Atlantic Association for Research in Mathematical Sciences
Memorial University of Newfoundland
Atlantic Algebra Centre

International Workshop

Groups, Rings, Lie and Hopf Algebras. II

August 13 - 17, 2007

Abstracts of the Workshop Presentations

Bonne Bay Marine Station of MUN

Norris Point, Newfoundland, Canada

Teymuraz Tvalavadze

Carleton University

Representations of toroidal Lie algebras and superalgebras

Toroidal Lie superalgebras are very natural multivariable generalizations of affine Kac – Moody algebras and superalgebras which in the last decade have attracted attention of physicists.

Let $\mathfrak{g}$ be a toroidal Lie algebra. Let B_χ denote the category of bounded $\mathfrak{g}$-modules with finite-dimensional weight spaces, with the central character χ. This category could be regarded as an analogue of the category of highest weight modules for affne Kac-Moody algebras. The difference from the affne case is that the highest weight space is not anymore one-dimensional. The structure of irreducible $\mathfrak{g}$-modules in category B_χ was completely determined by Y. Billig.

In this talk we will consider similar problems for the representations of toroidal Lie superalgebras.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Margaret Beattie

Co-Frobenius Hopf Algebras

Mount Allison University

Let H be a finite dimensional Hopf algebra over a field k. For example, H might be $k[G]$, the group algebra of a finite group with basis $x_g, g \in G$. The element $t = \sum_{g \in G} x_g$ has the property that for $x = \sum_{g \in G} \alpha_g x_g$, then $tx = xt = (\sum \alpha_g)t = \epsilon(x)t$ where ϵ denotes the augmentation map. The element t is called a left and right integral in $k[G]$. A Hopf algebra H contains a left or a right integral in H if and only if H is finite dimensional. In this case, H^* is also a finite dimensional Hopf algebra, and so contains left and right integrals. If $H = k[G]$, let p_h denote the element of H^* mapping x_h to 1 and all other basis elements to 0. Then p_e is a left and right integral in H^* since $p_g p_e = \delta_{g,e} p_e = p_g(1) p_e$.

If H is not finite dimensional, H may still have a left integral in H^*, i.e., an element $0 \neq \lambda \in H^*$ such that $h_1 \lambda(h_2) = \lambda(h)1$. If H is finite dimensional then this is an integral in the Hopf algebra H^*. Such a Hopf algebra is called co-Frobenius.

This talk will explain how many of the properties of finite dimensional Hopf algebras proved by using the integrals of the Hopf algebra and its dual, have analogues in the co-Frobenius case.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Ivan Dimitrov

Queen's University

Cup Product on Homogeneous Varieties and Generalized PRV Components

Let $G = GL(n, \mathbb{C})$ and let $B \subset G$ be a Borel subgroup with corresponding homogeneous variety $X = G/B$. If λ is a character of B, we denote by $\mathcal{L}_\lambda$ the corresponding line bundle on X. Given λ_1 and λ_2 we consider the cup product map

$$H^{q_1}(X, \mathcal{L}_{\lambda_1}) \otimes H^{q_2}(X, \mathcal{L}_{\lambda_2}) \to H^{q_1+q_2}(X, \mathcal{L}_{\lambda_1+\lambda_2}). \quad (1)$$

In this talk I will present necessary and sufficient conditions for (1) to be surjective. It turns out that whenever (1) is surjective, the G–module $H^{q_1+q_2}(X, \mathcal{L}_{\lambda_1+\lambda_2})$ is either trivial, or a generalized PRV component of the G–module $H^{q_1}(X, \mathcal{L}_{\lambda_1}) \otimes H^{q_2}(X, \mathcal{L}_{\lambda_2})$. However not every generalized PRV component of the tensor product of two G–modules appears in this way. Finally, I will explain what combinatorial problems relate to the properities of (1), e.g. multiplicities of the generalized PRV components, etc.

This talk is based on a joint work with Mike Roth.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Allen Herman

University of Regina, Regina, SK, Canada

Equivalence of Central Simple G-algebra

Let K be a field, G a finite group, and A a finite-dimensional G-algebra over K. A is called a *central simple* G-algebra over K if A has no proper G-invariant non-zero ideals and the fixed-point subalgebra $Z(A)^G$ is isomorphic to K.

One can define an equivalence relation on the class of central simple G-algebras over K in a manner similar to the definition of equivalence of central simple K-algebras that is used to define the Brauer group of K. However, the equivalence classes of central simple G-algebras over K do not generate a group. Instead, the most natural algebraic setting arises from an action of the group $H^2(G, K^\times) \times Br(K)$ on the classes.

A complete list of invariants for distinguishing the equivalence classes of central simple G-algebras over K is only known in certain cases. I will survey these situations and discuss some related open problems.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Kris Janssen

Free University of Brussels

Group Corings

An important result in Hopf-Galois theory states that the category of relative Hopf modules over a faithfully flat Hopf-Galois extension is equivalent to the category of modules over the coinvariants. So far, no such result is known in the framework of Hopf group coalgebras.

Group coalgebras and Hopf group coalgebras were introduced by V.G. Turaev in his work [1] on homotopy quantum field theories. In the case where the group under consideration is trivial one recovers the usual notions of coalgebra and Hopf algebra. A study of these structures from a purely algebraic point of view was carried out by A. Virelizier, M. Zunino and S.H. Wang.

We introduce group corings, and study functors between categories of comodules over group corings, and the relationship to graded modules over graded rings. Galois group corings are defined, and a Structure Theorem for the group comodules over a Galois group coring is given. We study (graded) Morita contexts associated to a group coring.

Our theory is applied to group corings associated to a comodule algebra over a Hopf group coalgebra, hence providing a Structure Theorem for Hopf group coalgebras.

This is a report on joint work [2] with S. Caenepeel and S.H. Wang.

References

[1] V. Turaev, Homotopy field theory in dimension 3 and crossed group-categories, preprint arXiv:math.GT/0005291 v1.

[2] S. Caenepeel, K. Janssen, S.H. Wang, Group corings, to appear in Applied Categorical Structures (preprint arXiv:math/0701931v1).

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Eric Jespers

Free University of Brussels

Primes of Height One and a Class of Noetherian Finitely Presented Algebras

The search for new concrete classes of finitely presented algebras (defined by monomial relations) that satisfy some important arithmetical properties (such as being a maximal order) recently gained a lot of interest. These algebras can be considered as semigroup algebras $K[S]$, where S is a monoid defined via a presentation as that of the algebra. In general, it remain unsolved problems to characterize when an arbitrary semigroup algebra $K[S]$ is Noetherian and when it is a prime Noetherian maximal order. The former question has been resolved for submonoids of polycyclic-by-finite groups. The second question has been completely solved for abelian monoids S (Chouinard) and for polycyclic-by-finite groups (Brown). Jespers and Okninski recently described when a semigroup algebra $K[S]$ of a submonoid S of a finitely generated abelian-by-finite group is a Noetherian maximal order that is a domain (so the group of fractions SS^{-1} is torsion free). In this case it turns out that: (1) the prime ideals of $K[S]$ intersecting S non-trivially are precisely the ideals $K[Q]$ with Q a prime ideal of S, and (2) the height one prime ideals of $K[S]$ intersecting S non-trivially are precisely the ideals $K[Q]$ with Q a minimal prime ideal of S.

In this lecture we report on recent results that give an answer to the question when $K[S]$ is a prime PI Noetherian maximal order in case S is a submonoid of a polycyclic-by-finite group (so SS^{-1} is abelian-by-finite but not necessarily torsion free). We show that the first property does not remain valid, however the second part on primes of height one still remains true. As a consequence, we establish going up and going down properties between prime ideals of S and prime ideals of $S \cap H$, where H is a subgroup of finite index in G. As an application it is shown that the classical Krull dimension of $K[S]$ is the sum of the prime dimension of S and the plinth length of the unit group of S. Also, a result of Schelter is extended to the monoid S: the prime dimension of S is the sum of the height and depth of any prime ideal of S. The information obtained on primes of height one then allows us to determine when a semigroup algebra $K[S]$ is a prime Noetherian maximal order provided that $G = SS^{-1}$ is a finitely generated abelian-by-finite group (that is, $K[S]$ satisfies a polynomial identity). The result reduces the problem to the structure of the monoid S (in particular S has to be a maximal order within its group of quotients G) and to that of G. It turns out that the action of G on minimal primes of some abelian submonoid of S is very important. We also give a useful criterion for verifying the maximal order property of such S. We finish with some examples illustrating the results.

$\infty \infty \infty \infty \infty \infty \infty \infty \infty \infty$

Yevgenia Kashina

DePaul University

Groups of Grouplike Elements of a Semisimple Hopf Algebra and its Dual

We are interested in the following question: How is the group of grouplike elements of a Hopf algebra connected to the group of grouplike elements of its dual? It is closely related to the problem of classifying semisimple Hopf algebras. We discuss this question in the case of cocentral abelian extensions. We then relate the number of the central grouplikes of a Hopf algebra to the number of grouplikes of its dual for cocentral abelian extensions of the cyclic group of order p. We also provide more specific results in the case of dimension 2^n.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Mikhail Kochetov

Memorial University

Group gradings on simple Lie algebras of type "A"

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Michael Lau

University of Windsor

Thin Coverings of Modules

Recent work on extended affine Lie algebras (EALAs) has shown that almost every EALA is obtained by adjoining vector fields and 1-forms to a (twisted) multiloop algebra. In joint work with Yuly Billig, we are developing techniques for "twisting" irreducible modules for (untwisted) toroidal EALAs into irreducible modules for more general ("twisted") EALAs.

The process relies on thin coverings, a method of constructing graded-simple modules from simple (ungraded) modules. In this talk, we define and classify thin coverings of quasifinite modules over graded associative algebras. Surprisingly, the representation theory of cyclotomic quantum tori will play an important role in the classification. The classification gives an explicit description of a large family of bounded highest weight modules for extended affine Lie algebras.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Yuanlin Li

Brock University

Strongly Clean Matrix Rings

An element of a ring R with identity is called strongly clean if it is the sum of an idempotent and a unit that commute, and R is called strongly clean if every element of R is strongly clean. In this talk, we determine when a 2×2 matrix A over a commutative local ring is strongly clean. Several equivalent criteria are given for such a matrix to be strongly clean. Consequently, we obtain several equivalent conditions for the 2×2 matrix ring over a commutative local ring to be strongly clean, extending a result of Chen, Yang, and Zhou.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Mitja Mastnak

University of Waterloo

Bialgebra Cohomology, Pointed Hopf Algebras, and Deformations

A Hopf algebra is called pointed when its coradical (a construction dual to that of Jacobson radical for algebras) is a group algebra. Over the last decade considerable progress has been made in construction and classification of pointed Hopf algebras. Andruskiewitch and Schneider have classified all finite dimensional pointed Hopf algebras having abelian coradical, under some mild restrictions on the size of the coradical. This important result is a culmination of a long series of papers. In the talk I will explain how deformation-theoretic approach can help in the understanding of the Andruskiewitch-Schneider classification. This is joint work with Sarah Witherspoon.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Jason McGraw

Memorial University of Newfoundland

Grading by Finite Groups on Lie Algebras of Type D_4

The gradings on simple Lie algebras of type D_l, $l > 4$, have been described by Y. Bahturin and M. Zaicev in *Gradings on simple Jordan and Lie algebras, J. Algebra 2005.* This was done by looking at gradings on the full matrix algebras and noting that for a realisation of a Lie algebra of type D_l, $l > 4$, as $K(M_{2l}, *)$, the skew-symmetric matrices with respect to a transpose involution $*$, the automorphisms of $K(M_{2l}, *)$ can be lifted to automorphisms of the full matrix algebra. Gradings by groups on finite dimensional Lie algebras L have corresponding Abelian subgroups of the automorphisms of L. The gradings on Lie algebras of type D_4 were not described because some of the automorphisms of these Lie algebras cannot be lifted to the full matrix algebra. I have determined which gradings are isomorphic to a grading whose corresponding subgroup of automorphism can be lifted. We described all gradings whose corresponding automorphism subgroups can be lifted using the results of *Gradings on simple Jordan and Lie algebras*. The corresponding subgroups of automorphism for the remaining gradings are of the form $\langle \rho \rangle \times A$ where ρ is the product of an outer automorphism of order 3 and an inner automorphism, and A is a subgroup of the inner automorphisms.

My idea to describe the remaining gradings on a Lie algebra L of type D_4 is to find all gradings by groups B on L^ρ, the subalgebra of L fixed by ρ, and, if possible, use this grading to find a new grading on L with corresponding automorphism $\langle g \rangle \times B$ where the induced grading by $\langle g \rangle$ has $\langle \rho \rangle$ as the corresponding subgroup of the automorphims of L. At the time the gradings for all of the Lie algebras L^ρ were not described. For example when ρ is an outer automorphism of order 3, L^ρ is a Lie algebra of type G_2. We used this technique successfully to describe all gradings on a Lie algebra L of type D_4 whose corresponding subgroups of automorphisms are of the form $\langle \sigma \rangle \times A$ where σ is an outer automorphism of order 2 and A is a subgroup of the inner automorphisms by using the gradings on L^σ which is a Lie algebra of type B_3.

$\infty \infty \infty \infty \infty \infty \infty \infty \infty \infty$

Erhard Neher

University of Ottawa

Lie Tori

Lie tori are mostly infinite-dimensional Lie algebras which have a grading by the root lattice of a root system and an additional compatible grading so that the joint homogeneous spaces are mostly 1-dimensional, i.e. an essentially fine grading. Example: the loop algebra of a finite-dimensional semisimple Lie algebra, or toroidal Lie algebras. Lie tori arise naturally in the theory of extended affine Lie algebras. There has recently been a lot of activity to work out their classification. My talk would describe (parts of) the classification of Lie tori, which requires nonassociative algebras (Jordan algebras, alternative algebras etc).

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Dmitry Nikshych

University of New Hampshire

Modular Categories, Their Group-Theoretical Properties, and Categorification of Manin Pairs

This is a report on a recent joint work with V.Drinfeld, S.Gelaki, and V.Ostrik. We show that nilpotent braided fusion categories can be constructed from finite groups in a very explicit way and that they admit an analogue of the Sylow decomposition. We also characterize a natural class of modular categories of prime power dimension as twisted Drinfeld doubles of finite p-groups. As a consequence, we obtain that semisimple quasi-Hopf algebras of prime power dimension are group-theoretical. Our techniques use a reconstruction of twisted group doubles from Lagrangian subcategories of modular categories, which is reminiscent to Drinfeld's characterization of doubles of quasi-Lie bialgebras in terms of Manin pairs.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

M. M. Parmenter

Memorial University of Newfoundland

Group Gradings and Graded Reversibility in Integral Group Rings

Recently, Yuri Bahturin and I investigated C_2-gradings of $\mathbb{Z}G$ for arbitrary finite groups G. If G has a subgroup H of index 2 and $g \in G \setminus H$, then an obvious C_2-grading is given by $\mathbb{Z}G = \mathbb{Z}H \oplus (\mathbb{Z}H)g$. Additionally, $\mathbb{Z}G = \gamma(\mathbb{Z}H) \oplus \gamma(\mathbb{Z}H)g$ is a C_2-grading for any automorphism γ of $\mathbb{Z}G$.

Question. *Do the above represent all possible C_2-gradings of $\mathbb{Z}G$?*

This has been answered affirmatively for finite abelian groups and finite nilpotent groups of odd order, but in general is still open.

An associative ring R is called reversible if $ab = 0$ implies $ba = 0$ when $a, b \in R$. Reversible rings were investigated by Cohn who showed that they satisfy the Köthe conjecture. Later, Gutan and Kisielewicz characterized reversible group algebras KG where G is a torsion group and K is a field, while Li and Parmenter investigated reversibility in more general group rings. When an S-algebra is graded by a group A we say R is graded reversible with respect to the grading if $ab = 0$ implies $ba = 0$ where a, b are homogeneous elements of R.

If $R = \mathbb{Z}G$ and $A = C_2$, the above open question indicates the possibility of restricting the gradings to those of the type listed. We will report on some preliminary joint work with Yuanlin Li on graded reversibility of $\mathbb{Z}G$, carried out in that setting.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Sudarshan Sehgal

University of Alberta

Hartley's Conjecture and Problems Arising Therefrom

Hartley conjectured that if the units of a group algebra KG of a torsion group G satisfy a group identity then KG satisfies a polynomial identity. This was solved affirmativley . The solution lead to lots of activity. The talk is a survey of this topic.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Vera Serganova

University of California - Berkeley

"On weight representations of sl(n)"
(joint work with D. Grantcharov)

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra. A $\mathfrak{g}$-module M is called a weight module if M is semisimple over $\mathfrak{h}$ and all $\mathfrak{h}$-eigenspaces have finite dimension. Weight modules were studied by Britten, Lemire, Benkart, Joseph, Fernando, Futorny and finally classified by Mathieu. A weight module M is called *cuspida*l if all $\mathfrak{h}$-eigen spaces have the same dimension and root elements of $\mathfrak{g}$ act bijectively between $\mathfrak{h}$-eigenspaces. Fernando proved that every irreducible weight module is isomorphic to a unique irreducible quotient of the module induced from some irreducible cuspidal module of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Clearly, this theorem reduces the problem of classifying irreducible weight modules to classification of cuspidal irreducible modules. Fernando also proved that if $\mathfrak{g}$ is simple then cuspidal modules exist only for $\mathfrak{g} = \mathrm{sl}(n)$ or $\mathrm{sp}(n)$. Mathieu classified cuspidal irreducible modules.

In our work we give a complete description of indecomposable cuspidal modules over $\mathrm{sl}(n)$. For $\mathfrak{g} = \mathrm{sp}(n)$ it was proven by Britten, Lemire, Khomenko and Mazorchuk that cuspidal modules are semisimple. To solve the problem we use the technique of quiver representations. The problem is closely related to the famous paper of Gelfand and Ponamorev, generalized later by Khoroshkin.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

M. Tvalavadze, T. Tvalavadze

University of Western Ontario, Carleton University

Group gradings on superalgebras

In this talk we discuss the structure of finite-dimensional graded superalgebras of different types such as associative, Lie and Jordan over an algebraically closed field of characteristic zero. First, we formulate a few theorems about the finite-dimensional simple associative superalgebras graded by finite abelian groups and equipped with a superinvolution compatible with the grading. Second, we apply these results to the classification of group gradings on finite-dimensional simple Lie and Jordan superalgebras of certain types.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

J. Vercruysse

Free University of Brussels

On Generalizations of Hopf-Galois Theory: A Unifying Approach

Initiated by Hopf-Galois theory (see e.g. [4] [5]) and following the recent evolution of generalizing algebraic structures, a wide variety of Galois theories surfaced during the last decades.

In this respect there were several attempts to lift the theory of finite Galois comodules [6] to an infinite theory [7], [9]. However the theory of [9] has another approach than [7] and the relation between both theories was not clear from the start.

Furthermore Hopf-Galois theory and the theory of Galois corings has been extended in different directions. In [1] equivalences between categories of comodules are described, a Galois theory for C-rings is being developed in [2] and [3] discusses a Galois theory for group corings, generalizing Hopf group coalgebras.

In a slightly different direction, there has been a similar evolution in (categorical) descent theory. Descent theory investigates the extension of scalars functor $-\otimes_B A : \mathcal{M}_B \to \mathcal{M}_A$ associated to a ring morphism $B \to A$. An important theorem in this respect, is the classical theorem of Beck that gives sufficient and necessary conditions for a functor with a right adjoint to be comonadic. Another interesting and more general result is the Joyal-Tierney theorem [8].

In this talk we propose a Galois theory for comonads in the general setting of bicategories. All variations on (Hopf-)Galois and descent theory discussed above can be obtained from our theory as special cases. Moreover our unifying framework provides the ability to relate the different theories and see how they are connected. In particular we obtain a transparant view on the interactions between the different approaches in the recent development on (infinite) Galois comodules and clarify the relation between the coring theory and the categorical descent theory. Another application of our theory is the development of a Galois theory over (dual) quasi-Hopf algebras, of which the octonians are an interesting example.

This is joint work with José Gómez-Torrecillas.

References

[1] T. Brzeziński, L. El Kaoutit and J. Gómez-Torrecillas The bicategories of corings *J. Pure Appl. Algebra* **205** (2006), 510–541.

[2] T. Brzeziński and R. B. Turner, The Galois theory of matrix C-rings, *Appl. Categ. Structures* **14**, 471-501 (2006).

[3] S. Caenepeel, K. Janssen and S. Wang, Group corings, to appear in *Appl. Categ. Structures*.

[4] S. U. Chase and M. E. Sweedler, Hopf algebras and Galois theory, Lecture Notes in Mathematics, Vol. 97, Springer-Verlag, Berlin, 1969.

[5] Y. Doi and M. Takeuchi, Hopf-Galois extensions of algebras, the Miyashita-Ulbrich action, and Azumaya algebras, *J. Algebra* **121** (1989), 488–516.

[6] L. El Kaoutit and J. Gómez-Torrecillas, Comatrix corings: Galois corings, Descent Theory, and a Structure Theorem for Cosemisimple corings, *Math. Z.* **244** (2003), 887–906.

[7] J. Gómez-Torrecillas and J. Vercruysse, Comatrix corings and Galois Comodules over firm rings, *Algebr. Represent. Theory*, **10** (3) (2007), 271–306.

[8] B. Mesablishvili, Pure morphisms of commutative rings are effective descent morphisms for modules–a new proof. *Theory Appl. Categ.* **7** (2000), 38–42.

[9] R. Wisbauer, On Galois comodules, *Comm. Algebra* **34** (2006), 2683–2711.

$\infty\ \infty\ \infty\ \infty\ \infty\ \infty\ \infty\ \infty\ \infty\ \infty$

Xiande Yang

Memorial University of Newfoundland

On Strongly Clean Matrix Rings

An element a in a ring R is called strongly clean if there exist an idempotent e and a unit u in R such that $a = e + u$ and $eu = ue$. The ring R is called strongly clean if every element of R is strongly clean. It is well known that the matrix ring over a strongly clean ring need not be strongly clean. A sufficient but not necessary condition for a matrix ring over a commutative ring to be strongly clean is given and a necessary condition for a matrix ring over any ring to be strongly clean is also given.

$\infty \infty \infty \infty \infty \infty \infty \infty \infty \infty$

Alexandre Zalesski

University of East Anglia

On eigenvalues of groups elements in representations of simple algebraic groups

Some conditions for the elements of simple algebraic groups will be provided to guarantee that the element fixes a non-zero vector in every representation of the group.

$\infty \infty \infty \infty \infty \infty \infty \infty \infty \infty$

Mikhail Zaicev

Moscow State University

Numerical invariants of algebras with polynomial identities

One of the main objectives of the talk is to show how one can combine methods of ring theory, combinatorics and representation theory of groups with analytical approach in order to study the polynomial identities satisfied by given algebra.

Recall that a PI-algebra is any associative algebra satisfying at least one nontrivial polynomial identity. This includes the polynomial rings in one or several variables, the Grassmann (or exterior) algebra, finite-dimensional algebras and many other algebras that occur naturally in mathematics. With any PI-algebra A one can associate numerical sequence $c_n(A)$ of codimensions of its identities. Asymptotic behavior of this sequence gives an important information about algebra itself and its polynomial identities.

In 1972 Regev have proved that codimension growth of any PI-algebra A is exponentially bounded. At the end of 80th Amitsur conjectured that the sequence of nth roots of $c_n(A)$ has limit and this limit is an integer. In 1999 Giambruno and speaker confirmed Amitsur's conjecture and called this limit the PI-exponent of A. Similarly, one can define codimensions, codimension growth and PI-exponent for any variety of associative algebras. Given positive integer n, a variety $\mathfrak{V}$ is said to be *minimal of PI-exponent* n if any its proper subvariety has a PI-exponent at most $n-1$ while the exponent of $\mathfrak{V}$ itself is exactly n. In 2003 all minimal varieties were completely described. In particular, it was shown that for any n there exist only finite number of minimal varieties.

In non-associative case codimensions behavior is completely different. First of all codimensions sequence is not exponentially bounded in general. Nevertheless for a wide class of algebras exponential upper bound exists. This class includes all inite-dimensional algebras, infinite-dimensional simple Lie algebras of Cartan type, affine Kac-Moody algebras and many other. But even if codimensions are exponentially bounded the integrality of PI-exponent does not hold. In 1999 Mishchenko and speaker have constructed first example of infinite-dimensional Lie algebra with fractional PI-exponent. In present time existence and integrality of PI-exponent have proved for all finite-dimensional Lie algebras, for affine Kac-moody algebras, for some finite-dimensional Lie superalgebras and for almost all finite-dimensional simple Jordan algebras. In general non-associative case it is shown the existence of algebras with any real PI-exponent.

Recall that a sequence has an intermediate growth if it grow faster than any polynomial function but slower than any exponential function (with the ratio of exponent greater than one). It is known that both in associative and Lie case there are no algebras with an intermediate codimension growth. Recently Giambruno, Mishchenko and speaker constructed the series of non-associative algebras with any (in some sense) ntermediate codimension growth.

Main tools of the theory of codimension growth are the representation theory of the symmetric group, combinatorics of Young diagrams, central polynomials, combinatorics of infinite words and structure theory of finite-dimensional algebras.