Modular data with few twists

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Pointed fusion categories

Finite integral group rings $\mathbb{Z}G$ are *categorified* by Vec_G^ω

Choice of scalars $(fg)h \xrightarrow{\sim} f(gh) \longleftrightarrow$ cohomological data ω of G

"Fusion categories where every simple has $\mathrm{FPdim}\ 1$"

This is the boundary of our understanding

Near-group: group + one simple which does not act by permutation

Theorem (Schopieray, 2023)

Near-group categories for elementary abelian 2-groups C_2^n (which are not just C_2- extensions) exist only for C_2 and C_2^2

Doubles of finite groups

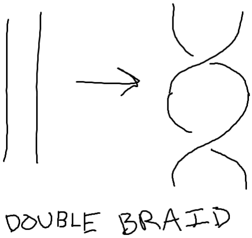
What elements of Vec_G^ω commute with all others?

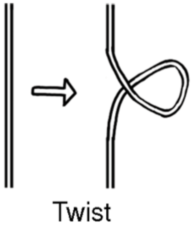
$$\mathcal{Z}(\mathrm{Vec}_G^\omega) \simeq \mathrm{Rep}(D^\omega(G))$$

This is what I'll call a *double* of the finite group G

Modular fusion categories

$\mathcal{Z}(\mathrm{Vec}_G^\omega)$ are *modular*:





Numerically: an S- and T-matrix (diagonal; finite order)

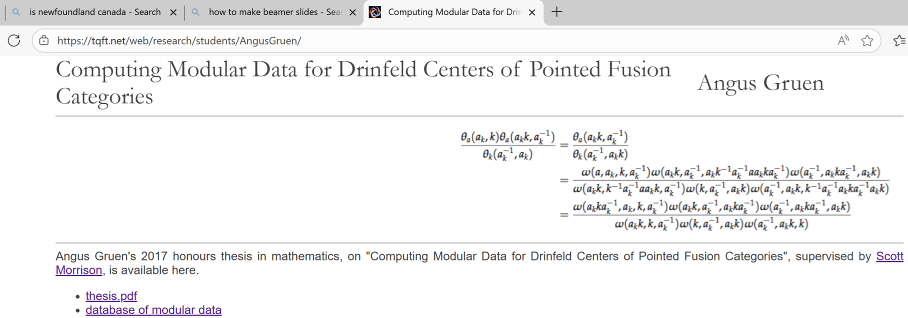
(projective representation of $\mathrm{SL}(2,\mathbb{Z})$)

Modular data of doubles $\mathcal{Z}(\mathrm{Vec}_G)$

$$S_{(a,\chi),(b,\chi')} = \frac{|G|}{|C_G(a)||C_G(b)|}\sum_g \chi(gbg^{-1})\chi'(g^{-1}ag)$$

$$T_{(a,\chi),(b,\chi')} = \delta_{(a,\chi),(b,\chi')}\frac{\chi(a)}{\chi(e)}.$$

A plaintext database



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https://tqft.net/web/research/students/AngusGruen/

Computing Modular Data for Drinfeld Centers of Pointed Fusion Categories

Angus Gruen

$$\frac{\theta_a(a_k,k)\theta_a(a_kk,a_k^{-1})}{\theta_k(a_k^{-1},a_k)} = \frac{\theta_a(a_kk,a_k^{-1})}{\theta_k(a_k^{-1},a_kk)}$$
$$= \frac{\omega(a,a_k,k,a_k^{-1})\omega(a_kk,a_k^{-1},a_kk^{-1}a_k^{-1}aa_kka_k^{-1})\omega(a_k^{-1},a_kka_k^{-1},a_kk)}{\omega(a_kk,k^{-1}a_k^{-1}aa_kk,a_k^{-1})\omega(k,a_k^{-1},a_kk)\omega(a_k^{-1},a_kk,k^{-1}a_k^{-1}a_kka_k^{-1}a_kk)}$$
$$= \frac{\omega(a_kka_k^{-1},a_k,k,a_k^{-1})\omega(a_kk,a_k^{-1},a_kka_k^{-1})\omega(a_k^{-1},a_kka_k^{-1},a_kk)}{\omega(a_kk,k,a_k^{-1})\omega(k,a_k^{-1},a_kk)\omega(a_k^{-1},a_kk,k)}$$

Angus Gruen's 2017 honours thesis in mathematics, on "Computing Modular Data for Drinfeld Centers of Pointed Fusion Categories", supervised by Scott Morrison, is available here.

- thesis.pdf
- database of modular data

Search: **Angus modular data**

Conductor/Frobenius-Schur exponent

The order of T is called the *conductor*
or *Frobenius-Schur exponent* FSexp

Importance: the S-matrix $\to$ algebraic integers in $\mathbb{Q}(\zeta_N)$, $N := \mathrm{FSexp}$.

Doubles of elementary abelian 2-groups

$\mathrm{FSexp}(\mathcal{Z}(\mathrm{Vec}_{C_2^n})) = 2$, and $\mathrm{FSexp}(\mathcal{Z}(\mathrm{Vec}_{C_2^n}^{\omega})) = 4$ if $\omega \neq 1$

For example: $\mathcal{Z}(\mathrm{Vec}_{C_2}^{\omega})$

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

Doubles of C_2^2

(just the twists)

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, i, i, -i, -i]$$

$$[1, 1, 1, 1, 1, 1, -1, -1, i, i, i, i, -i, -i, -i, -i]$$

$$[1, 1, 1, 1, i, i, i, i, i, i, -i, -i, -i, -i, -i, -i]$$

What do you notice?

Doubles of C_2^3

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, 1,$$
$$-1, 1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, 1, -1, 1, 1, -1, 1, -1, -1]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, -i, i]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, 1, 1, -1, -1, 1,$$
$$1, -1, -1, -1, -1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, i, -i, -i, i, i, -i, -i, i, i, -i, i, -i, i, -i, i, -i]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, -1, 1, -1, 1, -1, 1, -1, -i, i, -i, i, -i, i]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, i, -i,$$
$$i, -i, -i, i, -i, i, i, -i, -i, i, i, -i, -i, i, i, -i, -i, i, i, -i, -i, i, i, -i, i, -i, i, -i, i, -i]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, -1, 1, -1, 1, -1, -i, i, -i, i, -i, i, 1, -1]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -i, i, i, -i, -i, i, i, -i, 1, 1, -1, -1,$$
$$-1, -1, 1, 1, -i, i, -i, i, i, -i, i, -i, i, -i, -i, i, -i, i, i, -i, i, -i, i, -i, i, -i, i, -i]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, -1, 1, -1, -i, i, -i, i, -i, i, -i, i, -i, i]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -i, i, -i, i, i, -i, i, -i, -i, i, i, -i, -i, i, i, -i, -i, i, i, -i, i,$$
$$-i, -i, i, -i, i, -i, i, -i, i, -i, i, i, -i, -i, i, -i, i, i, -i, i, -i, i, -i, i, -i, i, -i]$$

$$[1, 1, 1, 1, 1, 1, 1, 1, -i, i, -i, i, -i, i, -i, i, -i, i, -i, i, -i, i]$$

Doubles of C_2^3

[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1]

[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, i, -i]

[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, i, i, i, i, i, i, i, i, -i, -i, -i, -i, -i, -i, -i, -i]

[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, i, i, i, -i, -i, -i]

[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, i, i, i, i, i, i, i, i, i, i, i, i, i, -i, i, -i, i, -i, i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i]

[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, i, i, i, -i, -i, -i]

[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i]

[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, i, i, i, i, i, -i, -i, -i, -i, -i]

[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, i, -i, -i, -i, -i, -i, -i, -i, -i, -i, i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i, -i]

[1, 1, 1, 1, 1, 1, 1, 1, i, i, i, i, i, i, i, -i, -i, -i, -i, -i, -i, -i]

Wrong.

Like near-group categories, there is a finiteness...

The only doubles $\mathcal{Z}(\mathrm{Vec}^{\omega}_{C_2^n})$ of elementary abelian 2-groups "missing" a twist are (one each of) C_2, C_2^2 and C_2^3.

A series of observations...

(1) If $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ has exactly one twist, then $\mathrm{FSexp}(\mathcal{Z}(\mathrm{Vec}_G^\omega)) = 1$.

(2) If $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ has exactly two twists, then $\mathrm{FSexp}(\mathcal{Z}(\mathrm{Vec}_G^\omega)) = 2$.

(3) If $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ has exactly three twists, then $\mathrm{FSexp}(\mathcal{Z}(\mathrm{Vec}_G^\omega)) = 3$... with exactly three exceptions: one each for C_2, C_2^2, and C_2^3.

$\vdots$

(N) Can you identify/classify $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ with exactly N twists and $\mathrm{FSexp}(\mathcal{Z}(\mathrm{Vec}_G^\omega)) \neq N$?

(a student could think about this)

Generalize to modular fusion categories $\mathcal{C}$

(1) If $\mathcal{C}$ has exactly one twist...then $\mathrm{FSexp}(\mathcal{C}) = 1$.
(only Vec)

(2) If $\mathcal{C}$ has exactly two twists...then $\mathrm{FSexp}(\mathcal{C}) = 2$,
(infinitely many; precisely [Wan-Wang, '21])

Are there any modular fusion categories with exactly two twists and FS-exponent different from 2?

Theorem (Schopieray, 2025)

There are exactly eight modular fusion categories $\mathcal{C}$ up to braided equivalence with exactly two distinct twists and $\mathrm{FSexp}(\mathcal{C}) \neq 2$.

(all of rank 2 + pointed rank 3)

Modular fusion categories with 3 twists

(3) If $\mathcal{C}$ has exactly three twists...then $\mathrm{FSexp}(\mathcal{C}) = 3$, (infinitely many)

Theorem (Schopieray, 2025)

There are exactly 27 modular fusion categories $\mathcal{C}$ up to braided equivalence with exactly three distinct twists and $\mathrm{FSexp}(\mathcal{C}) \neq 3$.

Organized by FS-exponent N...

- $(N = 4$; 3) Aforementioned doubles of C_2, C_2^2, and C_2^3
- $(N = 5$; 4) Pointed order 5, and (some) Fibonacci products
- $(N = 7$; 6) Transitive
- $(N = 8$; 6) Pointed cyclic of order 4
- $(N = 16$; 8) Ising

(range from rank 3 to rank 22)

The large one

$$S = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & -2 & -2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & -2 & -2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & -2 \\
2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & 4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 2 & -2 & -2 & 2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 \\
2 & -2 & 2 & -2 & -2 & 2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & -2 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 4 & 0 & 0 \\
2 & 2 & -2 & -2 & -2 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & 0 \\
2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 4 \\
2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4
\end{bmatrix}$$

$$T = \text{diagonal}(1, 1, 1, 1, 1, 1, 1, 1, i, i, i, i, i, i, i, -i, -i, -i, -i, -i, -i, -i)$$

Outlook

Conjecture: there are finitely many *prime* modular fusion categories with N distinct twists and FS-exponent different from N.

Why is this not (so) insane?

Thank you for listening.