Algorithms for p-Adic Group Rings

Florian Eisele
RWTH Aachen University

June 6, 2009

The setting

Notation

By $(k, \mathcal{O}, K)$ we will denote a p-modular system ($\mathcal{O}$ complete, k finite) and by Λ an $\mathcal{O}$-order such that $A := K \otimes \Lambda$ is semisimple.

The setting

Notation

By $(k, \mathcal{O}, K)$ we will denote a p-modular system ($\mathcal{O}$ complete, k finite) and by Λ an $\mathcal{O}$-order such that $A := K \otimes \Lambda$ is semisimple.

The problems

- Given Λ, compute its projective indecomposables from the knowledge of its irreducible representations.
- Compute a basic order for Λ (immediate from the above).
- Given Λ, find all (full) overorders of Λ that are self-dual w.r.t some trace bilinear form

Construction of projective indecomposables

Recognizing projectives

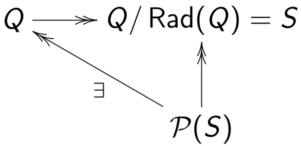
If Q is a Λ-lattice, then Q is isomorphic to the projective cover $\mathcal{P}(S)$ of the semisimple Λ-module S iff $Q/\operatorname{Rad} Q \cong S$ and $\operatorname{rank}_{\mathcal{O}} Q = \operatorname{rank}_{\mathcal{O}} \mathcal{P}(S)$.

Construction of projective indecomposables

Recognizing projectives

If Q is a Λ-lattice, then Q is isomorphic to the projective cover $\mathcal{P}(S)$ of the semisimple Λ-module S iff $Q/\operatorname{Rad} Q \cong S$ and $\operatorname{rank}_{\mathcal{O}} Q = \operatorname{rank}_{\mathcal{O}} \mathcal{P}(S)$.

Reason for that:

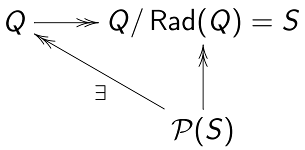


Construction of projective indecomposables

Recognizing projectives

If Q is a Λ-lattice, then Q is isomorphic to the projective cover $\mathcal{P}(S)$ of the semisimple Λ-module S iff $Q/\operatorname{Rad} Q \cong S$ and $\operatorname{rank}_{\mathcal{O}} Q = \operatorname{rank}_{\mathcal{O}} \mathcal{P}(S)$.

Reason for that:



So: We have to find **lattices with simple top** in arbitrary dimensions (in which they exist)!

Amalgams

Definition of an **amalgam**

Let L_1, L_2 be Λ-lattices. An **amalgam of** L_1 **and** L_2 is a Λ-sublattice $M \leqslant L_1 \oplus L_2$ s. t. the projections from M to L_1 and L_2 are onto.

Amalgams

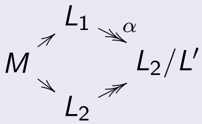
Definition of an **amalgam**

Let L_1, L_2 be Λ-lattices. An **amalgam of L_1 and L_2** is a Λ-sublattice $M \leqslant L_1 \oplus L_2$ s. t. the projections from M to L_1 and L_2 are onto.

Remark

An **amalgam** $M \leqslant L_1 \oplus L_2$ corresponds to a sublattice $L' \leqslant L_2$ ($L' = M \cap L_2$) and an epimorphism $\alpha : L_1 \twoheadrightarrow L_2/L'$.

Reconstruct M as pullback:



Amalgams

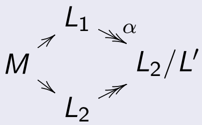
Definition of an amalgam

Let L_1, L_2 be Λ-lattices. An **amalgam of L_1 and L_2** is a Λ-sublattice $M \leqslant L_1 \oplus L_2$ s. t. the projections from M to L_1 and L_2 are onto.

Remark

An **amalgam** $M \leqslant L_1 \oplus L_2$ corresponds to a sublattice $L' \leqslant L_2$ ($L' = M \cap L_2$) and an epimorphism $\alpha : L_1 \twoheadrightarrow L_2/L'$.

Reconstruct M as pullback:



One can show: If there is no common torsionfree epic image of L_1 and L_2 then any descending chain of amalgams of L_1 and L_2 becomes stationary.

The main ingredient

Notation

For Λ-lattices L_1, L_2 denote by $\min.\operatorname{amal.}(L_1, L_2)$ the set of **minimal** (w.r.t. inclusion) **amalgams of** L_1 **and** L_2.

Theorem

Let S be a simple Λ-module. If L_1 and L_2 are Λ-lattices ***both with top*** *S. Then any $M \in \min.\operatorname{amal.}(L_1, L_2)$ has* ***top*** *S.*

The main ingredient

Notation

For Λ-lattices L_1, L_2 denote by $\min.\mathrm{amal}.(L_1, L_2)$ the set of **minimal** (w.r.t. inclusion) **amalgams of** L_1 **and** L_2.

Theorem

Let S be a simple Λ-module. If L_1 and L_2 are Λ-lattices ***both with top*** *S. Then any $M \in \min.\mathrm{amal}.(L_1, L_2)$ has* ***top*** *S.*

Corollary

If L_1 and L_2 are Λ-lattices as in the theorem, s. t. L_2 is irreducible and there is no epimorphism $L_1 \twoheadrightarrow L_2$, then we can construct an amalgam of L_1 and L_2 with simple top S.

Outline of the algorithm

Construction of PIM's

1. Let $\mathcal{L} = \mathcal{L}(\Lambda, S)$ denote the set of representatives of the isomorphism classes of **irreducible Λ-lattices with simple top** S.
2. Start with some $L_1 \in \mathcal{L}$.
   For $i = 1, 2, \ldots$: Pick an element $L' \in \mathcal{L}$ s. t. there is no epi $L_i \twoheadrightarrow L'$ and pick $L_{i+1} \in \text{min. amal.}(L_i, L')$.
3. When $\text{rank}_{\mathcal{O}} L_i = \text{rank}_{\mathcal{O}} \mathcal{P}(S)$, then $L_i \cong \mathcal{P}(S)$

Remark: To reach 3 it is crucial that there are "enough" non-isomorphic irreducible lattices with top S. This can in fact be shown.

Applications for $\Lambda = \mathcal{O}G$

Calculation of basic orders for

- Symmetric groups: $\ldots, \mathbb{Z}_2 S_7$, $\mathbb{Z}_2 S_8$, $\mathbb{Z}_2 S_9$
- Principal block of $\mathbb{Z}_2 M_{12}$

Applications for $\Lambda = \mathcal{O}G$

Calculation of basic orders for

- Symmetric groups: $\ldots, \mathbb{Z}_2 S_7$, $\mathbb{Z}_2 S_8$, $\mathbb{Z}_2 S_9$
- Principal block of $\mathbb{Z}_2 M_{12}$

If representations become too big / not all irreducible representations are known / $\ldots$:

- Calculate a basic order for $\varepsilon \mathcal{O}G$, where ε is a central idempotent in $A = KG$
- Calculate $\varepsilon e \mathcal{O}Ge$ for some idempotent $e \in \mathcal{O}G$ and ε as above.

For group rings, condensation is also an important tool to reduce the dimensions of the involved lattices.

Some random examples

$$\varepsilon^{(4,2,1^2)} e^{(8)} \mathbb{Z}_2 S_8 e^{(8)} \cong \mathbb{Z}_2 \left\langle \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$$\varepsilon^{(4,2,1^2)} e^{(6,2)} \mathbb{Z}_2 S_8 e^{(6,2)} \cong \mathbb{Z}_2 \left\langle \begin{pmatrix} 0 & 4 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 10 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 16 \end{pmatrix} \right\rangle$$

$$\varepsilon^{(4,2,1^3)} e^{(9)} \mathbb{Z}_2 S_9 e^{(9)} \cong$$
$$\mathbb{Z}_2 \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 16 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{pmatrix} \right\rangle$$

The problem: Finding candidates for $e\mathcal{O}Ge$

The setup

Λ as before. Let $u \in Z(A)^\times$. Associate to that u the trace bilinear form

$$T_u : A \times A \to K : (a, b) \mapsto \sum_{\varepsilon_i} \operatorname{tr.red.}_{\varepsilon_i A/K}(\varepsilon_i u \cdot ab)$$

$L \subset A$ an $\mathcal{O}$-lattice, then define $L^\sharp := \{a \in A \mid T_u(a, L) \subseteq \mathcal{O}\}$
Question: Are there any $\mathcal{O}$-orders $\Gamma \supseteq \Lambda$ s. t. $\Gamma^\sharp = \Gamma$?

The problem: Finding candidates for $e\mathcal{O}Ge$

The setup

Λ as before. Let $u \in Z(A)^\times$. Associate to that u the trace bilinear form

$$T_u : A \times A \to K : (a, b) \mapsto \sum_{\varepsilon_i} \operatorname{tr.red.}_{\varepsilon_i A/K}(\varepsilon_i u \cdot ab)$$

$L \subset A$ an $\mathcal{O}$-lattice, then define $L^\sharp := \{a \in A \mid T_u(a, L) \subseteq \mathcal{O}\}$

Question: Are there any $\mathcal{O}$-orders $\Gamma \supseteq \Lambda$ s. t. $\Gamma^\sharp = \Gamma$?

Motivation: For G a group, $e \in \mathcal{O}G$ an idempotent we know that $e\mathcal{O}Ge$ is selfdual w. r. t. $u = \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(1)}{|G|} \varepsilon_\chi \cdot e$. We want to determine $e\mathcal{O}Ge$ from the knowledge of a full suborder $\Lambda \subset e\mathcal{O}Ge$.

Finding **one** selfdual lattice

If Γ is an $\mathcal{O}$-order with

$$\Lambda \subseteq \Gamma = \Gamma^\sharp \subseteq \Lambda^\sharp$$

then Γ is a Λ-Λ-bilattice.
So: **Find selfdual Λ-Λ-bilattices $L \subset A$ with $L \supseteq \Lambda$**

Finding **one** selfdual lattice

If Γ is an $\mathcal{O}$-order with

$$\Lambda \subseteq \Gamma = \Gamma^\sharp \subseteq \Lambda^\sharp$$

then Γ is a Λ-Λ-bilattice.
So: **Find selfdual Λ-Λ-bilattices $L \subset A$ with $L \supseteq \Lambda$**

How to find **one** selfdual Λ-Λ-lattice containing Λ

Any chain of Λ-Λ-bilattices

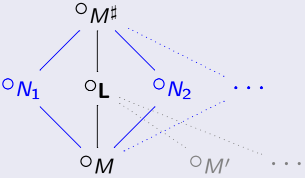
$$\Lambda^\sharp \supsetneq L_1 \supsetneq L_2 \supsetneq \dots$$

will eventually terminate with an L_k s. t. $\text{length}_{\mathcal{O}} L_k/L_k^\sharp$ is minimal among all Λ-Λ-bilattices L with $L \supseteq L^\sharp$.

Finding **all** selfdual lattices: Neighboring lattices

Definition of neighbors

Let L be a Λ-Λ-bilattice in A with $L = L^\sharp$
Then a Λ-Λ-bilattice $N \subset A$ s. t. $N = N^\sharp$ and $L/L \cap N$ is simple is called a **neighbor** of L.



The relation "L is a neighbor of L'" defines a graph with the selfdual lattices in A containing Λ as vertices.

Enumeration of selfdual overorders

Theorem (Kneser, Morales,...)

The neighboring graph is ***finite*** *and* ***connected****.*

Enumeration of selfdual overorders

Theorem (Kneser, Morales,...)

*The neighboring graph is **finite** and **connected**.*

Outline of the algorithm

1. Find a minimal Λ-Λ-bilattice $L \subseteq \Lambda^\sharp$ w.r.t. the condition $L^\sharp \supseteq L$ ($\rightsquigarrow$ this L is selfdual if any selfdual Λ-Λ-bilattices in A exist at all).
2. Search the neighboring graph (this requires calculation of maximal submodules and calculation of duals)

Remarks & applications

- $\mathrm{length}_{\mathcal{O}}\, \Lambda^{\sharp}/\Lambda$ should be small for this algorithm to work
- When determining basic algebras of group rings $\mathcal{O}G$ with decomposition numbers 0 and 1, this helps to find $e\mathcal{O}Ge$ for primitive idempotents $e \in \mathcal{O}G$ from the inclusion $eZ(\mathcal{O}G)e \subseteq e\mathcal{O}Ge$.
- The latter inclusion is often already an equality; in cases where it fails to be, it is often of small index (an example for that is the principal block of $\mathbb{Z}_3 S_{10}$).

Thank you for your attention!