Support theory for Hopf algebras and tensor categories

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ALGEBRA $\rightsquigarrow$ TOPOLOGY

ALGEBRA	$\rightsquigarrow$	TOPOLOGY
modules for Hopf algebras/ objects in tensor categories	$\rightsquigarrow$	support spaces

Finite tensor categories

k - field

Definition A **finite tensor category** $\mathcal{C}$ is a locally finite k-linear abelian category with finitely many simple objects, enough projectives, and a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying some conditions. There is a **unit object** 1 that is simple, and every object has both left and right duals.

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Examples

(1) $\mathcal{C} = A$-mod for a finite dimensional Hopf algebra A: $\quad \otimes = \otimes_k$,
if M, N are A-modules, then $M \otimes N$ is an A-module via the coproduct Δ,
and $\mathbf{1} = k$ is an A-module via the counit $\varepsilon : A \to k$

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(2) Benson-Etingof-Ostrik symmetric tensor categories, char$(k) > 0$

Some triangulated tensor categories

Definition The **stable category** of a finite tensor category $\mathcal{C}$ is the category with the *same* objects, and with morphisms
$\underline{\mathrm{Hom}}(U,V) := \mathrm{Hom}(U,V)/$(those factoring through projective objects)

Visually: Let $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ be a short exact sequence in $\mathcal{C}$.
In the stable category, this is replaced with a corresponding **triangle**

$$\begin{array}{ccc} & U & \\ {}^{[1]}\nearrow & & \searrow \\ W & \longleftarrow & V \end{array}$$

where $W \xrightarrow{[1]} U$ denotes the map $W \longrightarrow \Sigma U$ given by filling in the following diagram in $\mathcal{C}$, where Σ is called a **shift**:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & \Sigma U & \longrightarrow & 0 \end{array}$$

with X an injective (= projective) object

Tensor triangular geometry (Balmer support)
[Balmer '05, Buan-Krause-Solberg '07, Nakano-Vashaw-Yakimov '22, '24]

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Definition A (two-sided thick tensor) **ideal** I of $\mathcal{C}$ is a (full triangulated) subcategory that is **thick** (i.e. closed under taking direct summands) and $A \otimes U,\ U \otimes A \in I$ whenever $U \in I$ and $A \in \mathcal{C}$
An ideal P is **prime** if for all ideals I, J:

$$I \otimes J \subseteq P \quad \Rightarrow \quad I \subseteq P \text{ or } J \subseteq P$$

An ideal P is **completely prime** if for all $A, B \in \mathcal{C}$:

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Balmer spectrum:
Spc($\mathcal{C}$) $:= \{$prime ideals$\}$, a topological space with closed sets all $V(A) := \{P \in$ Spc($\mathcal{C}$) $\mid A \notin P\}$ for $A \in \mathcal{C}$, and arbitrary intersections
Call $V(A)$ the **Balmer support** of A

Properties of Balmer support

Recall: Spc($\mathcal{C}$) $:= \{$prime ideals$\}$,

$V(A) := \{P \in \text{Spc}(\mathcal{C}) \mid A \notin P\}$ for $A \in \mathcal{C}$

Then:

- $V(A \oplus B) = V(A) \cup V(B)$
- $V(\Sigma A) = V(A)$
- $V(A) \subseteq V(B) \cup V(C)$ for triangles $A \to B \to C \to \Sigma A$
- $\bigcup_{X \in \mathcal{C}} V(A \otimes X \otimes B) = V(A) \cap V(B)$

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Theorem [NVY '22] TFAE

(1) $V(A \otimes B) = V(A) \cap V(B)$ for all $A, B \in \mathcal{C}$

(2) Every prime ideal of $\mathcal{C}$ is completely prime

Example: Smash coproduct [G. I. Kac '68, Molnar '77]
G, L - finite groups, with L acting on G by automorphisms

$H := kG \otimes (kL)^*$ as an algebra,
comultiplication $\Delta(g \otimes p_\ell) = \sum_{x \in L} (g \otimes p_x) \otimes ((x^{-1} \cdot g) \otimes p_{x^{-1}\ell})$

Facts:
• There is a Hopf algebra isomorphism $H \cong ((kG)^* \# kL)^*$, and
$\mathcal{C} := H$-mod is equivalent to rep(G)⊠Vec$_L$ as an abelian category
• Generally $\exists$ H-modules A, B for which $V(A \otimes B) \neq V(A) \cap V(B)$;
not every prime tensor ideal of $\mathcal{C}$ is completely prime
• Yet still tensor ideals are classified, in terms of homological data (next)
[Benson-W '14]

Towards cohomological support

$\mathcal{C}$ - finite tensor category with tensor product $\otimes$ and unit object $\mathbf{1}$

Notation $\mathsf{H}(\mathcal{C}, \mathbf{1}) := \mathsf{Ext}^*_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) = \bigoplus_{n \geq 0} \mathsf{Ext}^n_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$

Fact: $\mathsf{H}(\mathcal{C}, \mathbf{1})$ is a graded commutative algebra and for all objects X of $\mathcal{C}$, the cohomology

$$\mathsf{H}(\mathcal{C}, X) := \mathsf{Ext}^*_{\mathcal{C}}(X, X)$$

is an $\mathsf{H}(\mathcal{C}, \mathbf{1})$-module (via "$- \otimes X$")

Hypothesis **(fgc)**: $\mathsf{H}(\mathcal{C}, \mathbf{1})$ is finitely generated, and $\mathsf{H}(\mathcal{C}, X)$ is a finitely generated $\mathsf{H}(\mathcal{C}, \mathbf{1})$-module for all $X \in \mathcal{C}$

From now on we will work with finite tensor categories $\mathcal{C}$ having **(fgc)**

Cohomological support varieties

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Define **support varieties**:

$V^c(\mathbf{1}) := \mathsf{Max}(\mathsf{H}(\mathcal{C}, \mathbf{1}))$,

the set of max. ideals of $\mathsf{H}(\mathcal{C}, \mathbf{1})$, with Zariski topology

(or homog. prime ideals, Proj, etc)

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$V^c(X) := \mathsf{Max}(\mathsf{H}(\mathcal{C}, \mathbf{1}) / \operatorname{Ann}_{\mathsf{H}(\mathcal{C},\mathbf{1})} \mathsf{H}(\mathcal{C}, X))$

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See also Buan-Krause-Snashall-Solberg ’20, Nakano-Vashaw-Yakimov ’22 for tensor triangulated categories; Maurer ’19 for Lie superalgebras via relative cohomology; also versions for module cat $\mathcal{M}$ over $\mathcal{C}$

Cohomological support varieties: example

$\mathcal{C} = kG$-mod, where $G = \mathbb{Z}/p\mathbb{Z}$ and k has prime characteristic p:

$\mathsf{H}(\mathcal{C}, \mathbf{1})$ is essentially $k[x]$, so $V^c(k)$ is a line.

More generally if X is the indecomposable kG-module with $\dim_k(X) = n$ and $n < p$, then $V^c(X)$ is a line.

Properties of support varieties

[Carlson, Feldvoss, Friedlander, Pevtsova, W; Bergh-Plavnik-W '21]

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(5) realization

Maps between these two types of support spaces

[Balmer, Benson, Carlson, Friedlander, Negron, Pevtsova, Rickard; NVY]

Under **(fgc)** and other conditions:

$$\begin{array}{ccc} \text{Balmer spec} & & \text{cohom spec} \\ \mathrm{Spc}(\mathcal{C}) & \longleftrightarrow & \mathrm{Proj}(\mathrm{H}(\mathcal{C},\mathbf{1})) \text{ or } \ldots \\ & & \\ P & \longrightarrow & (f \in \mathrm{H}(\mathcal{C},\mathbf{1}) \mid f \text{ homog, } \mathrm{cone}(f) \notin P) \\ \{X \mid \mathfrak{p} \notin V^c(X)\} & \longleftarrow & \mathfrak{p} \end{array}$$

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$\{X \mid \mathfrak{p} \notin V^c(X)\}$	$\longleftarrow$	$\mathfrak{p}$

Theorem
This is a homeo when $\mathcal{C}$ is stable cat of fin gen modules for a fin group, a fin dim cocommut Hopf algebra, or some other Hopf algs. Then, thick tensor ideals may be classified by cohomological data. For smash coproducts $kG \otimes (kL)^*$, this is a homeo after replacing $\mathsf{H}(\mathcal{C}, \mathbf{1})$ by its L-invariant subalg $\mathsf{H}(\mathcal{C}, \mathbf{1})^L$ (which is a “categorical centre”)

Categorical centre

Definition The **categorical centre** of $\mathsf{H}(\mathcal{C},1)$, with respect to a generating set $\mathcal{S}$ of $\mathcal{C}$, is the graded subalg gen by all

$$f \in \underline{\mathsf{Hom}}(\mathbf{1}, \Sigma^n \mathbf{1}) \cong \mathsf{H}^n(\mathcal{C}, \mathbf{1})$$

(and all n) such that for all $X \in \mathcal{S}$, the diagram commutes:

$$\begin{array}{ccccc}
\mathbf{1} \otimes X & \xrightarrow{\sim} & X & \xleftarrow{\sim} & X \otimes \mathbf{1} \\
\downarrow{\scriptstyle f \otimes \mathsf{id}_X} & & & & \downarrow{\scriptstyle \mathsf{id}_X \otimes f} \\
(\mathbf{\Sigma}^n \mathbf{1}) \otimes X & \xrightarrow{\sim} & \mathbf{\Sigma}^n X & \xleftarrow{\sim} & X \otimes (\mathbf{\Sigma}^n \mathbf{1})
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Theorem [NVY '24]
Let G be the group of tensor invertible objects in $\mathcal{C}$. The categorical centre, with respect to $\mathcal{S}$ for which $G \subseteq \mathcal{S}$, is contained in $\mathsf{H}(\mathcal{C}, \mathbf{1})^G$.

Tensor product property for cohomological support

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Remark This is known to be an equality when $\mathcal{C} = A$-mod for a cocommutative Hopf algebra A (Friedlander-Pevtsova '05) and for some quantum groups (Negron-Pevtsova '21, Nakano-Vashaw-Yakimov '24), and for $\mathcal{C}$ symmetric over k alg closed of char 0 (Drupieski-Kujawa '19, Bergh-Plavnik-W)

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It is known **not** to be an equality for some modules of some noncocommutative Hopf algebras (Benson-W ’14, Plavnik-W ’18, Bergh-Plavnik-W ’24), where modified tensor product property (NVY) holds instead.

More on tensor product properties for cohomological support

- $V^c(A \otimes B) \subseteq V^c(A)$ for all A per definition
- $V^c(A \otimes B) \subseteq V^c(A) \cap V^c(B)$ for all A, B in case $\mathcal{C}$ is braided or $\mathsf{H}(\mathcal{C}, \mathbf{1})$ is replaced by cat ctr
- $V^c(A \otimes B) = V^c(A) \cap V^c(B)$ for all A, B for many types of cats $\mathcal{C}$, false for some types of *nonbraided* cats [Benson-W '14, Plavnik-W '18, Bergh-Plavnik-W '23]
- $\bigcup_{X \in \mathcal{C}} V^c(A \otimes X \otimes B) = V^c(A) \cap V^c(B)$ often holds [NVY]

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• $\bigcup_{X \in \mathcal{C}} V^c(A \otimes X \otimes B) = V^c(A) \cap V^c(B)$ often holds [NVY]

Theorem [Bergh-Plavnik-W] Assume $\mathcal{C}$ is braided. TFAE:
(1) $V^c(A \otimes B) = V^c(A) \cap V^c(B)$ for all A, B
(2) $V^c(A \otimes B) = V^c(A) \cap V^c(B)$ for all A, B of complexity one

Open Questions

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(3) When are Balmer spec and cohomological spec homeomorphic?

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