Graded braided commutativity in Hochschild cohomology arising from duoids

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Hochschild cohomology

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the aforementioned cohomology is

$$\mathrm{H}^\bullet(A,M) = H\left(\mathrm{Hom}_{AA}(P_\bullet, M), d^*_\bullet\right).$$

Cup product

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Definition

Let $\varphi \in \mathrm{Hom}_{AA}(P_p, R)$, $\psi \in \mathrm{Hom}_{AA}(P_q, R)$. Their cup product $\varphi \cup \psi$ is the (convolution) composition

$$\begin{tikzcd} P_{p+q} \arrow[r, "\omega_{p,q}"] & P_p \otimes_A P_q \arrow[r, "\varphi\otimes_A\psi"] & R \otimes_A R \arrow[r, "\nu"] & R. \end{tikzcd}$$

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Well-defined up to homotopy $\Longrightarrow$ well-defined in cohomology:

$$\cup\colon\ \mathrm{H}^p(A,R) \otimes \mathrm{H}^q(A,R) \to \mathrm{H}^{p+q}(A,R).$$

Bar resolution

The bar resolution of A is

$$\cdots \xrightarrow{b'_3} B_3(A) \xrightarrow{b'_2} B_2(A) \xrightarrow{b'_1} B_1(A) \xrightarrow{b'_0} B_0(A) \xrightarrow{\mu} A \longrightarrow 0.$$

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Explicit calculations are extremely difficult.

Graded commutativity

The algebra $\mathsf{H}^\bullet(A, R)$ is graded commutative if

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Examples of interest: Nichols algebras, they are braided Hopf algebras in categories of Yetter-Drinfeld modules.

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When can we see Hochschild cohomology as a monoid in some $\mathcal{C}^{\mathbb{Z}}$?

A theorem for finite dimensional algebras

Theorem (Mastnak - Pevtsova - Schauenburg - Witherspoon)

Let H be a Hopf algebra with bijective antipode and let A be a bialgebra in ${}^H_H\mathcal{YD}$. If either A or H is finite dimensional, then $H^\bullet(A, \Bbbk)$ is a $\mathbb{Z}$-graded object in ${}^H_H\mathcal{YD}$, and its cup product is graded braided commutative.

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- Bar resolution + adjunction,

$$\mathrm{H}^\bullet(A, \Bbbk) = H(\mathrm{Hom}_\Bbbk(S_\bullet(A), \Bbbk), \partial_\bullet^*),$$

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- Inner hom objects $\mathrm{hom}(A^{\otimes n}, \Bbbk)$ that coincide as vector spaces with $\mathrm{Hom}_\Bbbk(A^{\otimes n}, \Bbbk)$.

Inner homs

Example

Let G be an abelian group and let $X, Y \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD} \simeq ({}_{\Bbbk G}\mathsf{Mod})^G$. Define $\mathsf{hom}(X, Y) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ as follows:

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In these cases, the evaluating and multiplying natural transformation

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Lax and braided lax monoidal functors

Definition

Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \times, J)$ be monoidal categories. A lax monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, φ, φ_0), where

- $F : \mathcal{C} \to \mathcal{D}$ is a functor,
- $\varphi_{X,Y} : F(X) \times F(Y) \to F(X \otimes Y)$ define a natural transformation,
- $\varphi_0 : J \to F(I)$ is a morphism,

which make associativity and unitality diagrams commute.

Associativity:

$$\begin{tikzcd}
F(X) \times F(Y) \times F(Z) \arrow[r, "\varphi_{X,Y} \times \mathbf{id}_Z"] \arrow[d, "\mathbf{id}_X \times \varphi_{Y,Z}"'] & F(X \otimes Y) \times F(Z) \arrow[d, "\varphi_{X \otimes Y, Z}"] \\
F(X) \times F(Y \otimes Z) \arrow[r, "\varphi_{X, Y \otimes Z}"'] & F(X \otimes Y \otimes Z).
\end{tikzcd}$$

Lax and braided lax monoidal functors

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Let $(\mathcal{C}, \otimes, I, c^{\mathcal{C}})$ and $(\mathcal{D}, \times, J, c^{\mathcal{D}})$ be braided monoidal categories. A lax monoidal functor (F, φ, φ_0) from $\mathcal{C}$ to $\mathcal{D}$ is braided if the following diagram commutes:

$$\begin{tikzcd}
F(X) \times F(Y) \arrow[r, "c^{\mathcal{D}}_{F(X),F(Y)}"] \arrow[d, "\varphi_{X,Y}"'] & F(Y) \times F(X) \arrow[d, "\varphi_{Y,X}"] \\
F(X \otimes Y) \arrow[r, "F\left(c^{\mathcal{C}}_{X,Y}\right)"'] & F(Y \otimes X).
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Lax and braided lax monoidal functors

Proposition (Aguiar - Mahajan)

- *A lax monoidal functor sends monoids to monoids.*
- *A braided lax monoidal functor sends commutative monoids to commutative monoids.*

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The complex $(S_\bullet(A), \mathit{deconcatenation})$ is a cocommutative comonoid in $\overline{\mathsf{Ch}}({}^H_H\mathcal{YD})$.

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An approach using duoidal categories

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- Interesting examples of Nichols algebras do not satisfy the previous finiteness conditions.

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- The product $\otimes$ is also present in bimodules over bialgebras. Still, not braided in this case...
- Despite not being braided, they are compatible with each other.

Duoidal categories

Definition

A duoidal category is a tuple $(\mathcal{C}, \diamond, I, \star, J, \zeta, \Delta_I, \mu_J, \zeta_0)$, where

- $(\mathcal{C}, \diamond, I)$ and $(\mathcal{C}, \star, J)$ are monoidal categories
- (I, Δ_I, ζ_0) is a comonoid in $(\mathcal{C}, \star, J)$.
- (J, μ_J, ζ_0) is a monoid in $(\mathcal{C}, \diamond)$
- $\zeta_{X,Y,Z,T} : (X \star Y) \diamond (Z \star T) \to (X \diamond Z) \star (Y \diamond T)$ define a natural transformation, called interchange law,

with a lot of compatibility diagrams.

Duoidal categories

Interchange law with associativity of $\diamond$:

$$\begin{tikzcd}
(X \star T) \diamond (Y \star U) \diamond (Z \star V) \arrow[r, "\mathbf{id}_{X \star T} \diamond \zeta_{Y,U,Z,V}"] \arrow[d, "\zeta_{X,T,Y,U} \diamond \mathbf{id}_{Z \star V}"'] & (X \star T) \diamond ((Y \diamond Z) \star (U \diamond V)) \arrow[d, "\zeta_{X,T,Y \diamond Z,U \diamond V}"] \\
((X \diamond Y) \star (T \diamond U)) \diamond (Z \star V) \arrow[r, "\zeta_{X \diamond Y,T \diamond U,Z,V}"'] & (X \diamond Y \diamond Z) \star (T \diamond U \diamond V).
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Interchange law with associativity of $\diamond$:

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(X \star T) \diamond (Y \star U) \diamond (Z \star V) \arrow[r, "\mathrm{id}_{X \star T} \diamond \zeta_{Y,U,Z,V}"] \arrow[d, "\zeta_{X,T,Y,U} \diamond \mathrm{id}_{Z \star V}"'] & (X \star T) \diamond ((Y \diamond Z) \star (U \diamond V)) \arrow[d, "\zeta_{X,T,Y \diamond Z,U \diamond V}"] \\
((X \diamond Y) \star (T \diamond U)) \diamond (Z \star V) \arrow[r, "\zeta_{X \diamond Y,T \diamond U,Z,V}"'] & (X \diamond Y \diamond Z) \star (T \diamond U \diamond V).
\end{tikzcd}$$

Example

From a braided monoidal category $(\mathcal{C}, \otimes, I, c)$ we define the duoidal category $(\mathcal{C}, \otimes, I, \otimes, I, \zeta^c, \mathrm{id}_I, \mathrm{id}_I, \mathrm{id}_I)$, where

$$\zeta^c_{X,Y,Z,T} = \mathrm{id}_X \otimes c_{Y,Z} \otimes \mathrm{id}_T\, .$$

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- Let ${}_A\mathcal{C}_A$ denote the category of A-bimodules. Then $({}_A\mathcal{C}_A, \otimes_A, A)$ is a monoidal category.
- If $\mathcal{C}$ is braided and A is a bimonoid, we can define left and right actions on $M \otimes N$. We denote this A-bimodule by $M \odot N$.
- We get a second monoidal structure, $({}_A\mathcal{C}_A, \odot, I)$.

Bimodules over a bimonoid

Proposition (Garner - López Franco)

Let $(A, \mu, \eta, \Delta, \varepsilon)$ *be a bimonoid in a braided monoidal category* $(\mathcal{C}, \otimes, I, c)$*, The tuple* $\left({}_A\mathcal{C}_A, \otimes_A, A, \odot, I, \zeta^A, \Delta, \mathrm{id}_I, \varepsilon\right)$ *is a duoidal category, where*

$$\zeta^A_{M,N,K,L} : (M \odot N) \otimes_A (K \odot L) \longrightarrow (M \otimes_A K) \odot (N \otimes_A L)$$

is the canonical projection of the map $\zeta^c_{M,N,K,L} = \mathrm{id}_M \otimes c_{N,K} \otimes \mathrm{id}_L$.

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Idea: It is a projection from a bigger quotient to a smaller one, and everything commutes before taking quotients.

(Co)chain complexes of bimodules

The monoidal structures are as usual:
Let $(C_\bullet, d^C_\bullet), (D_\bullet, d^D_\bullet) \in \mathsf{Coch}(\mathcal{C})$.

$$(C \diamond D)_n = \bigoplus_{p+q=n} C_p \diamond D_q$$

$$d^{(C\diamond D)}_{p+q}|_{C_p \diamond D_q} = d^C_p \diamond \mathrm{id}_{D_q} + (-1)^p \, \mathrm{id}_{C_p} \diamond d^D_q,$$

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The graded interchange law has a sign in similar way to the graded braiding from before:

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We can also take modulo homotopy.

Duoids and coduoids

Definition

Let $(\mathcal{C}, \diamond, I, \star, J, \zeta, \Delta_I, \mu_J, \zeta_0)$ be a duoidal category. A duoid in $\mathcal{C}$ is a tuple $(A, \mu, \eta, \nu, \iota)$, where

- (A, μ, η) is a monoid in $(\mathcal{C}, \diamond, I)$ and (A, ν, ι) is a monoid in $(\mathcal{C}, \star, I)$,
- the following diagrams commute:

$$\begin{tikzcd}
(A \star A) \diamond (A \star A) \arrow[rr, "\zeta_{A,A,A,A}"] \arrow[d, "\nu \diamond \nu"'] & & (A \diamond A) \star (A \diamond A) \arrow[d, "\mu \star \mu"] \\
A \diamond A \arrow[r, "\mu"'] & A & A \star A, \arrow[l, "\nu"]
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$$\begin{tikzcd}
I \arrow[r, "\Delta_I"] \arrow[d, "\eta"'] & I \star I \arrow[d, "\eta \star \eta"] \\
A & A \star A, \arrow[l, "\nu"]
\end{tikzcd}
\qquad
\begin{tikzcd}
J \diamond J \arrow[r, "\mu_J"] \arrow[d, "\iota \diamond \iota"'] & J \arrow[d, "\iota"] \\
A \diamond A \arrow[r, "\mu"'] & A,
\end{tikzcd}
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\begin{tikzcd}
I \arrow[rr, "\zeta_0"] \arrow[dr, "\eta"'] & & J \arrow[dl, "\iota"] \\
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Proposition (Eckmann - Hilton argument)

- *Duoids in a braided monoidal category are commutative monoids*

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Proposition (Eckmann - Hilton argument)

- *Duoids in a braided monoidal category are commutative monoids*
- *Coduoids in a braided monoidal category are cocommutative comonoids.*

Projective resolutions as coduoids

Theorem (C. - Solotar)

Let A be a bimonoid in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. Let $P_\bullet \xrightarrow{f} A$ be a chain complex in ${}_A\left({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right)_A$, which in ${}_A\operatorname{Mod}_A$ is a projective resolution of A.

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Take

- $\omega\colon P_\bullet \to (P \otimes_A P)_\bullet$ *the map that lifts* $A \simeq A \otimes_A A$,
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Then $(P_\bullet, \omega, f, \delta, \varepsilon \circ f)$ is a coduoid in the duoidal category $\left(\mathrm{Ch}\left({}_A\left({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right)_A\right), \otimes_A, A, \odot, \Bbbk, \left(\zeta^A\right)^{gr}\right)$, up to $\Bbbk$-linear homotopy.

Duoidal functors

Definition

Let $(\mathcal{C}, \diamond, I, \star, J, \zeta)$ and $(\mathcal{C}', \diamond', I', \star', J', \zeta')$ be duoidal categories. A duoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a tuple $(F, \varphi, \varphi_0, \gamma, \gamma_0)$, where

- (F, φ, φ_0) is a lax monoidal functor from $(\mathcal{C}, \diamond, I)$ to $(\mathcal{C}', \diamond', I')$.
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One of the compatibility diagrams:

$$\begin{tikzcd}
(F(X) \star' F(Y)) \diamond' (F(Z) \star' F(T)) \arrow[r, "\zeta'_{F(X),F(Y),F(Z),F(T)}"] \arrow[d, "\gamma_{X,Y} \diamond' \gamma_{Z,T}"'] & (F(X) \diamond' F(Z)) \star' (F(Y) \diamond' F(T)) \arrow[d, "\varphi_{X,Z} \star' \varphi_{Y,T}"] \\
F(X \star Y) \diamond' F(Z \star T) \arrow[d, "\varphi_{X \star Y, Z \star T}"'] & F(X \diamond Z) \star' F(Y \diamond T) \arrow[d, "\gamma_{X \diamond Z, Y \diamond T}"] \\
F((X \star Y) \diamond (Z \star T)) \arrow[r, "F(\zeta_{X,Y,Z,T})"'] & F((X \diamond Z) \star (Y \diamond T)),
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Let A be a bialgebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, with G abelian. Let M, N be A-bimodules. Define $\hom_{AA}(M, N) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ as follows:

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Proposition

If $M \simeq_{\mathsf{Vect}^G} A \otimes V \otimes A$ *and* $\dim V < \infty$, *then* $\mathrm{hom}_{AA}(M, N) = \mathrm{Hom}_{AA}(M, N)$.

The final theorem

Theorem (C. - Solotar)

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Let A *be a bimonoid in* ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$*, for* G *abelian. Let* $P_\bullet \to A$ *be a chain complex in* ${}_A\left({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right)_A$ *such that*

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2. *every* A*-bimodule* P_n *is isomorphic as* G*-graded space to a product* $A \otimes V_n \otimes A$*, with* V_n *finite dimensional.*

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Then, the cochain complex $\mathrm{Hom}_{AA}(P_\bullet, \Bbbk)$*, with the cup product, is graded braided commutative up to a* $\Bbbk$*-linear homotopy. In particular, its cohomology* $\mathrm{H}^\bullet(A, \Bbbk)$ *is graded braided commutative.*

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Sketch of proof:

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- The image of the coduoid $P_\bullet$ is a duoid.
- The codomain of the duoidal functor is braided monoidal, so by Eckmann - Hilton argument, $\mathsf{hom}_A A(P\_bullet, \Bbbk)$ is a commutative monoid.

Thanks - Merci - Gracias

Thank you for listening!