Finite free resolutions

Jürgen Herzog

In a series of 4 lectures we consider finite free resolutions from an algebraic and combinatorial point of view. By a famous theorem of Hilbert-Serre, each finitely generated module of a regular local ring or each finitely generated graded module of a polynomial ring which is defined over a field has a finite free resolution. The Betti numbers, respectively the graded Betti numbers of the module provide the numerical data of the resolution, and are a good measure for the complexity of the module. There is a conjectured lower bound for the Betti numbers, due to Buchsbaum-Eisenbud and Horrocks, and better understood upper bounds arising by passing to initial ideals with respect to monomial orders. In these lectures we first discuss the acyclicity criteria of Peskine-Szpiro and Buchsbaum-Eisenbud and describe some of the few known results regarding the lower bound conjecture. We then turn to graded resolutions and show how the multiplicity conjecture of Huneke and Srinivasan was solved by using Boij-Söderberg theory, which is an exciting new approach to graded finite resolutions. When passing to initial ideals the Betti numbers of the corresponding ideals can only increase. The extremal Betti mumbers however, as shown by Bayer, Charalambous and Popescu, remain unchanged for generic initial ideals with respect to the reverse lexicographic order. The possible position and values of extremal Betti numbers will be described, and those ideals will be characterized for which all Betti number remain unchanged when passing to the generic initial ideal. These are exactly the componentwise linear ideals. The possible sequences of Betti numbers for ideals with linear resolution will be characterized. A similar characterization for componentwise linear resolutions is open. These discussions reflect results by Varbaro, Sharifan, Hibi and myself. As a final topic we consider monomial ideals with linear resolution. While the ideals with 2-linear resolution are combinatorially characterized by Fröberg, such a uniform characterization for higher degree monomial ideals cannot be expected, though some nice results have been recently obtained in this direction. But if one requires that not only the monomial ideal but also all its monomial localizations have a linear resolution a classification seems to be possible. It is conjectured by Bandari and myself that these are exactly the polymatroidal ideals. The conjecture is true for squarefree monomial ideals, as well as in many other special cases that will be presented in this lecture.