The 12-dimensional Fomin-Kirilov algebra and its cousins

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Hopf algebras and related topics
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Motivation

More examples of Hopf algebras.

Explicit examples of finite dimensional Hopf algebras whose coradical is a Hopf algebra that is neither commutative nor cocommutative.

$$\vdots$$

$$A \leftrightarrow B$$

$$\Downarrow$$

$$(\text{Hopf algebras with coradical } A) \overset{?}{\leftrightarrow} (\text{Hopf algebras with coradical } B)$$

Coradical filtration

$U_0 = U_{(0)} = Cor(U) =$ sum of simple sub-coalgebras of $U = (J(U^*))^{\perp}$

$n > 0,\ U_{(n)} = \{h \in U : \Delta(h) \in U \otimes U_0 + U_{(n-1)} \otimes U\}$

We say that U has the Chevalley property if U_0 is a Hopf algebra.

In this case this filtration is a Hopf algebra filtration.

(If U_0 is a group algebra, then we say that H is pointed. If H is a dual of a group algebra, then we say that H is copointed.)

Lifting method

U - Hopf algebra with the Chevalley property

$H = \mathrm{Cor}(U)$ is a Hopf subalgebra

$B = \mathrm{gr}(U)$

$B = A \# H$

$A = B^{co\pi_0} = \{a \in B : (\mathrm{id} \otimes \pi_0)\Delta a = a \otimes 1\}$
is a graded connected Hopf algebra in ${}^H_H\mathcal{YD}$

If A is generated as an algebra by its degree one elements V, then
$R = \mathcal{B}(V)$ is a Nichols algebra.

Aim: describe all (finite dimensional) Hopf algebras whose coradical is a fixed Hopf algebra H

Hopf algebra $\Bbbk^N \rtimes_\beta \Bbbk Q$

Fix groups N and Q and a right action $N \times Q \to N$, $(u, x) \mapsto u^x$.

Induced left action $\Bbbk Q \otimes \Bbbk^N \to \Bbbk^N$, $x \otimes f \mapsto {}^x f$, $({}^x f)(u) = f(u^x)$

Fix $\beta \colon Q \times N \times N \to \Bbbk^\times$, a normalized Singer 2-cocycle:

$$\begin{aligned} \beta_x(u, 1_N) = \quad 1 \quad &= \beta_x(1_N, v) \\ \beta_x(v, w)\beta_x(uv, w) &= \beta_x(u, v)\beta_x(u, vw) \\ \beta_{1_G}(u, v) &= 1 \\ \beta_{xy}(u, v) &= \beta_x \beta_y(u^x, v^x) \end{aligned}$$

$(\beta_x = \beta(x, -, -) \in \Bbbk^{N \times N})$

Hopf algebra $\Bbbk^N \rtimes_\beta \Bbbk Q = \Bbbk\{p_u\widehat{x} : u \in N, x \in Q\}$.

Multplication:

$$(p_u\widehat{x})(p_v\widehat{y}) = \delta_{u,v^{x^{-1}}}\, p_u\widehat{xy}$$

Comultiplication:

$$\Delta(p_u\widehat{x}) = \sum_{v,w\in N, vw=u} \beta_x(v,w) p_v\widehat{x} \otimes p_w\widehat{x};$$

Abelian cocentral cleft extension:

$$\Bbbk^N \to \Bbbk^N \rtimes_\beta \Bbbk Q \to \Bbbk Q$$

Hopf algebra K_n

Described by G. I. Kac (ring groups).

Assume that $n > 1$ is odd. Fix ξ, a primitive n-th root of 1.

$$N = C_n \times C_n = \langle a, b : a^n, b^n \rangle, \qquad \text{and} \qquad Q = C_2 = \langle x : x^2 \rangle,$$

Action: $a^x = b, b^x = a$

Cocycle β:

$$\beta_x(a^i b^j, a^k b^\ell) = \xi^{i\ell - jk}.$$

(For $f \in \Bbbk^N$ and $g \in \Bbbk^{N \times N} \simeq \Bbbk^N \otimes \Bbbk^N$ we abbreviate $f(i,j) = f(a^i b^j)$ and $g((i,j),(k,\ell)) = g(a^i b^j \otimes a^k b^\ell)$.)

Structure of K_n

Set $p_{i,j} = p_{a^i b^j}$ and $f_{i,j} = p_{i,j}\widehat{x}$ for all $i, j \in \mathbb{Z}_n$. Then $\{p_{i,j}, f_{i,j} : i, j \in \mathbb{Z}_n\}$ is a basis for K_n. The algebra structure in terms of this basis is as follows:

$$p_{ij}p_{ij} = p_{ij}, \qquad p_{ij}f_{ij} = f_{ij}, \qquad f_{ij}p_{ji} = f_{ij}, \qquad f_{ij}f_{ji} = p_{ij},$$

where all other products of two basis elements are zero. The coalgebra structure is given by:

$$\begin{aligned}
\Delta(p_{ij}) &= \sum_{i'+i''=i, j'+j''=j} p_{i'j'} \otimes p_{i''j''}, \qquad \varepsilon(p_{ij}) = \delta_{i,0}\delta_{j,0}, \\
\Delta(f_{ij}) &= \sum_{i'+i''=i, j'+j''=j} \xi^{i'j''-j'i''} f_{i'j'} \otimes f_{i''j''}, \qquad \varepsilon(f_{ij}) = \delta_{i,0}\delta_{j,0}, \\
\Delta(\widehat{x}) &= \sum_{i,j,k,\ell\in\mathbb{Z}_n} \xi^{i\ell-jk} p_{ij}\,\widehat{x} \otimes p_{k\ell}\widehat{x}, \qquad \varepsilon(\widehat{x}) = 1.
\end{aligned}$$

The antipode is as follows:

$$S(p_{ij}) = p_{-i,-j}, \qquad S(f_{ij}) = f_{-j,-i}, \qquad S(\widehat{x}) = \widehat{x}.$$

Simple Yetter-Drinfeld modules over K_n

They are of dimensions 1, 2, and n.

1 and 2 dimensional modules: Little groups of Wigner and Mackey

n dimensional modules: they are all simple as comodules. They can be obtained by tensoring a fixed Yetter-Drinfeld module W_0 with one dimensional $\mathcal{YD}$ modules

Little groups of Wigner and Mackey

Assume $B = \Bbbk^N \rtimes_\beta \Bbbk Q$ and assume that β is a bicharacter.

Consider the action of Q on $N \times \widehat{N}$ given by

$$x * (a, \chi) = (a^{x^{-1}}, \beta_x(-, a^{x^{-1}})\beta_x^{-1}(a^{x^{-1}}, -)({}^x\chi))$$

Let $(a_1, \chi_1), \dots, (a_k, \chi_k)$ be a fixed set of representatives of distinct orbits under this action.

For each $i = 1, \dots, k$, let $Q_i = \mathrm{Stab}_Q(a_i, \chi_i)$ and let U be an irreducible representation of Q_i.

The induced $\Bbbk Q$-module $\Theta(U, a_i, \chi_i) = \Bbbk Q \otimes_{\Bbbk Q_i} U \in^{\widehat{N}}_{B} \mathcal{YD} \subseteq^{B}_{B} \mathcal{YD}$

Little groups continuted

The induced $\Bbbk Q$-module $\Bbbk Q \otimes_{\Bbbk Q_i} U$ becomes an element in ${}^{\widehat{N}}_{B}\mathcal{YD}$ as follows: for all $x, y \in Q$, $f \in \Bbbk^N$ and $u \in U$ we set

$$\begin{aligned} (f\widehat{x}) \cdot (y \otimes_{\Bbbk Q_i} u) &= f(a_i^{xy})(xy \otimes_{\Bbbk Q_i} u), \\ \delta(y \otimes_{\Bbbk Q_i} u) &= \beta_y(-, a_i^{y^{-1}})\beta_y^{-1}(a_i^{y^{-1}}, -)({}^y\chi_i) \otimes (y \otimes_{\Bbbk Q_i} u). \end{aligned}$$

In particular,

$$\begin{aligned} (f\widehat{x}) \cdot (1 \otimes_{\Bbbk Q_i} u) &= f(a_i^{x})(x \otimes_{\Bbbk Q_i} u) \\ \delta(1 \otimes_{\Bbbk Q_i} u) &= \chi_i \otimes (1 \otimes_{\Bbbk Q_i} u) \end{aligned}$$

One and two dimensional $\mathcal{YD}$ modules over K_n

For $m, \ t \in \mathbb{Z}_n$, let $\chi_{m,t} \in \widehat{N}$ be given by $\chi_{m,t}(a^i b^j) = \xi^{mi+tj}$.

Then

$$\begin{aligned} {}^{x}\chi_{m,t} &= \chi_{t,m} \\ x * (a^i b^j, \chi_{m,t}) &= (a^j b^i, \chi_{t+2i,m-2j}) \qquad \text{for all } i,j,m,t \in \mathbb{Z}_n. \end{aligned}$$

The orbits under the action of Q are as follows:

1. Orbits of size one: $\{(a^i b^i, \chi_{m,m-2i})\}$ for $i, m \in \mathbb{Z}_n$.
2. Orbits of size two:
   (a) $\{(a^i b^i, \chi_{m,t}), (a^i b^i, \chi_{t+2i,m-2i}\}$, where $i, m, t \in \mathbb{Z}_n$ and $t \neq m - 2i$.
   (b) $\{(a^i b^j, \chi_{m,t}), (a^j b^i, \chi_{t+2i,m-2j})\}$, where $i, j, m, t \in \mathbb{Z}_n$ and $i \neq j$.

In the case (b), it is impossible to have $(m, t) = (t + 2i, m - 2j)$.

$\mathbf{V}_{\mathbf{i,m}}^{\epsilon}$

For $\epsilon = \pm 1$ and $i, m \in \mathbb{Z}_n$, the objects $V_{i,m}^{\epsilon} \in {}^{\widehat{N}}_{K_n}\mathcal{YD}$ are one-dimensional vector spaces generated by $v \neq 0$ where

- the coaction is given by $\delta(v) = \chi_{m,m-2i} \otimes v$;
- the action of is given by $(f\widehat{x}^k) \cdot w = f(a^i b^i)\epsilon^k w$ for $f \in \mathbb{k}^N$ and $k = 0, 1$;
- the braiding is given by $c(v \otimes v) = \xi^{2i(m-i)} v \otimes v$.

$\mathbf{U_{i,j,m,t}}$

For $i, j, m, t \in \mathbb{Z}_n$, the objects $U_{i,j,m,t}$ are two-dimensional vector spaces spanned by non-zero vectors u_1, u_2 where

- $\delta(u_1) = \chi_{m,t} \otimes u_1$, $\delta(u_2) = \chi_{t+2i,m-2j} \otimes u_2$.
- $f \cdot u_1 = f(a^i b^j) \cdot u_1$, $f \cdot u_2 = f(a^j b^i) \cdot u_2$ for $f \in \Bbbk^N$ and $\widehat{x} \cdot u_1 = u_2$, $\widehat{x} \cdot u_2 = u_1$.
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$$\begin{aligned} c(u_1 \otimes u_1) &= \xi^{mi+tj} u_1 \otimes u_1, & c(u_1 \otimes u_2) &= \xi^{it+mj} u_2 \otimes u_1, \\ c(u_2 \otimes u_1) &= \xi^{it+mj+2(i^2-j^2)} u_1 \otimes u_2, & c(u_2 \otimes u_2) &= \xi^{mi+tj} u_2 \otimes u_2. \end{aligned}$$

$U_{i,j,m,t} \simeq U_{i',j',m',t'} \iff (i', j', m', t') \in \{(i, j, m, t), (j, i, t+2i, m-2j)\}$.
$U_{i,j,m,t}$ is reducible $\iff i = j$ and $t = -2i + m$.
In this case $U_{i,i,m,-2i+m} \simeq V^+_{i,m} \oplus V^-_{i,m}$.

Yetter-Drinfled modules that are simple as comodules

$\Bbbk^N\widehat{x} \simeq \mathcal{M}_n(\Bbbk)^*$ as coalgebras:

For $i, j \in \mathbb{Z}_n$, we define in K_n

$$e_{ij} = \sum_{k \in \mathbb{Z}_n} \xi^{-2(i+j)k} f_{k+i-j,k-i+j}.$$

These elements are linearly independent and

$$\Delta(e_{ij}) = \sum_r e_{ir} \otimes e_{rj} \qquad \text{and} \qquad \varepsilon(e_{ij}) = \delta_{i,j}.$$

W_0

The comodule $W_0 = span\{e_{r0} : r \in \mathbb{Z}_n\}$ is invariant under the adjoint action of K_n, i.e., it is a Yetter-Drinfeld submodule of the regular Yetter-Drinfeld module K_n.

Its structure is given by:

- $\delta(e_{r0}) = \sum_k e_{rk} \otimes e_{k0}$;
- $\widehat{x} \rightharpoonup e_{r0} = e_{-r0}$ and $f \rightharpoonup e_{r0} = f(2r, -2r)e_{r0}$ for all $f \in \Bbbk^N$.

$W_{i,m}^{\pm}$

Now, for $i, m \in \mathbb{Z}_n$ and $\epsilon \in \{\pm 1\}$ we define the Yetter-Drinfeld modules

$$W_{i,m}^{\epsilon} := V_{i,m}^{\epsilon} \otimes W_0.$$

Abbreviate $w_k = v \otimes e_{k,0}$. The ${}^H_H\mathcal{YD}$ structure is as follows:

$$p_{\ell,s} \cdot_{i^\epsilon} w_r = \begin{cases} \epsilon\, \xi^{2i(r-\ell)} w_r, & s = \ell + 2r \\ 0, & \text{otherwise} \end{cases}$$

$$f_{pq} \cdot_{i^\epsilon} w_r = \begin{cases} \epsilon\, \xi^{4ir} w_{-r}, & p = i - 2r, q = i + 2r \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{i,m}(w_r) = \sum_k \chi_{m,m-2i}\, e_{rk} \otimes w_k$$

These ${}^H_H\mathcal{YD}$ modules are pairwise non-isomorphic and every ${}^H_H\mathcal{YD}$ module whose coaction is inside $\Bbbk^N\widehat{x}$ is one of these modules.

Fusion Rules

The fusion ring $\mathcal{F}$ of ${}^{K_n}_{K_n}\mathcal{YD}$ is the commutative ring generated by the elements $v^{\varepsilon}_{i,m}$, $u_{i,j,m,t}$, $w^{\varepsilon}_{i,m}$ with $\varepsilon = \pm 1$, $i, j, m, t \in \mathbb{Z}_n$ and $t \neq m - 2i$ when $i = j$, satisfying the following relations: (set $w^{+}_{0,0} = w_0$)

$$\begin{aligned}
v^{\varepsilon_1}_{i_1,m_1} v^{\varepsilon_2}_{i_2,m_2} &= v^{\varepsilon_1\varepsilon_2}_{i_1+i_2,m_1+m_2}, \\
v^{\varepsilon}_{i,m} w_0 &= w^{\varepsilon}_{i,m}, \\
v^{\varepsilon}_{i_1,m_1} u_{i_2,j_2,m_2,t_2} &= u_{i_1+i_2,i_1+j_2,m_1+m_2,-2i_1+m_1+t_2}, \\
u_{i_1,j_1,m_1,t_1} u_{i_2,j_2,m_2,t_2} &= u_{i_1+i_2,j_1+j_2,m_1+m_2,t_1+t_1} + u_{i_1+j_2,j_1+i_2,m_1+2i_2+t_2,t_1-2j_2+m_2} \\
u_{i,j,m,t} w_0 &= w^{+}_{\frac{i+j}{2},\frac{2i+m+t}{2}} + w^{-}_{\frac{i+j}{2},\frac{2i+m+t}{2}}, \\
w_0 \otimes w_0 &= v^{+}_{0,0} + \sum_{\substack{[r,k]\in\mathcal{Z}_n \\ (r,k)\neq(0,0)}} u_{-4k,4k,4k-\frac{1}{2}r,4k+\frac{1}{2}r},
\end{aligned}$$

where $\mathcal{Z}_n$ is the set of isomorphism classes in $\mathbb{Z}_n \times \mathbb{Z}_n$ given by the relation $(r, k) \sim \pm(r, k)$. □

Some finite-dimensional Nichols algebras

The braided vector spaces $V^{\pm}_{i,m}$, $U_{i,j,m,t}$ and their direct sums are of diagonal type. The finite dimensional ones can be classified and described in terms of generators and relations using the results of Heckenberger and Angiono.

The Nichols algebras of $W^{\pm}_{i,m}$ can be studied using the theory of set-theoretical solutions to the braid equation and racks. As braided vectors spaces these modules are t-equivalent to dihedral racks.

(Braided vector spaces U, V are t-equivalent if for every $n \geq 2$, the representations $U^{\otimes n}$, $V^{\otimes n}$ of $\mathbb{B}_n$ are isomorphic. Nichols algebras of t-equivalent braided vector spaces are isomorphic as graded braided vector spaces.)

Fomin-Kirilov algebras

$\mathcal{E}_n$ is the quadratic algebra with $\binom{n}{2}$ generators x_{ij}, $1 \leq i < j \leq n$ (convention $x_{ji} = -x_{ij}$) subject to (i, j, k, ℓ pairwise distinct):
- $x_{ij}^2 = 0$
- $x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0$
- $x_{ij}x_{k\ell} = x_{k\ell}x_{ij}$

Connections with Schubert calculus, cluster algebras, Nichols algebras,...

For $n = 3, 4, 5$ they are finite dimensional and Nichols algebras of S_n module $V_n = span\{x_{ij} : 1 \leq i < j \leq n\}$,
- $\delta x_{ij} = (ij) \otimes x_{ij}$
- $\sigma \cdot x_{ij} = x_{\sigma(i)\sigma(j)}$

Famous Conjectures: For $n \geq 6$, $\dim \mathcal{E}_n = \infty$ and $\mathcal{E}_n = \mathcal{B}(V_n)$.

New Hopf algebras of dimension 216

Assume for the rest of the talk that $n = 3$.

Using the theory of racks and some recent results of HMV we get that $\mathfrak{B}(W^{\varepsilon}_{i,m})$ is finite dimensional if and only if

$$\varepsilon = -1$$

and either

- $i = 0$: V_3 and $W^{-}_{0,m}$ are iso. as braided vector spaces, or
- $m = i$: V_3 and $W^{-}_{m,m}$ are t-equivalent

$\mathcal{E}_3$ and cousins

As braided graded algebras we have

$$\mathfrak{B}(W^-_{i,m}) = \mathbb{k}\langle w_0, w_1, w_2\rangle / \mathcal{I}^-_{i,m},$$

where:

$$\mathcal{I}^-_{0,m} = \langle w_0^2, w_1 w_2, w_2 w_1, \quad w_1^2 + w_0 w_2 + w_2 w_0, \quad w_2^2 + w_0 w_1 + w_1 w_0 \rangle$$
$$\mathcal{I}^-_{1,1} = \langle w_0^2, w_1 w_2, w_2 w_1, \quad w_1^2 + \xi w_0 w_2 + \xi^2 w_2 w_0, \quad w_2^2 + \xi w_0 w_1 + \xi^2 w_1 w_0 \rangle$$
$$\mathcal{I}^-_{2,2} = \langle w_0^2, \ w_1 w_2, w_2 w_1, \quad w_1^2 + \xi^2 w_0 w_2 + \xi w_2 w_0, \quad w_2^2 + \xi^2 w_0 w_1 + \xi w_1 w_0 \rangle$$

$\mathfrak{B}(W^-_{0,m})$ are isomorphic to $\mathcal{E}_3$ as graded braided Hopf algebras.

$\mathfrak{B}(W^-_{1,1})$, $\mathfrak{B}(W^-_{2,2})$ are only isomorphic to $\mathcal{E}_3$ as braided graded vector spaces, but **not** as algebras.

$\mathfrak{B}(W^{-}_{0,m})$, $m = 0, 1, 2$, are isomorphic as graded braided Hopf algebras, but are not isomorphic as objects in ${}^{K_3}_{K_3}\mathcal{YD}$.

Hopf algebras $\mathfrak{B}(W^{-}_{0,0})\#K_3$, $\mathfrak{B}(W_{0,1})\#K_3$ are not isomorphic

(they have different truncated GS cohomology)

Hopf algebras $\mathfrak{B}(W^{-}_{0,1})\#K_3$, $\mathfrak{B}(W^{-}_{0,2})\#K_3$ are isomorphic
(isomorphism is given by $w_1 \leftrightarrow w_2$
and the automorphism $\chi_{i,j} \mapsto \chi_{-i,-j}$, $\widehat{x} \mapsto \widehat{x}$ on K_3)

$\mathfrak{B}(W_{1,1}^{-})$, $\mathfrak{B}(W_{2,2}^{-})$ are isomorphic as graded braided Hopf algebras (via $w_1 \leftrightarrow w_2$), but are not isomorphic as objects in ${}^{K_3}_{K_3}\mathcal{YD}$

Hopf algebras $\mathfrak{B}(W_{1,1}^{-})\#K_3$, $\mathfrak{B}(W_{2,2}^{-})\#K_3$ are isomorphic
(isomorphism is given by $w_1 \leftrightarrow w_2$
and the automorphism $\chi_{i,j} \mapsto \chi_{-i,-j}$, $\widehat{x} \mapsto \widehat{x}$ on K_3)

Liftings

Liftings correspond to formal graded bialgebra deformations

Deformation theory is governed by the **truncated** Gerstenhaber-Schack biaglebra cohomology

Untruncated version of the GS bialgebra cohomology is the Hochchild cohomology of the Drienfeld double with trivial coefficients (there is also an infinite dimensional analogue of this statement). Result is implicit in GS, proven in greater generality by Taileffer (different formulation), explicit isomorphism given in MW.

Computing truncated GS cohomology

For $W = W^-_{i,m}$ with either $i = 0$ or $i = m$ we have

$$\mathfrak{B}(W) = T(W)/I(W)$$

where $I(W)$ is generated by a $\mathcal{YD}$-submodule $R(W)$ of $W \otimes W$ satisfying $R(W) \cap (T^+(W)I(W) + I(W)T^+(W)) = 0$ (hence $R(W) \simeq I(W)/(T^+(W)I(W) + I(W)T^+(W))$).

Since no simple summand of $W \otimes W$ is isomorphic to W we have that

$$\mathcal{YD}(R(W), W) = 0$$

and hence there is, for every $\ell < 0$, a surjective homomorphism

$$\partial \colon \operatorname{Hom}_{\Bbbk}(R(W), \Bbbk)^{K_3}_{(\ell)} \simeq \mathrm{H}^2_{\varepsilon}(\mathfrak{B}(W)\#K_3, \Bbbk)_{(\ell)} \to \widehat{\mathrm{H}}^2_b(\mathfrak{B}(W)\#K_3)_{(\ell)}.$$

(Clearly $\operatorname{Hom}_{\Bbbk}(R(W), \Bbbk)$ is concentrated in degree -2.)

Liftings

- $\mathfrak{B}(W^{-}_{1,1})\#K_3 \simeq \mathfrak{B}(W^{-}_{2,2})\#K_3$, has no nontrivial liftings

- $\mathfrak{B}(W^{-}_{0,0})\#K_3$ has a one-parameter family of liftings (described explicitly)

- $\mathfrak{B}(W^{-}_{0,1})\#K_3 \simeq \mathfrak{B}(W^{-}_{0,2})\#K_3$ has a two-parameter family of liftings (described explicitly)

Liftings of $\mathfrak{B}(W_{0,1}^{-})\#K_3$

$\mathcal{L}_{0,1}^{-}(\lambda,\mu)$ is generated as an algebra by K_3, w_0, w_1, w_2 subject to:

relations in K_3 and action of K_3 on w_0, w_1, w_2

$$w_0^2 + w_1 w_2 + w_2 w_1 = \lambda(1 - \chi_{2,2})$$
$$w_0^2 + \xi w_1 w_2 + \xi^2 w_2 w_1 = \mu(1 - \chi_{0,1})$$
$$w_0^2 + \xi^2 w_1 w_2 + \xi w_2 w_1 = \mu(1 - \chi_{1,0})$$
$$w_1^2 + w_0 w_2 + w_2 w_0 = 0$$
$$w_2^2 + w_0 w_1 + w_1 w_0 = 0$$

Comultiplication is induced by that of K_3 and

$$\begin{aligned}
\Delta(w_0) &= w_0 \otimes 1 + \chi_{1,1} e_{0,0} \otimes w_0 + \chi_{1,1} e_{0,1} \otimes w_1 + \chi_{1,1} e_{0,2} \otimes w_2, \\
\Delta(w_1) &= w_1 \otimes 1 + \chi_{1,1} e_{1,0} \otimes w_0 + \chi_{1,1} e_{1,1} \otimes w_1 + \chi_{1,1} e_{1,2} \otimes w_2, \\
\Delta(w_2) &= w_2 \otimes 1 + \chi_{1,1} e_{2,0} \otimes w_0 + \chi_{1,1} e_{2,1} \otimes w_1 + \chi_{1,1} e_{2,2} \otimes w_2.
\end{aligned}$$