma,\mathfrak{l}})$ and if $\delta : N \to \widehat{\Gamma}$ ia a group map we have

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(GL_n) & \xrightarrow{\iota} & \mathcal{O}_{\alpha,\beta}(GL_n) & \xrightarrow{\pi} & \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{gl}_n)^* & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_{\alpha,\beta}(L) & \xrightarrow{\pi_L} & \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & \downarrow {}^t\sigma & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\bar{\iota}} & A_{\alpha,\beta,\sigma,\mathfrak{l}} & \xrightarrow{\bar{\pi}} & \hat{\mathbf{u}}_{\alpha,\beta}(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\tilde{\iota}} & A_{\mathcal{D}} & \xrightarrow{\tilde{\pi}} & H & \longrightarrow & 1,
\end{array}
$$

where $A_{\mathcal{D}} = A_{\alpha,\beta,\sigma,\mathfrak{l}}/(\psi(\bar{\chi}) - \delta(\chi) \mid \chi \in N)$.

Definition

A subgroup datum is a collection $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$

- $I_+, I_- \subseteq \{1, \dots, n-1\}$. *They determine an algebraic subgroup* L *of* GL_n *consisting in block matrices whose nonzero blocks are in the diagonal.*
- N *is a subgroup of* $\widehat{\mathbb{T}}$.
- Γ *algebraic group.*
- $\sigma : \Gamma \to L$ *injective morphism of algebraic groups.*
- $\delta : N \to \widehat{\Gamma}$ *group homomorphism.*

Theorem

There is a bijection between

(a) *Hopf algebra quotients* $q : \mathcal{O}_{\alpha,\beta}(GL_n) \twoheadrightarrow A$.

(b) *Subgroup data.*

Properties

Let $\mathbf{T}$ be the diagonal torus of GL_n.

Proposition

(a) *If $A_{\mathcal{D}}$ is pointed, then $I_+ \cap I_- = \emptyset$ and Γ is a subgroup of the group of upper triangular matrices of some size. In particular, if Γ is finite, then it is abelian.*

(b) *$A_{\mathcal{D}}$ is semisimple if and only if $I_+ \cup I_- = \emptyset$ and Γ is finite.*

(c) *If $\dim A_{\mathcal{D}} < \infty$ and $A_{\mathcal{D}}^*$ is pointed, then $\sigma(\Gamma) \subseteq \mathbf{T}$.*

(d) *If $A_{\mathcal{D}}$ is co-Frobenius then Γ is reductive.*

(e) *If $\mathcal{HZ}(A_D) \ncong \mathcal{HZ}(A_{D'})$ then A_D and $A_{D'}$ are not twist equivalent.*

Here $\mathcal{HZ}(A)$ is the maximal commutative Hopf subalgebra contained in A.

Theorem

Let $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$ be a finite subgroup datum such that $I_+ \cap I_- \neq \emptyset$ and $\sigma(\Gamma) \nsubseteq \mathbf{T}$. Then $A_{\mathcal{D}}$ is non-semisimple, non-pointed and its dual is also non-pointed. If moreover, $I = I_+ = I_-$ and I is connected, then A is not a quotient of $\mathcal{O}_\epsilon(G)$.

The Rings of Generalized Groups

Allen Herman

University of Regina

June 7, 2009

What do we mean by the Rings of Generalized Groups?

The Rings of Generalized Groups include:

- *Double Coset Algebras*: i.e. Hecke algebras of the form $e_H KGe_H$, where K is a field of characteristic 0, G is a finite group, H is a subgroup of G, and $e_H = \frac{1}{|H|} \sum_{h \in H} h$ is the idempotent corresponding to the trivial character of H;
- *Schur rings*: Rank d subalgebras of KG spanned by the characteristic functions of the elements of an inverse-closed partition of G having size d;
- *Scheme rings*: Rank d subrings of $M_n(K)$ spanned by a collection of d $(0,1)$-matrices $S = \{\sigma_0 = I_n, \sigma_1, \dots, \sigma_{d-1}\}$ for which S is closed under the transpose and the sum of all elements of S is the $n \times n$ all 1's matrix.

We will refer to the set S as a *association scheme* (of order n and rank d) and denote its scheme ring over K by KS.

The Basic Matrix of a Scheme Ring

The easiest way to visualize a small association scheme is by means of its basic matrix. Here is an example of the basic matrix of a scheme S of order 12 and rank 8:

$$\sum_{i=0}^{d-1} i\sigma_i = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 \\ 1 & 0 & 3 & 2 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 \\ 3 & 2 & 0 & 1 & 6 & 6 & 7 & 7 & 4 & 4 & 5 & 5 \\ 2 & 3 & 1 & 0 & 6 & 6 & 7 & 7 & 4 & 4 & 5 & 5 \\ 4 & 4 & 7 & 7 & 0 & 1 & 6 & 6 & 5 & 5 & 2 & 3 \\ 4 & 4 & 7 & 7 & 1 & 0 & 6 & 6 & 5 & 5 & 3 & 2 \\ 5 & 5 & 6 & 6 & 7 & 7 & 0 & 1 & 2 & 3 & 4 & 4 \\ 5 & 5 & 6 & 6 & 7 & 7 & 1 & 0 & 3 & 2 & 4 & 4 \\ 7 & 7 & 4 & 4 & 5 & 5 & 3 & 2 & 0 & 1 & 6 & 6 \\ 7 & 7 & 4 & 4 & 5 & 5 & 2 & 3 & 1 & 0 & 6 & 6 \\ 6 & 6 & 5 & 5 & 3 & 2 & 4 & 4 & 7 & 7 & 0 & 1 \\ 6 & 6 & 5 & 5 & 2 & 3 & 4 & 4 & 7 & 7 & 1 & 0 \end{bmatrix}.$$

When K is a field of characteristic 0,

Group algebras $\equiv$ rings of thin schemes

$\bigcap$

Double Coset Algebras $\underset{\uparrow}{=}$ Schur rings $\equiv$ rings of Schurian schemes

$\bigcap$ (up to fusion)

Scheme rings

Embedding Scheme Rings into Group Rings

Lemma

Every matrix σ_s in a scheme S can be written as a sum of $n \times n$ permutation matrices.

Proof. This is a consequence of the fact each σ_s in S has a constant row and column sum (its *valency*), so it is a scalar multiple of a doubly stochastic matrix. The result then follows because the doubly stochastic matrices are the convex hull of the collection of permutation matrices. □

This Lemma implies that, for any scheme S of order n, there is a unital embedding of its scheme ring $\mathbb{Q}S$ into the group ring $\mathbb{Q}[S_n]$.

An Example: Consider the association scheme arising from the Petersen Graph, which is strongly regular of valency 3, so its adjacency matrix σ_1 generates an association scheme of rank 3: $S = \{\sigma_0, \sigma_1, \sigma_2\}$, where

$$\sum i\sigma_i = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 2 & 2 & 0 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 & 1 & 2 & 0 & 2 & 1 \\ 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 \end{bmatrix}.$$

Suppose we try to write σ_1 as the sum of permutation matrices.

An Example: Consider the association scheme arising from the Petersen Graph, which is strongly regular of valency 3, so its adjacency matrix σ_1 generates an association scheme of rank 3: $S = \{\sigma_0, \sigma_1, \sigma_2\}$, where

$$\sum i\sigma_i = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 2 & 2 & 0 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 & 1 & 2 & 0 & 2 & 1 \\ 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 \end{bmatrix}.$$

Suppose we try to write σ_1 as the sum of permutation matrices. (i.e. We edge-colour the directed graph!)

Here is one way to accomplish it in this case:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

$\sigma_1 = a + b + c$, where
$a = (1,5,4,3,2)(6,9,7,10,8)$, $b = (1,2,3,4,5)(6,8,10,7,9)$, and
$c = (1,6)(2,7)(3,8)(4,9)(5,10)$.

Embedding Scheme Rings into Group Rings, IV

Note that $G = \langle a, b, c \rangle \cong (C_5 \times C_5) \rtimes C_2$ is a subgroup of S_{10} of order 50, and σ_0 and $\sigma_1 = a + b + c$ generate a subring of KG of rank 3 whose support involves 10 elements of G whose corresponding permutation matrices sum to the 10×10 all 1's matrix.

Fact: σ_1 is the smallest scheme matrix that cannot be written as a sum of *commuting* permutation matrices. Related to this issue is

The Cyclotomic Character Values Conjecture

Let S be a scheme and suppose that χ is an irreducible character of $\mathbb{C}S$. Then there exists a root of unity ζ_m such that for all $\sigma_s \in S$, $\chi(\sigma_s) \in \mathbb{Q}(\zeta_m)$.

Remark: Every scheme matrix that I know of has cyclotomic eigenvalues.

Using Schemes to compute Schur indices

Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. We would like to compute the Schur index $m_{\mathbb{Q}}(\chi)$, and, if possible, the isomorphism type of the block $\mathbb{Q}(\chi)Ge_\chi$.

Let H be a subgroup of G that is maximal with respect to the property $(\chi_H, 1_H) \neq 0$.

Then the double coset algebra $e_H\mathbb{Q}Ge_H$ has a block that is Morita equivalent to the block of $\mathbb{Q}G$ corresponding to χ.

Recall that $e_H\mathbb{Q}Ge_H$ is the ring of a Schurian scheme.

Theorem (Criteria for a scheme to be Schurian (Zieschang, 2005))

If S is a schurian scheme of order n, then $\mathbb{Q}S$ is (combinatorially) isomorphic to $e_D\mathbb{Q}Ce_D$, where C is the combinatorial automorphism group of S (the centralizer of S in the group of all $n \times n$ permutation matrices), and D is the stabilizer of 1 in G.

Using Schemes to compute Schur indices, II

This means we can replace the pair G and H with the pair C and D with $[C : D] = [G : H]$, and C is the smallest group that can generate this scheme ring as one of its double coset algebras.
An Example: Let W be a Coxeter group of type $H4$. W has a rational-valued irreducible character χ of degree 48 that is the only example of a non-parabolic irreducible character of a finite Coxeter group. W has a subgroup G of order 144 that has a rational valued irreducible character φ for which $(\chi_G, \varphi) = 1$. G has a nonabelian subgroup H of order 6 for which $(\chi_H, 1_H)$ and $(\varphi_H, 1_H) \neq 0$, so it follows that $e_H \mathbb{Q} G e_H$ has a block that is Morita equivalent to $\mathbb{Q} W e_\chi$.
Since $[G : H] = 24$, there is a scheme S of order 24 (and rank 14) whose scheme ring is isomorphic to this double coset algebra. The scheme is as24-657 in the library on the *Data for Small Association Schemes* website.

Using Schemes to compute Schur indices, III

What can you do with S?

- You can compute the combinatorial automorphism group C of S using `nauty` (provided you have GAP installed on a UNIX system). It is the SmallGroup(48,41) in the GAP library.
- You can use `wedderga` to obtain a presentation for the corresponding simple component of $\mathbb{Q}C$:
  `[1, Rationals, 12, [[2,5,0],[2,7,6],[[6]]]]` $=$
  $\mathbb{Q}(\zeta_{12})\langle u, v : \zeta^u = \zeta^5, u^2 = 1, \zeta^v = \zeta^7, v^2 = -1, vu = -uv\rangle$.

If you want to know the Schur index of χ, or better yet, identify the division algebra part of this crossed product algebra, you still have some work left to do.

Using Schemes to compute Schur indices, IV

- Since the group C is small, you can find a representation $\mathcal{X}$ affording the appropriate character of C. (using `IrreducibleRepresentationsDixon(C, Irr(C)[15]);`)
- Let `D := Stabilizer(C,1)`, and compute generators for the span of $\mathcal{X}(e_D x e_D)$ where x runs through a set of (D, D)-double coset representatives.

The algebra turns out to be generated by:

$$\mathcal{X}(e_D) = E = \frac{1}{2}\begin{bmatrix} 1 & \zeta_3 & 0 & 0 \\ \zeta_3^2 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \end{bmatrix},$$

$$X = \frac{1}{2}\begin{bmatrix} 0 & 0 & \zeta_3^2 & \zeta_{12}^{11} \\ 0 & 0 & \zeta_3 & \zeta_{12}^{7} \\ -\zeta_3 & -\zeta_3^2 & 0 & 0 \\ \zeta_{12}^{7} & \zeta_{12}^{11} & 0 & 0 \end{bmatrix}, \text{ and}$$

$$Y = \frac{1}{4}\begin{bmatrix} \sqrt{3} & \sqrt{3}\zeta_3 & 0 & 0 \\ -\sqrt{3}\zeta_3 & \sqrt{3} & 0 & 0 \\ 0 & 0 & -\sqrt{3} & -\sqrt{3}i \\ 0 & 0 & \sqrt{3}i & -\sqrt{3} \end{bmatrix}.$$

Isolating the Division Algebra

These matrices all have rank 2, and they satisfy the relations

$$X^2 = -E, Y^2 = \frac{3}{4}E, \text{ and } XY = -YX.$$

It follows therefore that the matrices X and $2Y$ generate a copy of the generalized quaternion algebra whose symbol is $(\frac{-1,3}{\mathbb{Q}})$.

This quaternion algebra is a central division algebra over $\mathbb{Q}$ with local index 2 at precisely the primes 2 and 3.

Note: This division algebra does not have a cyclotomic crossed product algebra presentation. Using the Hecke algebra allowed us to obtain a presentation of this division algebra by itself.

Comparing Finite Groups and Schemes

Many finite group notions have been extended to schemes:

- *Structural:* closed subsets $\equiv$ (normal) subgroups, quotient schemes, direct products, primitivity ($\equiv$ simplicity), Sylow theory (for p-valenced schemes).
- *Categorical:* scheme homomorphisms, kernel, isomorphism theorems, exact sequences, extensions.
- *Scheme Ring:* valency map ($\equiv$ augmentation map) Semisimplicity, character tables, Frobenius reciprocity, Schur indices (Frobenius-Schur indicator, Scheme Schur subgroup, Clifford theory.)

Many notions of group theory and group ring theory remain to be investigated for schemes.

Sylow theory for the scheme S

We can illustrate some of the above properties with the above scheme S of order 12:

- Sylow theory: Let S be the scheme of order 12 shown previously.
  S is 2-valenced (but not 3-valenced), so it has at least one Sylow 2-closed subset of order 4.

  The Sylow 2-closed subsets of S are $P_1 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, $P_2 = \{\sigma_0, \sigma_1, \sigma_4\}$, and $P_3 = \{\sigma_0, \sigma_1, \sigma_5\}$.

  All of these are conjugate in the scheme sense (i.e. the support of $\sigma_6 P_1 \sigma_{6^*}$ is P_2). Both the index of its normalizer in S and the number of these Sylow subsets has to be congruent to 1 mod 2.

- Quotient schemes and scheme homormorphisms:
  P_1 is not normal in S, but we can still obtain a scheme (of order $\frac{12}{4}$) from its collection of double cosets:
  $S/\!\!/P_1 = \{P_1 \sigma P_1 : \sigma \in S\} = \{P_1, P_1\sigma_4 P_1\}$.

  This is also the way scheme homomorphisms arise.

  Remark: When P is a Sylow p-subset of a p-valenced scheme S, then $S/\!\!/P$ is a p-valenced scheme of order coprime to p whose thin radical (unique maximal subgroup) is $N_S(P)/\!\!/P$.

Clifford theory for schemes

- Consider the scheme S of order 16 whose basic matrix is shown:

$$\begin{bmatrix}
0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 8 \\
1 & 0 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 8 \\
2 & 2 & 0 & 1 & 4 & 4 & 3 & 3 & 7 & 7 & 8 & 8 & 5 & 5 & 6 & 6 \\
2 & 2 & 1 & 0 & 4 & 4 & 3 & 3 & 7 & 7 & 8 & 8 & 5 & 5 & 6 & 6 \\
4 & 4 & 3 & 3 & 0 & 1 & 2 & 2 & 6 & 6 & 7 & 7 & 8 & 8 & 5 & 5 \\
4 & 4 & 3 & 3 & 1 & 0 & 2 & 2 & 6 & 6 & 7 & 7 & 8 & 8 & 5 & 5 \\
3 & 3 & 4 & 4 & 2 & 2 & 0 & 1 & 8 & 8 & 5 & 5 & 6 & 6 & 7 & 7 \\
3 & 3 & 4 & 4 & 2 & 2 & 1 & 0 & 8 & 8 & 5 & 5 & 6 & 6 & 7 & 7 \\
6 & 6 & 8 & 8 & 5 & 5 & 7 & 7 & 0 & 1 & 4 & 4 & 2 & 2 & 3 & 3 \\
6 & 6 & 8 & 8 & 5 & 5 & 7 & 7 & 1 & 0 & 4 & 4 & 2 & 2 & 3 & 3 \\
5 & 5 & 7 & 7 & 8 & 8 & 6 & 6 & 3 & 3 & 0 & 1 & 4 & 4 & 2 & 2 \\
5 & 5 & 7 & 7 & 8 & 8 & 6 & 6 & 3 & 3 & 1 & 0 & 4 & 4 & 2 & 2 \\
8 & 8 & 6 & 6 & 7 & 7 & 5 & 5 & 2 & 2 & 3 & 3 & 0 & 1 & 4 & 4 \\
8 & 8 & 6 & 6 & 7 & 7 & 5 & 5 & 2 & 2 & 3 & 3 & 1 & 0 & 4 & 4 \\
7 & 7 & 5 & 5 & 6 & 6 & 8 & 8 & 4 & 4 & 2 & 2 & 3 & 3 & 0 & 1 \\
7 & 7 & 5 & 5 & 6 & 6 & 8 & 8 & 4 & 4 & 2 & 2 & 3 & 3 & 1 & 0
\end{bmatrix}.$$

S has a strongly normal closed subset $T = \{\sigma_0, \sigma_1, \sigma_2\}$ of order 4. Being strongly normal, all of its double cosets have order 4 : $T\sigma_3 T = \{\sigma_3, \sigma_4\}$, $T\sigma_5 T = \{\sigma_5, \sigma_7\}$, $T\sigma_6 T = \{\sigma_6, \sigma_8\}$.

$KS = \oplus_{T\sigma T} K(T\sigma T)$ is graded by the group $S/\!\!/T$.
But it is not strongly graded because $K(T\sigma T)$ does not contain a unit if $\sigma \notin T$.
So Clifford theory for schemes is a particular case of Dade's group-graded Clifford Theory:

Theorem (Clifford Theory for Schemes)

Suppose T is a strongly normal closed subset of a scheme S. Let M be an irreducible KT-module and let $U/\!\!/T$ be the stabilizer of M in $S/\!\!/T$. Then $N \mapsto N \otimes_{KU} KS$ defines an equivalence of categories between $\mathrm{Mod}(KU|M)$ and $\mathrm{Mod}(KS|M)$.

Scheme Schur Subgroup

Definition of $S_{\mathcal{A}}(K)$

Let K be a number field. The *Scheme Schur Subgroup* $S_{\mathcal{A}}(K)$ is the collection of Brauer classes of simple components of scheme rings over $\mathbb{Q}$ that have center isomorphic to K.

Question

Is $S_{\mathcal{A}}(K)$ always equal to the ordinary Schur subgroup $S(K)$?

Scheme Schur Subgroup

Definition of $S_{\mathcal{A}}(K)$

Let K be a number field. The *Scheme Schur Subgroup* $S_{\mathcal{A}}(K)$ is the collection of Brauer classes of simple components of scheme rings over $\mathbb{Q}$ that have center isomorphic to K.

Question

Is $S_{\mathcal{A}}(K)$ always equal to the ordinary Schur subgroup $S(K)$?

Matthias Künzer

A Fourier-Hopf inversion and spectral sequences

1. Fourier-Hopf inversion

Maps on the right : $A \xrightarrow{f} B$, $a \mapsto af = [a]f$.

DATA

- R : commutative ring
- H : Hopf algebra free over R, with involutive antipode ; so

$$H \xrightarrow{\varepsilon} R \quad \text{counit}$$

$$H \xrightarrow{\Delta} H \otimes_R H \quad \text{comultiplication}$$
$$x \mapsto \sum_i xu_i \otimes xv_i$$

$$H \xrightarrow{S} H \quad \text{antipode, } S^2 = \mathrm{id}$$

- $K \subseteq H$: normal Hopf subalgebra free over R ; i.e. $K\Delta \subseteq K \otimes_R K$, $KS \subseteq K$, and

$$\left.\begin{array}{l} \sum_i xu_i \cdot k \cdot xv_i S \in K \\ \sum_i xu_i S \cdot k \cdot xv_i \in K \end{array}\right\} \text{ for } k \in K,\ x \in H$$

- M : left H-module

Write $K^+ := \mathrm{Kern}(K \xrightarrow{\varepsilon} R)$.

Then $K^+H = HK^+ \subseteq H$ is a Hopf ideal, so $\bar{H} := H/HK^+$ is a Hopf algebra.

Write $\bar{x} := x + HK^+$ for $x \in H$, etc.

Examples

- G group, $N \trianglelefteqslant G$, $H = RG$, $K = RN$, $\bar{H} = R(G/N)$ ($\rightsquigarrow$ Lyndon-Hochschild-Serre).
- $\mathfrak{g}$ Lie algebra free$/R$, $\mathfrak{n} \trianglelefteqslant \mathfrak{g}$ such that $\mathfrak{n}$ and $\mathfrak{g}/\mathfrak{n}$ free$/R$, $H = \mathcal{U}(\mathfrak{g})$, $K = \mathcal{U}(\mathfrak{n})$, $\bar{H} = \mathcal{U}(\mathfrak{g}/\mathfrak{n})$ ($\rightsquigarrow$ Hochschild-Serre).

Two $\bar{H}$-modules

(a) $\mathrm{Hom}_K(H, M)$ is a left $\bar{H}$-module via

$$[h](\bar{x} \cdot f) \;:=\; \textstyle\sum_i xu_i \cdot [xv_iS \cdot h]f$$

for $f \in \mathrm{Hom}_K(H, M)$, h, $x \in H$.

(b) $\mathrm{Hom}_R(\bar{H}, M)$ is a left $\bar{H}$-module via

$$[\bar{h}](\bar{x} \cdot g) \;:=\; [\bar{h} \cdot \bar{x}]g$$

for $g \in \mathrm{Hom}_R(\bar{H}, M)$, h, $x \in H$.

With help of G. Carnovale :

<u>Lemma</u> (Fourier-Hopf inversion $=:$ FH)

There exist mutually inverse isos of $\bar{H}$-modules :

$$f \longmapsto \left(f\Phi : \bar{h} \mapsto \textstyle\sum_i hu_i \cdot [hv_iS]f\right)$$

$$\mathrm{Hom}_K(H, M) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Hom}_R(\bar{H}, M) \quad (\sim)$$

$$\left(g\Psi : h \mapsto \textstyle\sum_i hv_i \cdot [\overline{hu_iS}]g\right) \longmapsfrom g$$

2. Spectral sequence comparisons

Assume : H, K, $\bar{H}$ free over R, and H free over K

Write

$$\begin{array}{ccccl} (H\text{-Mod})^\circ & \times & H\text{-Mod} & \xrightarrow{U} & \bar{H}\text{-Mod} \\ (X & , & X') & \mapsto & U(X,X') := \underbrace{\mathrm{Hom}_K(X,X')}_{\text{cf. 1.(a)}} \end{array}$$

$$\begin{array}{ccccl} (\bar{H}\text{-Mod})^\circ & \times & \bar{H}\text{-Mod} & \xrightarrow{V} & R\text{-Mod} \\ (Y & , & Y') & \mapsto & V(Y,Y') := \mathrm{Hom}_{\bar{H}}(Y,Y') \end{array}$$

Three spectral sequences

(a) The composition

$$H\text{-Mod} \xrightarrow[K\text{-fixed points}]{U(R,-)} \bar{H}\text{-Mod} \xrightarrow[\bar{H}\text{-fixed points}]{V(R,-)} R\text{-Mod}$$

gives rise to the Grothendieck sp. seq.

$$\mathrm{E}_{U(R,-),V(R,-)}(M) .$$

(b) The composition

$$(H\text{-Mod})^\circ \xrightarrow{U(-,M)} \bar{H}\text{-Mod} \xrightarrow{V(R,-)} R\text{-Mod}$$

gives rise to the Grothendieck sp. seq.

$$\mathrm{E}_{U(-,M),V(R,-)}(R)$$

But . . .

…existence of this sp. seq. needs : P proj.$/H \overset{!}{\Rightarrow}$ $U(P,M)$ acyclic w.r.t. $V(R,-) = \mathrm{Hom}_{\bar{H}}(R,-)$.

Suffices $P := H$, so $U(P,M) = U(H,M)$. Now

$$U(H,M) \overset{\text{def}}{=} \mathrm{Hom}_K(H,M) \overset{\boxed{\text{FH}}}{\simeq} \mathrm{Hom}_R(\bar{H},M) ,$$

and

$$\mathrm{Ext}^i_{\bar{H}}\big(R, \mathrm{Hom}_R(\bar{H},M)\big) \simeq \mathrm{Ext}^i_R(\underbrace{\bar{H} \otimes_{\bar{H}} R}_{=\ R}, M) \simeq 0$$

for $i \geqslant 1$. So existence is ok.

(c) B : proj. res. of R over H.
$\tilde{B}$: proj. res. of R over $\bar{H}$.
Replacing R by B resp. by $\tilde{B}$, we obtain a double complex

$$V(\tilde{B}, U(B,M)) \simeq \mathrm{Hom}_H(\tilde{B} \otimes_R B, M) ,$$

whence a Hochschild-Serre type sp. seq.

$$\mathrm{E}\big(\mathrm{Hom}_H(\tilde{B} \otimes_R B, M)\big) .$$

Theorem We have

$$\begin{array}{ll} \mathrm{E}_{U(R,-),V(R,-)}(M) & \text{(a), abstract, using inj.} \\ \wr\| & \\ \mathrm{E}_{U(-,M),V(R,-)}(R) & \text{(b), intermediate, } \boxed{\text{FH}}\,! \\ \wr\| & \\ \mathrm{E}\big(\operatorname{Hom}_H(\tilde{B}\otimes_R B, M)\big) & \text{(c), concrete, using proj.} \end{array}$$

Note : both (a) and (c) a priori have

$$\mathrm{E}_2^{p,q} \;=\; \operatorname{Ext}^p_{\bar{H}}\big(\operatorname{Ext}^q_K(R,M)\big)$$

and converge to

$$\operatorname{Ext}^{p+q}_H(R,M)\;.$$

The problem in comparing them is the differentials.

Related work

- Beyl : case of groups
- Haas : naturality of spectral sequences
- Barnes : comparison theorem in different setup

p-adic integral group rings.

Gabriele Nebe

Lehrstuhl D für Mathematik

St. Johns, June 5, 2009

Setup

- G finite group.
- $R \supseteq \mathbb{Z}_p$ (complete) discrete valuation ring.
- $K = \text{frac}(R)$ field of fractions.
- π uniformizer of R, $F = R/\pi R$.

We want to understand the representation theory of RG.
First step: Understand

$$\mathfrak{L}_V(G) := \{L \subset V \mid L \text{ is an } RG\text{-lattice in } V\}$$

for all simple KG-modules V.
For the talk assume that R splits RG.

Intersection of maximal orders.

- V a simple KG-module.
- $\epsilon_V \in KG$ the associated central primitive idempotent.
- S a simple FG-module.
- The decomposition number $d_{V,S}$ is the multiplicity of the composition factor S in $L/\pi L$ for any $L \in \mathfrak{L}_V(G)$.

Fact.

$$\epsilon_V RG \subseteq \bigcap_{L \in \mathfrak{L}_V(G)} \mathrm{End}_R(L)$$

with equality if $d_{V,S} \in \{0, 1\}$ for all S.

Example S_3.

$G = S_3$ the symmetric group of degree 3, $R = \mathbb{Z}_3$, $\dim(V) = 2$.

$$\mathcal{L}_V(G) = \{3^n L_1, 3^n L_2 \mid n \in \mathbb{Z}\}$$

where

$$L_1 := \langle b_1, b_2 \rangle \supset L_2 = \langle 3b_1, b_2 \rangle \supset 3L_1$$

$L_1/L_2 \cong \mathbb{F}_3$ trivial module, $L_2/3L_1 \cong \mathbb{F}_3$ sign module.

$$\mathrm{End}_R(L_1) = \begin{pmatrix} R & R \\ R & R \end{pmatrix}, \ \mathrm{End}_R(L_2) = \begin{pmatrix} R & 3R \\ 3^{-1}R & R \end{pmatrix}$$

and

$$\epsilon_V RG = \mathrm{End}_R(L_1) \cap \mathrm{End}_R(L_2) = \begin{pmatrix} R & 3R \\ R & R \end{pmatrix}.$$

Example D_8.

$G = D_8$ the dihedral group of order 8, $R = \mathbb{Z}_2$, $\dim(V) = 2$.

$$\mathfrak{L}_V(G) = \{2^n L_1, 2^n L_2 \mid n \in \mathbb{Z}\}$$

where

$$L_1 := \langle b_1, b_2 \rangle \supset L_2 = \langle 2b_1, b_2 \rangle \supset 2L_1$$

and $L_1/L_2 \cong L_2/2L_1 \cong \mathbb{F}_2$.

$$\mathrm{End}_R(L_1) \cap \mathrm{End}_R(L_2) = \begin{pmatrix} R & 2R \\ R & R \end{pmatrix}$$

and

$$\epsilon_V RG = \{\begin{pmatrix} a & 2b \\ c & d \end{pmatrix} \mid a, b, c, d \in R, a \equiv d \pmod 2\}$$

Exponent matrices.

Definition.

For $M \in \mathbb{Z}^{k\times k}$, $(n_1,\ldots,n_k) \in \mathbb{N}^k$, $n := \sum_{i=1}^k n_i$ let

$$\Lambda(n_1,\ldots,n_k;M) := \{(X_{ij}) \in R^{n\times n} \mid X_{ij} \in \pi^{m_{ij}} R^{n_i\times n_j}\}$$

be the graduated order with exponent matrix M.

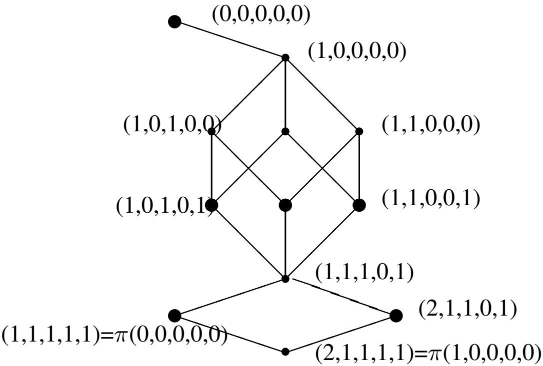
$G = S_3$, $\dim(V) = 2$ then $\epsilon_V \mathbb{Z}_3 G = \Lambda(1,1;\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$.

Theorem.

$\epsilon_V RG$ is graduated $\Leftrightarrow d_{V,S} \in \{0,1\}$ for all S.
Then $(\mathcal{L}_V(G), \subset)$ is a distributive lattice.

From exponent matrices to lattices.

$$\Lambda = \Lambda(M) \text{ with } M = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$



Obvious properties of exponent matrices.

- Since $\Lambda := \Lambda(n_1, \ldots, n_k; M)$ is an order we have

$$m_{ii} = 0 \text{ and } m_{ij} + m_{j\ell} \geq m_{i\ell} \text{ for all } i, j, \ell$$

- If
$e_1 = \operatorname{diag}(\underbrace{1, \ldots, 1}_{n_1}, 0 \ldots, 0), \ldots, e_k = \operatorname{diag}(0, \ldots, 0, \underbrace{1, \ldots, 1}_{n_k})$
are lifts of the central primitive idempotents of $\Lambda / J(\Lambda)$, then

$$J(\Lambda(n_1, \ldots, n_k; M)) = \Lambda(n_1, \ldots, n_k; M + I_k)$$

and hence $m_{ij} + m_{ji} > 0$ for $i \neq j$.
- W.l.o.g. write matrices with respect to a suitable basis of $L := \Lambda \operatorname{diag}(1, 0 \ldots, 0)$. Then

$$m_{i1} = 0 \text{ for all } i \text{ and } m_{ij} \geq 0 \text{ for all } i, j.$$

Duality.

$< x, y >:= \frac{1}{|G|} \mathrm{trace}_{reg}(xy)$ is an associative non degenerate symmetric bilinear form on KG so that RG is self-dual

$$RG = RG^{\#} = \{x \in KG \mid < x, RG > \subset R\}.$$

$$\Gamma^{\#} \subset RG \subset \Gamma := \bigoplus_V \epsilon_V RG.$$

Remark.

If $\epsilon_V RG = \Lambda(n_1, \ldots, n_k; M) =: \Lambda$, then

$$\Lambda^{\#} = \Lambda(n_1, \ldots, n_k; aJ - M^{\mathrm{tr}}) \subset \Lambda$$

with $a := v_\pi(|G|) - v_\pi(\dim(V))$ and $J = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \ldots & 1 \end{pmatrix}$.

So $a - m_{ij} \geq m_{ji}$ or equivalently

$$m_{ij} + m_{ji} \leq a.$$

Involution.

RG is an R-order with a canonical R-linear involution

$$\circ : RG \to RG, g \mapsto g^{-1}.$$

If $\epsilon_V = \epsilon_V^\circ$ then choose $0 \neq \phi = \phi^{tr} \in K^{n\times n}$ such that

$$g\phi g^{\mathrm{tr}} = \phi, \text{ so } g^{-1} = \phi g^{\mathrm{tr}}\phi^{-1} \text{ for all } g \in G.$$

Then

$$\epsilon_V RG = (\epsilon_V RG)^\circ = \{\phi X^{\mathrm{tr}}\phi^{-1} \mid X \in \epsilon_V RG\} \subset R^{n\times n}.$$

Remark.

If $\epsilon_V RG = \Lambda(n_1, \ldots, n_k; M)$ and all simple modules S are selfdual, then ϕ can be chosen as

$$\phi = \mathrm{diag}(f_1, \pi^{a_2} f_2, \ldots, \pi^{a_k} f_k) \text{ with } a_i \in \mathbb{N}, f_i \in \mathrm{GL}_{n_i}(R)$$

and $m_{ij} - m_{ji} = a_j - a_i$.

Summary: Properties of exponent matrices.

Theorem.

Let V be a simple KG-module such that $\epsilon_V RG = \Lambda(n_1, \ldots, n_k; M)$ is a graduated order.
Let $a := v_\pi(|G|) - v_\pi(\dim(V))$. Then w.r.t. a suitable basis of L as above for all $i, j, \ell \in \{1, \ldots, k\}$

- **wlog** $m_{i1} = 0$, $m_{ij} \geq 0$.
- **order** $m_{ii} = 0$, $m_{ij} + m_{j\ell} \geq m_{i\ell}$
- **rad** $m_{ij} + m_{ji} > 0$ if $i \neq j$.
- **dual** $m_{ij} + m_{ji} \leq a$
- **invo** $m_{ij} - m_{ji} = m_{1j} - m_{1i} = a_j - a_i$ if $\epsilon_V^\circ = \epsilon_V$ and all simple FG-modules S with $d_{V,S} > 0$ are selfdual.

Symmetric groups.

- Irreducible representations in characteristic 0 are parametrized by the partitions λ of n.
- $S^\lambda \le 1_{S_{\lambda_1}\times\ldots\times S_{\lambda_s}}^{S_n}$ Specht lattice.
- Irreducible representations D^λ in characteristic p are parametrized by the p-regular partitions λ of n.
- $D^\lambda = FS^\lambda/(FS^\lambda)^\perp$
- Jantzen-Schaper-formula: multiplicity of D^μ as composition factor in $(S^\lambda)^\#/S^\lambda$ if decomposition matrix is known.
- This formula yields the exponents $a_2, \ldots, a_k$ of the invariant form ϕ and hence the first row of the exponent matrices.

The decomposition matrix of the principal block of $\mathbb{Z}_3 S_6$.

	(6)	$(5,1)$	$(4,1^2)$	(3^2)	$(3,2,1)$
(6)	1	.	.	.	.
$(5,1)$	1	1	.	.	.
$(4,1^2)$	.	1	1	.	.
(3^2)	.	1	.	1	.
$(3,2,1)$	1	1	1	1	1
$(3,1^3)$	.	.	1	.	1
(2^3)	1	.	.	.	1
$(2,1^4)$	.	.	.	1	1
(1^6)	.	.	.	1	.

Example $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6$.

- Jantzen-Schaper yields:
  $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6 = \Lambda((3,2,1),(3^2),(4,1^2),(5,1),(6);M)$ where

$$M = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & a & b & c \\ 0 & a' & 0 & d & e \\ 0 & b' & d' & 0 & f \\ 0 & c' & e' & f' & 0 \end{pmatrix}$$

- For all x we have

$$0 < x + x' \leq 2, \text{ so } x + x' \in \{1, 2\}$$

- Invariance under the involution $\circ$ yields

  $a - a' = c - c' = e - e' = 0$ and $b - b' = d - d' = f' - f = 1$

  hence $a = a' = c = c' = e = e' = 1$, $b = d = f' = 1$, and $b' = d' = f = 0$.

Example $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6$.

- Jantzen-Schaper yields:
  $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6 = \Lambda((3,2,1),(3^2),(4,1^2),(5,1),(6);M)$ where

$$M = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

- For all x we have

$$0 < x + x' \leq 2, \text{ so } x + x' \in \{1, 2\}$$

- Invariance under the involution $\circ$ yields

  $a - a' = c - c' = e - e' = 0$ and $b - b' = d - d' = f' - f = 1$

  hence $a = a' = c = c' = e = e' = 1$, $b = d = f' = 1$, and
  $b' = d' = f = 0$.

Symmetric groups of degree 2p, exponent matrices.

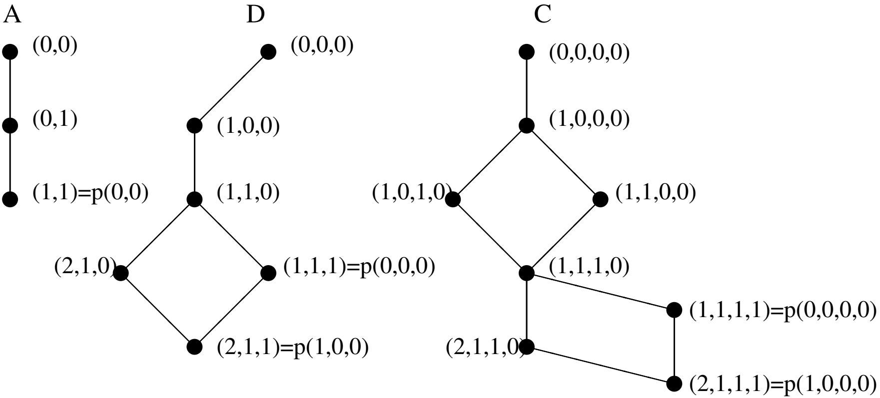
Theorem.

Let $G = S_{2p}$, $R = \mathbb{Z}_p$, V a simple KG-module. Then $\epsilon_V RG$ is graduated with exponent matrix X, A, B, C, or D:

$$X := \begin{pmatrix} 0 \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B := \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Symmetric groups of degree 2p, lattices.



Second step: Description of RG.

$$RG/J(RG) = \bigoplus_{i=1}^{s} \underbrace{\overline{e}'_i RG/J(RG)}_{M_{n_i}(F)}$$

$P_1, \ldots, P_s$ the projective indecomposable RG-lattices.
Morita equivalence:

$$RG \sim \mathrm{End}_{RG}(\bigoplus_{i=1}^{s} P_i) =: \Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) == \bigoplus_{i,j=1}^{s} \underbrace{e_j \Delta e_i}_{\Delta_{ji}}$$

where $\Delta/J(\Delta) = \bigoplus_{i=1}^{s} \underbrace{\overline{e}_i \Delta/J(\Delta)}_{F}$ and the e_i are orthognal primitive idempotents in Δ that lift the $\overline{e}_i$.

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$. Then

- Δ_{ii} are self-dual local R-orders.
- Δ_{ij} is a $\Delta_{ii} - \Delta_{jj}$-bimodule.
- $\Delta_{ij}\Delta_{jk} \subseteq \Delta_{ik}$.
- $\Delta_{ij}\Delta_{ji} \subseteq J(\Delta_{ii})$ if $i \neq j$.
- $\Delta_{ij}^{\#} = \Delta_{ji}$.
- $(e_i \Delta e_j)^{\circ} = e_j^{\circ} \Delta e_i^{\circ}$
- $\Delta_{ij} \subset \bigoplus_V \epsilon_V \Delta_{ij}$.
- If $d_{V,S_i} \in \{0,1\}$ for all V, then Δ_{ii} is commutative and

$$\bigoplus_{V, d_{V,S_i}=1} \epsilon_V \Delta_{ii} \cong \bigoplus_{V, d_{V,S_i}=1} R$$

  is the unique maximal order in $K\Delta_{ii}$.

Results.

Theorem.

We know RG for

- blocks with cyclic defect groups (Plesken, 1983)
- $G = S_{2p}$, $R = \mathbb{Z}_p$.
- $G = \mathrm{SL}_2(p^2)$.
- $G = \mathrm{SL}_2(2^f)$, $f \leq 6$.
- $G = S_n$, $n \leq 9$.
- some other examples.

Theorem.

We know $\Gamma = \bigoplus_V \epsilon_V RG$ for $\mathrm{SL}_2(p^f)$.

The decomposition matrix of the principal block of $\mathbb{Z}_3 S_6$.

$\chi(1)$		(6)	$(5,1)$	$(4,1^2)$	(3^2)	$(3,2,1)$
1	(6)	1	.	.	.	.
5	$(5,1)$	1	1	.	.	.
10	$(4,1^2)$	.	1	1	.	.
5	(3^2)	.	1	.	1	.
16	$(3,2,1)$	1	1	1	1	1
10	$(3,1^3)$	.	.	1	.	1
5	(2^3)	1	.	.	.	1
5	$(2,1^4)$	.	.	.	1	1
1	(1^6)	.	.	.	1	.

Example $\mathrm{End}(P_{(6)})$.

$\chi(1)$	1	5	16	5
$\chi(1) \pmod 3$	1	-1	1	-1
	1	1	1	1
	0	3	0	-3
	0	0	3	3
	0	0	0	9

On group rings of finite simple groups
-
and related topics

Wolfgang Kimmerle

Universität Stuttgart
Fachbereich Mathematik

St.Johns, 4. Juni 2009

Notations

G	finite group
RG	group ring of G over the ring R
$\mathrm{U}(RG)$	group of units of RG
$\mathrm{V}(RG)$	group of normalized units of RG, i.e.

$$\mathrm{V}(RG) = \left\{ \sum_{g \in G} u_g g \in \mathrm{U}(RG) : \sum_{g \in G} u_g = 1 \right\}$$

Motivating Question

1963, Brauer's Problem 2*

Let G and H be groups. Assume that $KG \cong KH$ for all fields K. Are then G and H isomorphic ?

- In general the answer is negative. (E.Dade 1968) The groups of the counterexample are metabelian.
- Even if one assumes that the integral groups rings $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic it does not follow in general that G and H are isomorphic (M.Hertweck 1998).
- However for many finite groups Brauer's question has an affirmative answer.

Results for finite simple groups

Theorem (K.2002)

Suppose that G is a finite simple group. Then

$$KG \cong KH \text{ for all fields } K \Longrightarrow G \cong H.$$

The proof consists of the following steps

1. Let G be simple of Lie type with defining characteristic p and let K be a field of chracteristic p , then $KG \cong KH$ implies that H is simple.
2. Assume that G is sporadic or alternating simple. If $\mathbb{C}G$ maps onto $\mathbb{C}Q$ and Q is simple then $G \cong Q$.
3. By steps 1 and 2 we get that

$$KG \cong KH \ \forall \text{ fields } K \quad \Longrightarrow H \text{ is simple and } |H| = |G|.$$

Now use

Artin's theorem on the order of the finite simple groups [KLST 1990]

Let G and H be finite simple groups. Then

$$|G| = |H| \Longrightarrow G \cong H$$

or

$\{G, H\} = \{B_n(q), C_n(q)\}$ with $n \geq 3, q$ odd or $\{G, H\} = \{A_8, A_2(4)\}$

[KLST] W.Kimmerle, R.Lyons, R.Sandling and D.Teague, Composition factors from the group ring and Artins' theorem on the order of the finite simple groups, Proc.London Math Soc. 1990

Proof continued

By Artin's theorem on the order of the finite simple groups it follows that $G \cong H$ except $\{G, H\} = \{A_8, A_2(4)\}$ or $= \{B_n(q), C_n(q)\}$.

4 Clearly $\mathbb{C}A_8 \ncong \mathbb{C}A_2(4)$. This follows immediately from a look at the character table of these groups.

5 $\mathbb{C}B_n(q) \ncong \mathbb{C}C_n(q)$ because their smallest ordinary character degrees $\neq 1$ are different (Tiep and Zalesski).

Ultimate problems

The proof indicates that much more may be true - at least for simple groups.

Question 1.

Is a finite simple group determined by its complex group algebra ?

Question 2.

Let K be a prime field. Is a finite simple group determined by its group algebra over K?

Character degrees

cdG = set of the degrees of the ordinary irreducible characters
$\mathrm{cd_mG}$ = set of the degrees of the ordinary irreducible characters including their multiplicities

Huppert's Conjecture HC (2000):

Let G and H be finite groups. Assume that G is simple and that
$\mathrm{cdG} = \mathrm{cdH}$.
Then $H \cong G \times A$, where A is abelian.

Weaker version of HC :

Let G and H be finite groups. Assume that G is simple and that
$\mathrm{cd_mG} = \mathrm{cd_mH}$.
Is then $H \cong G$?

Results on Character Degrees

Remark

$\mathrm{cd_m G} = \mathrm{cd_m H}$ if and only if $\mathbb{C}G \cong \mathbb{C}H$.
Therefore Question 1 has an affirmative answer provided Huppert's conjecture holds.

- B.Huppert established his conjecture for all minimal simple groups, some other simple groups of Lie type of small Lie rank, A_7, A_8, A_9, A_{10}, for 18 of the sporadic simple groups.
- HC holds for ${}^2G_2(3^{2m+1})$ (T.Wakefield 2008)
- The weaker version of HC is established for
  all sporadic simple groups, all alternating groups A_n with $n \geq 5$ (T.Haeberlen - K. 2002)
  all simple groups with abelian Sylow 2 - subgroups (M.Nagl 2007)
- All simple groups of Lie type $\Delta_n(p), p$ a prime (M.Nagl 2009).

Conjugacy Classes

clG = set of the lengths of the conjugacy classes of G
$\mathrm{cl_m G}$ = set of the lengths of the conjugacy classes of G including their multiplicities

Thompson's Conjecture TC (1988):

Let G and H be finite groups. Assume G is simple, $Z(H) = 1$ and $\mathrm{clG} = \mathrm{clH}$.
Then $H \cong G$.

Remark. From the view of character tables Thompson's Conjecture is dual to that one of Huppert. Conjugacy classes label the columns, irreducible characters the rows of $\mathrm{CT(G)}$. The multiplicity of classes with length 1 determines the order of the centre. The multiplicity of irreducible characters of degree 1 the order of the abelianized group. In both situations, if the multiplicities are known, the group order of G is known.

Known results on TC

- TC is valid for the sporadic simple groups (Guiyun Chen 1992)
- TC is valid for all simple groups G, whose Gruenberg-Kegel graph $\Gamma(G)$ has at least three connecting components (Guiyun Chen 1996)

As in the case of HC it is obvious to study also the following

Weaker version of TC :

Let G and H be finite groups. Assume that G is simple and that
$\mathrm{cl_m G = cl_m H}$.
Then $H \cong G$.

Characterizations of alternating groups A_n

Denote by $\Pi(X)$ the set of element orders of the finite group X (also called the spectrum of X).

Wujie Shi, Bi Jianxing 1992

Let H_n be a finite group with $|H_n| = |A_n|$. Assume that $\Pi(H_n)$ and $\Pi(A_n)$ coincide. Then $H_n \cong A_n$.

Diplomarbeit M. Borgart 2008

Let H_n be as above. $H_n \cong A_n$ provided for pairwise different primes p_i

1. $p_1 \cdot \ldots \cdot p_k \notin \Pi(H_n)$, if $\sum p_i > n$.
2. $p_1 \cdot \ldots \cdot p_k \in \Pi(H_n)$, if $\sum p_i = n$, all p_i odd.
3. $2 \cdot 3 \cdot 5 \notin \Pi(H_{10})$.
4. $5 \cdot 7 \cdot 13 \in \Pi(H_{27})$.

Proposition (M. Borgart - K. 2009)

The weaker version of TC is valid for A_n .

Composition factors

Problem 20 (Collection, cf. Oberwolfach Reports Vol.4 no.4 No.55/2007)

Let G be a finite group and let U be a finite subgroup of $V(\mathbb{Z}G)$. Are the composition factors of U isomorphic to sections of G ?

- If U is contained in a group basis the answer is positive (KLST).
- In particular finite simple groups are determined by $\mathbb{Z}G$.
- In the case of a group basis for some series of simple groups group bases are conjugate within $\mathbb{Q}G$ to G, i.e. ZC2 holds in this case. (F.Bleher 1999, F.Bleher-G.Hiss-K. 1996, F.Bleher-K.2000).
- ZC2 is still a wide open question for simple groups.

Results on composition factors

Theorem 1

(C.Höfert 2007) Let G be a minimal finite simple group. Then a finite subgroup of $V(\mathbb{Z}G)$ of order strictly smaller than $|G|$ is solvable.

Theorem 2

(M.Hertweck - C.Höfert - K. 2008) For $G = \mathrm{PSL}(2, q)$, each composition factor of a finite subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of G.

Why are integral group rings of simple groups of major interest ?

„ZC3 “

Is every finite subgroup $H \leq V(\mathbb{Z}G)$ isomorphic to a subgroup of G?

Subgroup isomorphism problem

Classify all finite groups H with the following property.

$$H \leq V(\mathbb{Z}G) \Longrightarrow H \text{ is isomorphic to a subgroup of } G$$

Hertwecks's counterexample to the isomorphism problem of group bases shows that in general the answer is no!

A result of Cohn - Livingstone gives a positive answer to the subgroup isomorphism problem for cyclic p -groups.
Till 2006 this result was more or less all what was known on the subgroup isomorphism problem

Open questions

In particular the following cases are open.

- cylic subgroups
- abelian subgroups
- p - groups

!

In order to find an answer these questions have to be settled in particular for simple groups.

Marciniak's question

2006 on the ICM satellite conference on Noncommutative Algebra in Granada Z.Marciniak posed the following question:

Let H be a Kleinian fourgroup. Assume that $H \leq V(\mathbb{Z}G)$. Is H isomorphic to a subgroup of G?

The question demonstrates that in the case of a general finite group we know almost nothing on the torsion structure of $V(\mathbb{Z}G)$.

First results on non-cyclic subgroups

- Marciniak's question has a positive answer, i.e. the subgroup isomorphism problem has an affirmative answer if $H \cong C_2 \times C_2$. . (K.2006)

- Let p be an odd prime. $V(\mathbb{Z}G)$ contains subgroups isomorphic to $C_p \times C_p$ if, and only if, G has such a subgroup. (M.Hertweck 2007)

2-subgroups, elementary abelian subgroups

Theorem 3 (M.Hertweck - C.Höfert - K. 2008)

Let $G = PSL(2, p^f)$. Then every finite 2-subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of G. Each elementary abelian subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of G.

Conclusion

In the case when G is a minimal simple group we know in the mean time quite a couple of results. Nevertheless there are still open questions. Let me conclude with two precise questions on $PSL(2, p^f)$.

- If the order of a torsion unit in $V(PSL(2, p^f)$ is divisible by p, has it order p?
- If p is odd, are finite p - subgroups of $V(PSL(2, p^f)$ elementary abelian ?