

On the finite generation of cohomology of Hopf algebras

Nicolás Andruskiewitsch
CIEM-CONICET, Córdoba, Argentina

International Workshop

Hopf Algebras and Related Topics
Gros Morne, August 11 to 15, 2025.

Joint work with Iván Angiono, Julia Pevtsova, Sarah Witherspoon,
with Sonia Natale,
with David Jaklitsch, Van C. Nguyen, Amrei Oswald, Julia Plavnik,
Anne V. Shepler, Xingting Wang

I. Antecedents. Let \mathbb{k} be a field; we assume that $\mathbb{k} = \overline{\mathbb{k}}$. Let G be a finite group and $H = \mathbb{k}G$ the group algebra.

Theorem. (Maschke, 1898)

If $\text{char } \mathbb{k} \nmid |G|$, then $H = \mathbb{k}G$ is semisimple.

When $\text{char } \mathbb{k} \mid |G|$, H is not semisimple and one would need to compute the groups

$$\text{Ext}_H^n(N, M)$$

for any finitely generated H -modules N and M , any $n \in \mathbb{N}$. Note:

$$\text{Ext}_H^n(\mathbb{k}, M \otimes N^*) \simeq \text{Ext}_H^n(N, M).$$

Theorem (Golod, 1959; Evens, 1961; Venkov, 1959).

The following properties hold:

(fgc-i) The cohomology ring

$$H(H, \mathbb{k}) = \text{Ext}_H^\bullet(\mathbb{k}, \mathbb{k}) = \bigoplus_{n \in \mathbb{N}_0} \text{Ext}_H^n(\mathbb{k}, \mathbb{k})$$

is finitely generated, and

(fgc-ii) For any finitely generated H -module M ,

$$H(H, M) = \text{Ext}_H^\bullet(\mathbb{k}, M) = \bigoplus_{n \in \mathbb{N}_0} \text{Ext}_H^n(\mathbb{k}, M)$$

is a finitely generated $H(H, \mathbb{k})$ -module.

If $\text{char } \mathbb{k} = 0$, any cocommutative finite-dimensional Hopf algebra is a group algebra.

If $\text{char } \mathbb{k} > 0$, there are more cocommutative finite-dimensional Hopf algebras (difficult to classify).

Example. The restricted enveloping algebra $u(\mathfrak{g})$ of a fin.-dim. restricted Lie algebra \mathfrak{g} is a fin.-dim. cocommutative Hopf alg.

Theorem (Friedlander & Parshall, 1983; cf. Wilkerson, 1981).

Let $H = u(\mathfrak{g})$, \mathfrak{g} a finite-dimensional restricted Lie algebra.

(fgc-i) The cohomology ring $H(H, \mathbb{k})$ is finitely generated.

(fgc-ii) For any finitely generated H -module M , $H(H, M)$ is a finitely generated module over $H(H, \mathbb{k})$.

Assume that $\text{char } \mathbb{k} = p > 0$.

Let \mathfrak{g} be a finite-dimensional complex simple Lie algebra. The nilpotent cone of \mathfrak{g} is $\mathcal{N} = \{x \in \mathfrak{g} : (\text{ad } x)^n = 0, n \gg 0\}$.

Let \mathfrak{G} be the associated restricted Lie algebra over \mathbb{k} , $H = \mathfrak{u}(\mathfrak{G})$.

Theorem (Friedlander & Parshall, 1986).

The cohomology ring $H(H, \mathbb{k})$ is isomorphic to the graded ring of polynomial functions over \mathcal{N} .

Remark. These three categories are equivalent:

{cocommutative finite-dimensional Hopf algebras},
{commutative finite-dimensional Hopf algebras}^{op},
{finite group schemes}.

Theorem (Friedlander & Suslin, 1997).

Let H be a cocommutative finite-dimensional Hopf algebra (that is, a finite group scheme).

(fgc-i) The cohomology ring $H(H, \mathbb{k})$ is finitely generated.

(fgc-ii) For any finitely generated H -module M , $H(H, M)$ is a finitely generated module over $H(H, \mathbb{k})$.

In the same paper, Friedlander & Suslin observe that the cohomology ring of a finite-dimensional *commutative* Hopf algebra is easily seen to be finitely generated using the structure and add:

We do not know whether it is reasonable to expect finite generation of the cohomology of an arbitrary finite-dimensional Hopf algebra.

Definition. We say that a finite-dimensional augmented algebra H has finite generation of the cohomology **(fgc)** if

(fgc-i) The cohomology ring $H(H, \mathbb{k})$ is finitely generated.

(fgc-ii) For any finitely generated H -module M , $H(H, M)$ is a finitely generated module over $H(H, \mathbb{k})$.

Theorem (Ginzburg & Kumar 1993).

($\text{char } \mathbb{k} = 0$). Let H be the Frobenius-Lusztig kernel (aka small quantum group) $u_q(\mathfrak{g})$, \mathfrak{g} a simple Lie algebra, $q \in \mathbb{G}_\infty$ with restrictions on the order.

Then H has **(fgc)**.

Actually they prove that $H(H, \mathbb{k})$ is isomorphic to the algebra of rational functions on the nilpotent cone of \mathfrak{g} .

Note: Similar, more restricted result by Verbitsky & Kazhdan.

Conjecture. (Etingof & Ostrik, 2005) A finite tensor category \mathcal{C} (e.g. $\mathcal{C} = \text{rep } H$, H finite-dimensional Hopf algebra) has fgc:

(fgc-i) The cohomology ring $\text{Ext}_{\mathcal{C}}^{\bullet}(1, 1)$ is finitely generated.

(fgc-ii) If $M \in \mathcal{C}$, $\text{Ext}_{\mathcal{C}}^{\bullet}(1, M)$ is a fin. gen. $\text{Ext}_{\mathcal{C}}^{\bullet}(1, 1)$ -module.

Known in many cases, for instance:

- (Gordon 2000). ($\text{char } \mathbb{k} = 0$). $H = u_q(\mathfrak{g})^*$, \mathfrak{g} a simple Lie algebra, $q \in \mathbb{G}_{\infty}$ with restrictions on the order, has **(fgc)**.
- (Drupieski 2011). ($\text{char } \mathbb{k} > 0$). $H = u_q(\mathfrak{g})$, \mathfrak{g} a simple Lie algebra, $q \in \mathbb{G}_{\infty}$ with restrictions on the order, has **(fgc)**.
- (Drupieski 2016). $\text{char } \mathbb{k} > 0$: Finite supergroup schemes have **(fgc)**.

- (Mastnak-Petvsova-Schauenburg-Witherspoon 2010) ($\text{char } \mathbb{k} = 0$).
 H pointed, $G(H)$ abelian, $(|G(H)|, 210) = 1$, has **(fgc)**..
- (A-Angiono-Petvsova-Witherspoon 2022) ($\text{char } \mathbb{k} = 0$).
 If H is pointed, $G(H)$ abelian and the associated Nichols algebra comes in families, then $D(H)$ has **(fgc)**.
- (Stefan & Vay 2016). ($\text{char } \mathbb{k} = 0$).
 $H = \mathcal{B}(V) \#_{\mathbb{k}} \mathbb{S}_3$, where $\mathcal{B}(V) \simeq \text{FK}_3$, $\dim \mathcal{B}(V) = 12$, has **(fgc)**.
- (Nguyen, Wang & Witherspoon; Erdmann, Solberg & Wang 2018).
 $(\text{char } \mathbb{k} = p > 0)$. (Some) pointed Hopf algebras of $\dim p^3$ have **(fgc)**.
- (Friedlander & Negron, 2019; Negron, 2021) If H is cocommutative, then $D(H)$ has **(fgc)**.

II. Morita equivalence.

Let $H = (H, m, \Delta)$ be a finite-dim. Hopf algebra.

- $H^* = (H, \Delta^t, m^t)$
- $H^F = (H, m, \Delta^F)$
- $H_\sigma = (H, m_\sigma, \Delta)$
- $D(H) =$ Drinfeld double

Questions:

Let $H = (H, m, \Delta)$ be a finite-dim. Hopf algebra.
If H has **(fgc)**, does...

- $H^* = (H, \Delta^t, m^t)$ have **(fgc)**?
- $H^F = (H, m, \Delta^F)$ has **(fgc)**.
- $H_\sigma = (H, m_\sigma, \Delta)$ have **(fgc)**?
- $D(H) =$ Drinfeld double have **(fgc)**?

Let H and U be finite-dimensional Hopf algebras.

We say that H and U are *Morita equivalent*, $H \sim_{\text{Mor}} U$, if there exists an equivalence of braided tensor categories between the Drinfeld centers $\mathcal{Z}(\text{rep } H)$ and $\mathcal{Z}(\text{rep } U)$ (Müger, Etingof-Nikshych-Ostrik). Equivalently, $D(H) \simeq D(U)^G$, G a twist.

Example. $H \sim_{\text{Mor}} H^*$, $H \sim_{\text{Mor}} H^F$, $H \sim_{\text{Mor}} H_\sigma$.

Remark. This is not the same as Morita equivalence of algebras.

Lemma. [AAPW, Negron-Plavnik] Let R be an augmented subalgebra of a finite-dimensional augmented algebra A .

Suppose that A is projective as a right R -module under multiplication. If A has fgc, then so does R .

Corollary. [AAPW, Negron-Plavnik] Let H and U be finite-dim. Morita equivalent Hopf algebras. If $D(H)$ has fgc, then U has fgc.

Indeed, $D(H)$ has fgc $\implies D(U)$ has fgc $\implies U$ has fgc.

In particular, if $D(H)$ has fgc, then H^* , H^F , H_σ have fgc.

Bosonization [Radford, Majid].

- K a finite-dimensional Hopf algebra.
- R a finite-dimensional Hopf algebra in ${}^K_K\mathcal{YD}$.

The bosonization $R\#K = R \otimes K$ with smash multiplication and smash comultiplication \rightsquigarrow a new Hopf algebra

Question [A-Natale]: If K and R have fgc, does the bosonization $R\#K$ also have fgc?

Partial results: let G be a finite group.

- [AAPW,A-Natale] $K = \mathbb{k}G$, $\text{char } \mathbb{k} \nmid |G|$; and R has fgc.
- [AAPW,A-Natale], $K = \mathbb{k}^G := \text{Hom}(\mathbb{k}G, \mathbb{k})$; and R has fgc.
- [Negron] K cocommutative and R is its \mathbb{k} -dual K^* where K acts on K^* by the coadjoint action. Indeed, $K^* \# K \cong D(K)$.
- [AIM team] K semisimple and R admits a deformation sequence (notion introduced in [Negron-Pevtsova]).
- [AIM team] K cocommutative & R admits a K -equivariant deformation sequence.
- [AIM team] K admits a deformation sequence \mathfrak{C} and R admits a \mathfrak{C} -equivariant deformation sequence.

Let H be a finite dimensional pointed Hopf algebra with $G(H)$ abelian & infinitesimal braiding V . Assume that V comes in families, cf. [Heckenberger].

Theorem [AAPW] $D(H)$ has (fgc).

Outline of the proof. One starts by

Theorem A. [AAPW] Let U be a braided vector space of diagonal type such that the Nichols algebra $\mathcal{B}(U)$ has finite dimension. Assume that U comes in families. Then $\mathcal{B}(U)$ has fgc.

Next we need

Theorem B. [Angiono, Angiono-García Iglesias] We have

$$H \sim_{\text{Morita}} \text{gr } H \simeq \mathcal{B}(V) \#_{\mathbb{k}} G(H).$$

Theorem A $\implies \mathcal{B}(V), \mathcal{B}(V^*)$ have fgc $\xrightarrow{\star} \text{gr } H, (\text{gr } H)^*$ have fgc

$\Downarrow \heartsuit$

H has fgc $\xleftarrow{\clubsuit} D(H)$ has fgc $\xleftarrow{\spadesuit} D(\text{gr } H)$ has fgc

★ By the Lemma on bosonization.

♡ A cohomological result on twisted tensor product.

♠ Theorem B.

♣ since $D(H)$ is free over H .

III. Extensions.

Consider an extension of finite-dimensional Hopf algebras:

$$\mathbb{k} \rightarrow K \rightarrow H \rightarrow L \rightarrow \mathbb{k} \quad (*)$$

As we have seen, If H has fgc, then so does K .

Question. [A-Natale] If H has fgc, does L also have fgc?

Lemma. [AAPW] K semisimple, L has fgc, then so does H .

Question. [A-Natale] If K and L have fgc, does H also have fgc?

Consider an *abelian* extension of finite-dim. Hopf algebras:

$$\mathbb{k} \rightarrow K \rightarrow H \rightarrow L \rightarrow \mathbb{k}, \quad (*)$$

that is, K is commutative and L is cocommutative.

Thus, K and L have fgc (and H if $\text{char } \mathbb{k} = 0$; assume $\text{char } \mathbb{k} > 0$).

Split extensions. [G. I. Kac, Majid] The following are equivalent:

exact factorizations \leftrightarrow matched pairs \leftrightarrow split extensions

$$S = G \cdot L \longleftrightarrow (G, L, \triangleright, \triangleleft) \longleftrightarrow (L^*, G, \leftarrow, \rho)$$

$$S = G \bowtie L \longleftarrow (G, L, \triangleright, \triangleleft) \longleftrightarrow L^* \hookrightarrow L^* \# G \twoheadrightarrow G.$$

Note: S cocommutative $\Leftrightarrow L^* \hookrightarrow L^* \# G \twoheadrightarrow G$ abelian extension.

Note: Any abelian extension is like $L^* \hookrightarrow L^{*\tau} \#_{\sigma} G \twoheadrightarrow G$ for suitable 2-cocycles τ and σ .

Consider an *abelian* extension of finite-dim. Hopf algebras:

$$\mathbb{k} \rightarrow K \rightarrow H \rightarrow L \rightarrow \mathbb{k}, \quad (*)$$

Definition. We say that H is *quasi-split* if it is Morita equivalent to the split extension: $H \sim_{\text{Mor}} K \# L$.

Theorem. (Schauenburg) If $(*)$ is a split abelian extension, then there is a cocommutative Hopf algebra U such that $H \sim_{\text{Mor}} U$ (actually, $U \simeq L \bowtie K^*$).

Theorem. (A.-Natale) If H is a *quasi-split* abelian extension, then $D(H)$, hence H and any Hopf algebra $U \sim_{\text{Mor}} H$, have fgc.

Proof: Negron + Schauenburg.

IV. Applications. Assume in this Section that $\text{char } \mathbb{k} > 2$.

A class of braided vector spaces \mathcal{V} was introduced in (A-Angiono-Heckenberger); they decompose as direct sums of Jordan blocks, super Jordan blocks and labelled points. Their Nichols algebras are finite-dimensional.

Let \mathcal{V}_+ be the subclass with only Jordan blocks and points labelled with 1; these depend on a family of parameters:

$$\Lambda \ni (\mathfrak{q}, \mathfrak{a}) \rightsquigarrow \mathcal{V}(\mathfrak{q}, \mathfrak{a}).$$

Let $(q, a) \in \Lambda$.

Theorem. (A.-Natale) For a suitable finite abelian group Γ ,

- the bosonization $H = \mathcal{B}(\mathcal{V}(1, a)) \# \mathbb{k}\Gamma$ fits into a *split* abelian exact sequence, hence $D(H)$, H and $\mathcal{B}(\mathcal{V}(1, a))$ have fgc;
- for a general q , $\mathcal{B}(\mathcal{V}(q, a)) \# \mathbb{k}\Gamma$ is a cocycle deformation of H , hence $D(\mathcal{B}(\mathcal{V}(q, a)) \# \mathbb{k}\Gamma)$, $\mathcal{B}(\mathcal{V}(q, a)) \# \mathbb{k}\Gamma$ and $\mathcal{B}(\mathcal{V}(q, a))$ have fgc too.

As illustration, we give details for two simple examples.

The *Jordan block* $\mathcal{V}(1, 2)$ is the braided vector space with basis $\{x, y\}$ such that

$$\begin{aligned} c(x \otimes x) &= x \otimes x, & c(y \otimes x) &= x \otimes y, \\ c(x \otimes y) &= (y + x) \otimes x, & c(y \otimes y) &= (y + x) \otimes y. \end{aligned}$$

Lemma. [Cibils-Lauve-Witherspoon]

The Nichols algebra $\mathcal{B}(\mathcal{V}(1, 2))$ (called the *restricted Jordan plane*) is generated by x, y with relations

$$yx - xy + \frac{1}{2}x^2, \quad x^p, \quad y^p.$$

$\{x^a y^b : 0 \leq a, b < p\}$ is a basis of $\mathcal{B}(\mathcal{V}(1, 2)) \Rightarrow \dim \mathcal{B}(\mathcal{V}(1, 2)) = p^2$.

The minimal bosonization.

Let $\Gamma = \mathbb{Z}/p = \langle g \rangle$. We realize $\mathcal{V}(1, 2)$ in ${}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma}\mathcal{YD}$ by

$$g \cdot x = x, \quad g \cdot y = y + x, \quad \deg x = \deg y = g.$$

Thus the Hopf algebra $H = \mathcal{B}(\mathcal{V}(1, 2)) \# \mathbb{k}\Gamma$ has dimension p^3 .

Let $K = \mathbb{k}\langle x, g \rangle \subset H$ and $L = \mathbb{k}[\zeta]/(\zeta^p)$ with ζ primitive.

Lemma. [A-Natale] H fits into a split abelian extension

$$\mathbb{k} \rightarrow K \xrightarrow{\iota} H \xrightarrow{\pi} L \rightarrow \mathbb{k},$$

ι is the inclusion & π is defined by $\pi(x) = 0$, $\pi(g) = 1$, $\pi(y) = \zeta$.

Remark. [A-Peña Pollastri] The Drinfeld double of H fits into an abelian exact sequence $\mathbb{k} \rightarrow \mathbf{R} \rightarrow D(H) \rightarrow u(\mathfrak{sl}_2(\mathbb{k})) \rightarrow \mathbb{k}$, where \mathbf{R} is a local commutative Hopf algebra.

Proposition.

(i) [Nguyen-Wang-Witherspoon] The Hopf algebra $\mathcal{B}(\mathcal{V}(1, 2)) \# \mathbb{k}\Gamma$ and the Nichols algebra $\mathcal{B}(\mathcal{V}(1, 2))$ have fgc.

(ii) [A-Natale] The Drinfeld double $D(\mathcal{B}(\mathcal{V}(1, 2)) \# \mathbb{k}\Gamma)$ has fgc.

Question: Let F be another finite group such that $\mathcal{V}(1, 2)$ admits a realization in ${}_{\mathbb{k}F}^{\mathbb{k}F}\mathcal{YD}$. Does $\mathcal{B}(\mathcal{V}(1, 2)) \# \mathbb{k}F$ have fgc?

Proposition. [A-Natale] $(\mathcal{B}(\mathcal{V}(1, 2)) \# \mathbb{k}F)^*$ has fgc.

The Nichols algebra $\mathcal{B}(\mathfrak{L}_q(1, \mathcal{G}))$. Let $q \in \mathbb{k}^\times$, $a \in \mathbb{F}_p^\times$ and

$$r \in \{1 - p, 2 - p, \dots, -2, -1\} \quad \text{such that} \quad r \equiv 2a \pmod{p}.$$

The *ghost* is $\mathcal{G} := -r \in \{1, \dots, p - 1\}$; since p is odd, \mathcal{G} gives a .

The braided vector space $\mathfrak{L}_q(1, \mathcal{G})$ has basis $\mathfrak{b} = \{x_1, y_1, x_2\}$ and

$$(c(\mathfrak{b} \otimes \mathfrak{b}'))_{\mathfrak{b}, \mathfrak{b}' \in \mathfrak{b}} = \begin{pmatrix} x_1 \otimes x_1 & (y_1 + x_1) \otimes x_1 & q x_2 \otimes x_1 \\ x_1 \otimes y_1 & (y_1 + x_1) \otimes y_1 & q x_2 \otimes y_1 \\ q^{-1} x_1 \otimes x_2 & q^{-1} (y_1 + ax_1) \otimes x_2 & x_2 \otimes x_2 \end{pmatrix}.$$

Thus $V_1 := \mathbb{k}x_1 + \mathbb{k}y_1 \simeq \mathcal{V}(1, 2)$ and $V_2 := \mathbb{k}x_2$ satisfy

$$c : V_i \otimes V_j = V_j \otimes V_i, \quad i, j \in \{1, 2\}.$$

Hence V_1 and V_2 are braided subspaces of V .

Set $z_0 := x_2$, $z_{n+1} := y_1 z_n - q z_n y_1$, $n > 0$.

Lemma. [A-Angiono-Heckenberger] The Nichols algebra $\mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))$ is generated by x_1, y_1, x_2 with relations

$$\begin{aligned} y_1 x_1 - x_1 y_1 + \frac{1}{2} x_1^2, & & x_1^p, y_1^p. \\ x_1 x_2 = q x_2 x_1, & & \\ z_{1+\mathcal{G}} = 0, & & \\ z_t z_{t+1} = q^{-1} z_{t+1} z_t, & & 0 \leq t < \mathcal{G}, \\ z_t^p = 0, & & 0 \leq t \leq \mathcal{G}. \end{aligned}$$

It has $\dim \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) = p^{\mathcal{G}+3}$; indeed a PBW-basis is

$$B = \{x_1^{m_1} y_1^{m_2} z_{\mathcal{G}}^{n_{\mathcal{G}}} \dots z_1^{n_1} z_0^{n_0} : 0 \leq m_i, n_j < p\}.$$

The minimal bosonization. Assume that q is a root of 1. Set $d := \text{ord } q$; then $(d, p) = 1$. Fix $f \in \mathbb{Z}_{>0}$ pd . Let

$$\Gamma = \mathbb{Z}/f \times \mathbb{Z}/f = \langle g_1 \rangle \oplus \langle g_2 \rangle, \quad \text{where } \text{ord } g_1 = \text{ord } g_2 = f.$$

Let $\Gamma = \mathbb{Z}/p = \langle g \rangle$. We realize $\mathfrak{L}_q(1, \mathcal{G})$ in $\frac{\mathbb{k}\Gamma}{\mathbb{k}\Gamma} \mathcal{YD}$ by

$$\begin{aligned} g_1 \cdot x_1 &= x_1, & g_1 \cdot y_1 &= y_1 + x_1, & g_1 \cdot x_2 &= qx_2, \\ g_2 \cdot x_1 &= q^{-1}x_1, & g_2 \cdot y_1 &= q^{-1}(y_1 + ax_1), & g_2 \cdot x_2 &= x_2, \end{aligned}$$

$$\text{deg } x_1 = g_1, \quad \text{deg } y_1 = g_1, \quad \text{deg } x_2 = g_2.$$

The Hopf algebra $H = \mathcal{B}(\mathfrak{L}_q(1, \mathcal{G})) \# \mathbb{k}\Gamma$ has dimension $p^{\mathcal{G}+3}f^2$.

Let $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let $V(\mathcal{G})$ be a $\mathfrak{sl}(2)$ -module of highest weight \mathcal{G} , with a basis $(v_n)_{n \in \mathbb{I}_{0, \mathcal{G}}}$ such that

$$E \cdot v_n = v_{n+1}, \quad 0 \leq n < \mathcal{G}, \quad E \cdot v_{\mathcal{G}} = 0.$$

Let $\mathfrak{l} = V(\mathcal{G}) \rtimes \mathbb{k}E$, a restricted Lie subalgebra of $V(\mathcal{G}) \rtimes \mathfrak{sl}(2)$.

Lemma. [A-Natale] $\mathcal{B}(\mathcal{L}_1(1, \mathcal{G})) \# \mathbb{k}\Gamma$ fits into a split abelian extension

$$\mathbb{k} \rightarrow K \xrightarrow{\iota} \mathcal{B}(\mathcal{L}_1(1, \mathcal{G})) \# \mathbb{k}\Gamma \xrightarrow{\pi} \mathfrak{u}(\mathfrak{l}) \rightarrow \mathbb{k},$$

where $K = \mathbb{k}\langle x_1, g_1, g_2 \rangle$, ι is the inclusion and π is defined by

$$\begin{aligned} \pi(x_1) &= 0, & \pi(y_1) &= E, & \pi(x_2) &= v_0, \\ \pi(g_1) &= 1, & \pi(g_2) &= 1. \end{aligned}$$

Theorem. [A-Natale]

(i) The Drinfeld double $D(\mathcal{B}(\mathcal{L}_1(1, \mathcal{G})) \# \mathbb{k}\Gamma)$ has fgc.

Therefore, the Hopf algebra $\mathcal{B}(\mathcal{L}_1(1, \mathcal{G})) \# \mathbb{k}\Gamma$ and the Nichols algebra $\mathcal{B}(\mathcal{L}_1(1, \mathcal{G}))$ have fgc.

(ii) The Drinfeld double $D(\mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \# \mathbb{k}\Gamma)$ has fgc.

Therefore, the Hopf algebra $\mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \# \mathbb{k}\Gamma$ and the Nichols algebra $\mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))$ have fgc.

Indeed, $\mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \# \mathbb{k}\Gamma$ is a cocycle deformation of $\mathcal{B}(\mathcal{L}_1(1, \mathcal{G})) \# \mathbb{k}\Gamma$.

Some references.

N. Andruskiewitsch, S. Natale, “On the finite generation of the cohomology of abelian extensions of Hopf algebras”, *Documenta Math.*, to appear, arXiv:2407.05881 [math.QA] (2024)

N. Andruskiewitsch, I. Angiono, J. Pevtsova, S. Witherspoon, “Cohomology rings of finite-dimensional pointed Hopf algebras over abelian groups”. *Res. Math. Sci.* 9, No. 1, Paper No. 12, 132 p. (2022)

N. Andruskiewitsch, D. Jaklitsch, V. C. Nguyen, A. Oswald, J. Plavnik, Julia and Shepler, Anne V. and Wang, Xingting et al., “On the finite generation of the cohomology of bosonizations”, Preprint, arXiv:2506.05267 [math.QA] (2025)

Thank you for your attention!