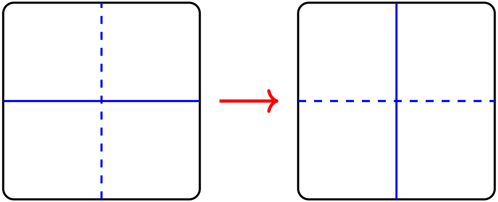
Introduction to Duoidal Categories



Marcelo Aguiar, Cornell University

Hopf Workshop, Bonne Bay Station, August 11-15, 2025

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THE ECKMANN-HILTON ARGUMENT

The Eckmann-Hilton argument

Consider a set with two binary operations $+$ and $\bullet$ and two elements 0 and 1 such that

$$(a + b) \bullet (c + d) = (a \bullet c) + (b \bullet d),$$
$$a + 0 = a = 0 + a,$$
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Then:

$$0 = 1, \quad + = \bullet,$$

and this operation is both commutative and associative.

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Let $\mu(a, c) = a \bullet c$. The axiom also says

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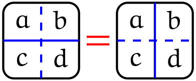
Each operation is a morphism with respect to the other.

The Eckmann-Hilton argument

Write the axiom as 

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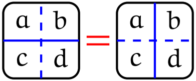
Write the axiom as 

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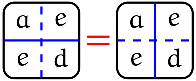
If we're able to show $0 = 1$, then it's easy!
Write e instead of 0 or 1. Then:



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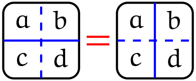
Write the axiom as 

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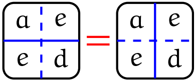

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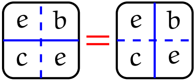

$$\Rightarrow (e + b) \bullet (c + e) = (e \bullet c) + (b \bullet 0)$$

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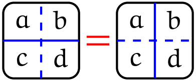
Write the axiom as 

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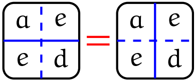

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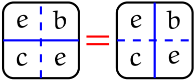

$$\Rightarrow (e + b) \bullet (c + e) = (e \bullet c) + (b \bullet 0)$$

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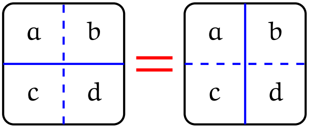
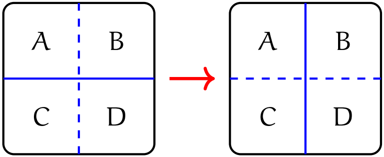
Exercise: Show $0 = 1$ and associativity.



A duoidal category is a context in which this equality is replaced by coherent maps.



Beno Eckmann
(1917-2008)

Peter Hilton
(1923-2010)

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Doc note, I dissent. A fast never prevents a fatness. I diet on cod.

What is special about this phrase by Peter Hilton?

MONOIDAL CATEGORIES AND FUNCTORS

Monoidal categories

A category $\mathscr{C}$ is monoidal if it is equipped with a functor

$$\mathscr{C} \times \mathscr{C} \to \mathscr{C} \qquad (A, B) \mapsto A \cdot B,$$

an object I, and natural isomorphisms

$$(A \cdot B) \cdot C \cong A \cdot (B \cdot C) \qquad \text{and} \qquad A \cdot I \cong A \cong I \cdot A.$$

It is braided if it is equipped with natural isomorphisms

$$\beta_{A,B} : A \cdot B \to B \cdot A.$$

$\cdot$ is the monoidal operation, I is the unit object, and β is the braiding.

The associativity and unit constraints, and the braiding, are subject to various coherence conditions.

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Examples: Set under cartesian product $X \times Y$.
$\text{Mod}_{\Bbbk}$ under tensor product $M \otimes_{\Bbbk} N$, where $\Bbbk$ is a commutative ring.

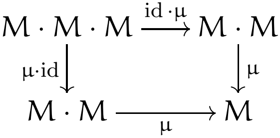
Monoids and comonoids

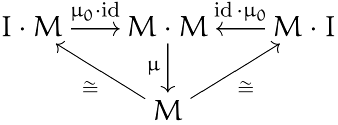
Let $\mathscr{C}$ be a monoidal category.

A monoid in $\mathscr{C}$ is an object M with morphisms

$$\mu : M \cdot M \to M \quad \text{and} \quad \mu_0 : I \to M$$

such that





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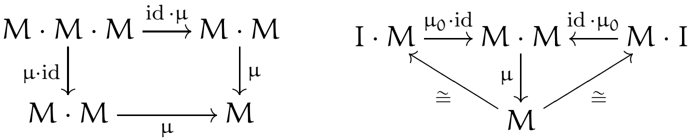

A comonoid in $\mathscr{C}$ carries morphisms

$$\delta : C \to C \cdot C \quad \text{and} \quad \delta_0 : C \to I$$

satisfying dual axioms.

Let $\mathscr{C}$ be a braided monoidal category.

A monoid M in $\mathscr{C}$ is commutative if

$$\begin{array}{ccc} M \cdot M & \xrightarrow{\beta} & M \cdot M \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & M & \end{array}$$

Cocommutative comonoids are defined similarly.

Examples

A monoid in $(\mathrm{Set}, \times)$ is an ordinary monoid.
Any set X carries a unique comonoid structure in $(\mathrm{Set}, \times)$:

$$\delta : X \to X \times X,\ \delta(x) = (x, x) \quad \text{and} \quad \delta_0 : X \to \{*\},\ \delta_0(x) = *.$$

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More generally, let $\mathscr{C}$ be a category with finite products. Then:

- $\mathscr{C}$ is monoidal with $A \cdot B$ a chosen product of A and B, and I a chosen terminal object.

Such monoidal categories are called cartesian.

- Every object in $\mathscr{C}$ carries a unique comonoid structure, and this comonoid is cocommutative.

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A (co)monoid in $(\mathsf{Mod}_{\Bbbk}, \otimes_{\Bbbk})$ is a $\Bbbk$-(co)algebra.

Example: spans

Let X be a set. A span over X is a set A with maps $A \overset{s}{\underset{t}{\rightrightarrows}} X$.

Spans are also called digraphs or quivers with vertex set X:

$$s(a) \xrightarrow{a} t(a) .$$

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The category Span_X is monoidal with unit X and tensor product

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This construction can be carried out in any finitely complete category instead of Set.

Example: bimodules

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The category Bimod_K of K-bimodules is monoidal with unit K and tensor product

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- A comonoid in Bimod_K is a K-coring (by definition).

Bimonoids

Let $\mathscr{C}$ be a braided monoidal category.

A bimonoid in $\mathscr{C}$ is an object B with a monoid structure (μ, μ_0) and a comonoid structure (δ, δ_0) such that

$$\begin{array}{ccccc} B \cdot B \cdot B \cdot B & & \xrightarrow{\mathrm{id}\cdot\beta\cdot\mathrm{id}} & & B \cdot B \cdot B \cdot B \\ \uparrow{\scriptstyle \delta\cdot\delta} & & & & \downarrow{\scriptstyle \mu\cdot\mu} \\ B \cdot B & \xrightarrow[\mu]{} & B & \xrightarrow[\delta]{} & B \cdot B \end{array}$$

$$\begin{array}{ccc} B \cdot B & \xrightarrow{\delta_0\cdot\delta_0} & I \cdot I \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \cong} \\ B & \xrightarrow[\delta_0]{} & I \end{array} \qquad \begin{array}{ccc} & B & \\ {\scriptstyle \mu_0}\nearrow & & \searrow{\scriptstyle \delta_0} \\ I & = = = & I \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{\mu_0} & B \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \delta} \\ I \cdot I & \xrightarrow[\mu_0\cdot\mu_0]{} & B \cdot B \end{array}$$

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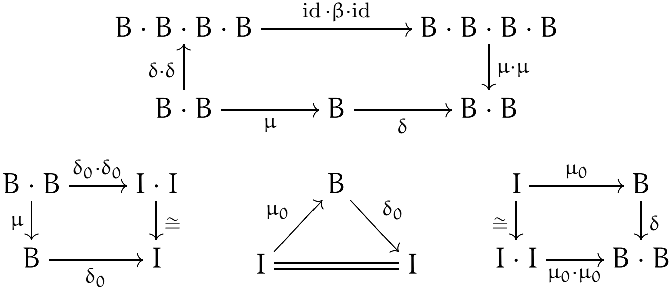

When $\mathscr{C}$ is braided,

$$\mathsf{Mon}(\mathscr{C}), \quad \mathsf{Comon}(\mathscr{C}), \quad \text{and} \quad \mathsf{Bimon}(\mathscr{C})$$

are monoidal categories, and

$$\mathsf{Mon}(\mathsf{Comon}(\mathscr{C})) \cong \mathsf{Bimon}(\mathscr{C}) \cong \mathsf{Comon}(\mathsf{Mon}(\mathscr{C})).$$

Examples

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In $(\mathsf{Bimod}_{\mathsf{K}}, \otimes_{\mathsf{K}})$, a bimonoid is $\ldots$?

Monoidal functors

Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories.
A functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is lax monoidal if it is equipped with natural morphisms

$$\varphi_{A,B} : \mathscr{F}A \cdot \mathscr{F}B \to \mathscr{F}(A \cdot B)$$

and a morphism $\varphi_0 : I \to \mathscr{F}I$, such that

$$\begin{array}{ccc}
\mathscr{F}A \cdot \mathscr{F}B \cdot \mathscr{F}C & \xrightarrow{\mathrm{id} \cdot \varphi_{B,C}} & \mathscr{F}A \cdot \mathscr{F}(B \cdot C) \\
{\scriptstyle \varphi_{A,B} \cdot \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \varphi_{A,B \cdot C}} \\
\mathscr{F}(A \cdot B) \cdot \mathscr{F}C & \xrightarrow[\varphi_{A \cdot B,C}]{} & \mathscr{F}(A \cdot B \cdot C)
\end{array}$$

$$\begin{array}{ccc}
I \cdot \mathscr{F}A & \xleftarrow{\cong} & \mathscr{F}A \\
{\scriptstyle \varphi_0 \cdot \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \cong} \\
\mathscr{F}I \cdot \mathscr{F}A & \xrightarrow[\varphi_{I,A}]{} & \mathscr{F}(I \cdot A)
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{F}A \cdot I & \xleftarrow{\cong} & \mathscr{F}A \\
{\scriptstyle \mathrm{id} \cdot \varphi_0} \big\downarrow & & \big\downarrow {\scriptstyle \cong} \\
\mathscr{F}A \cdot \mathscr{F}I & \xrightarrow[\varphi_{A,I}]{} & \mathscr{F}(A \cdot I)
\end{array}$$

Similarly, a colax monoidal functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is equipped with natural morphisms

$$\psi_{A,B} : \mathscr{F}(A \cdot B) \to \mathscr{F}A \cdot \mathscr{F}B \qquad \text{and} \qquad \psi_0 : \mathscr{F}I \to I.$$

Similarly, a colax monoidal functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is equipped with natural morphisms

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If φ is invertible, and $\psi = \varphi^{-1}$, then

$$(\mathscr{F}, \varphi) \text{ is lax monoidal } \iff (\mathscr{F}, \psi) \text{ is colax monoidal.}$$

In this case, we say $\mathscr{F}$ is strong monoidal.

Bilax monoidal functors

Let $\mathscr{C}$ and $\mathscr{D}$ be braided monoidal categories.
A functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is bilax monoidal if it is equipped with a lax structure φ and a colax structure ψ such that

$$\begin{array}{ccccc}
 & & \mathscr{F}(A\cdot B)\cdot\mathscr{F}(C\cdot D) & & \\
 & \swarrow{\scriptstyle \varphi_{A\cdot B,C\cdot D}} & & \searrow{\scriptstyle \psi_{A,B}\cdot\psi_{C,D}} & \\
\mathscr{F}(A\cdot B\cdot C\cdot D) & & & & \mathscr{F}A\cdot\mathscr{F}B\cdot\mathscr{F}C\cdot\mathscr{F}D \\
\downarrow{\scriptstyle \mathscr{F}(\mathrm{id}\cdot\beta\cdot\mathrm{id})} & & & & \downarrow{\scriptstyle \mathrm{id}\cdot\beta\cdot\mathrm{id}} \\
\mathscr{F}(A\cdot C\cdot B\cdot D) & & & & \mathscr{F}A\cdot\mathscr{F}C\cdot\mathscr{F}B\cdot\mathscr{F}D \\
 & \searrow{\scriptstyle \psi_{A\cdot C,B\cdot D}} & & \swarrow{\scriptstyle \varphi_{A,C}\cdot\varphi_{B,D}} & \\
 & & \mathscr{F}(A\cdot C)\cdot\mathscr{F}(B\cdot D) & &
\end{array}$$

$$\begin{array}{ccccc}
I & \xrightarrow{\varphi_0} & \mathscr{F}I & \xrightarrow{\cong} & \mathscr{F}(I\cdot I) \\
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I\cdot I & & \xrightarrow[\varphi_0\cdot\varphi_0]{} & & \mathscr{F}I\cdot\mathscr{F}I
\end{array}
\qquad
\begin{array}{ccc}
 & \mathscr{F}I & \\
\nearrow{\scriptstyle\varphi_0} & & \searrow{\scriptstyle\psi_0} \\
I & = & I
\end{array}
\qquad
\begin{array}{ccccc}
I & \xleftarrow{\psi_0} & \mathscr{F}I & \xleftarrow{\cong} & \mathscr{F}(I\cdot I) \\
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I\cdot I & & \xleftarrow[\psi_0\cdot\psi_0]{} & & \mathscr{F}I\cdot\mathscr{F}I
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Alternative terminology

lax monoidal	monoidal
colax monoidal	comonoidal
bilax monoidal	bimonoidal

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Warning: in certain contexts, monoidal is used for strong monoidal.

Braided monoidal functors

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- A monoidal functor $(\mathscr{F}, \varphi) : \mathscr{C} \to \mathscr{D}$ is braided if

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- Braided comonoidal functors $(\mathscr{F}, \psi)$ are defined similarly.

Braided monoidal functors

Let $\mathscr{C}$ and $\mathscr{D}$ be braided monoidal categories.

- A monoidal functor $(\mathscr{F}, \varphi) : \mathscr{C} \to \mathscr{D}$ is braided if

$$\begin{array}{ccc} \mathscr{F}A \cdot \mathscr{F}B & \xrightarrow{\varphi_{A,B}} & \mathscr{F}(A \cdot B) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\mathscr{F}\beta} \\ \mathscr{F}B \cdot \mathscr{F}A & \xrightarrow[\varphi_{B,A}]{} & \mathscr{F}(B \cdot A) \end{array}$$

- Braided comonoidal functors $(\mathscr{F}, \psi)$ are defined similarly.
- A bimonoidal functor $(\mathscr{F}, \varphi, \psi) : \mathscr{C} \to \mathscr{D}$ is braided if both $(\mathscr{F}, \varphi)$ and $(\mathscr{F}, \psi)$ are braided.

Preservation under monoidal functors

Proposition. Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories.
Let $(\mathscr{F}, \varphi) : \mathscr{C} \to \mathscr{D}$ be a monoidal functor.
If (M, μ) is a monoid in $\mathscr{C}$, then $\mathscr{F}M$ is a monoid in $\mathscr{D}$ with

$$\mathscr{F}M \cdot \mathscr{F}M \xrightarrow{\varphi_{M,M}} \mathscr{F}(M \cdot M) \xrightarrow{\mathscr{F}\mu} \mathscr{F}M \quad \text{and} \quad I \xrightarrow{\varphi_0} \mathscr{F}I \xrightarrow{\mathscr{F}\mu_0} \mathscr{F}M.$$

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If $\mathscr{C}$ and $\mathscr{D}$ are braided, the monoid M is commutative, and the functor $\mathscr{F}$ is braided monoidal, then the monoid $\mathscr{F}M$ is commutative.

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If $\mathscr{C}$ and $\mathscr{D}$ are braided, the monoid M is commutative, and the functor $\mathscr{F}$ is braided monoidal, then the monoid $\mathscr{F}M$ is commutative.

Similarly, comonoidal functors preserve comonoids, and bimonoidal functors (between braided monoidal categories) preserve bimonoids.

Composition of monoidal functors

Proposition. Let $\mathscr{C}, \mathscr{D}$, and $\mathscr{E}$ be monoidal categories.
Let $(\mathscr{F}, \varphi) : \mathscr{C} \to \mathscr{D}$ and $(\mathscr{G}, \gamma) : \mathscr{D} \to \mathscr{E}$ be monoidal functors.
Then $\mathscr{G}\mathscr{F} : \mathscr{C} \to \mathscr{E}$ is monoidal with

$$\mathscr{G}\mathscr{F}A \cdot \mathscr{G}\mathscr{F}B \xrightarrow{\gamma_{\mathscr{F}A,\mathscr{F}B}} \mathscr{G}(\mathscr{F}A \cdot \mathscr{F}B) \xrightarrow{\mathscr{G}(\varphi_{A,B})} \mathscr{G}\mathscr{F}(A \cdot B),$$

$$I \xrightarrow{\gamma_0} \mathscr{G}I \xrightarrow{\mathscr{G}(\varphi_0)} \mathscr{G}\mathscr{F}I.$$

Similarly, comonoidal and bimonoidal functors can be composed.

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Similarly, comonoidal and bimonoidal functors can be composed.

A monoid in $\mathscr{C}$ is the same thing as a monoidal functor $\mathscr{I} \to \mathscr{C}$, where $\mathscr{I}$ is the unit category.

Similarly, (co,bi)monoids are (co,bi)monoidal functors $\mathscr{I} \to \mathscr{C}$.

Therefore, preservation is a consequence of composition.

Digression: pseudomonoids

- Monoidal categories provide a context for monoids.

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Let Cat be the bicategory consisting of categories, functors, and natural transformations.

- The bicategory Cat is monoidal under cartesian product.
- A pseudomonoid in Cat is a monoidal category.
- A lax morphism in Cat is a lax monoidal functor.
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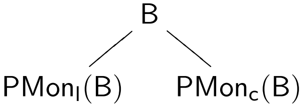
- A lax morphism in Cat is a lax monoidal functor.
- A colax morphism in Cat is a colax monoidal functor.

Later, we will consider pseudomonoids in bicategories whose objects are monoidal categories.

Let B be a bicategory.
Let $\mathrm{PMon}_l(B)$ be the category of pseudomonoids and lax morphisms.
Let $\mathrm{PMon}_c(B)$ be the category of pseudomonoids and colax morphisms.



Example.
$\mathrm{PMon}_l(\mathrm{Cat})$ consists of monoidal categories and monoidal functors,
$\mathrm{PMon}_c(\mathrm{Cat})$ consists of monoidal categories and comonoidal functors.

THE CUP PRODUCT

The cup product

Let $\Bbbk$ a commutative ring.

Let X be a topological space.
Fact. The cup product on the singular cohomology $\mathrm{H}(X, \Bbbk)$ is commutative:

$$f \smile g = (-1)^{|f|\,|g|} g \smile f.$$

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Reason.

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- $\mathrm{H}(-, \mathbb{k})$ is contravariant, monoidal, and braided.

Let G be a group.
Fact. The cup product on the group cohomology $\mathrm{H}(G, \mathbb{k})$ is commutative.
Reason. Let $X = K(G, 1)$. Then $\mathrm{H}(G, \mathbb{k}) = \mathrm{H}(X, \mathbb{k})$.

Group cohomology is a special case of Hochschild cohomology:

$$\mathrm{H}(\mathrm{G}, \mathbb{k}) = \mathrm{H}(\mathbb{k}\mathrm{G}, \mathbb{k}).$$

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$$\mathrm{H}(\mathrm{G}, \Bbbk) = \mathrm{H}(\Bbbk\mathrm{G}, \Bbbk).$$

Let A be an augmented $\Bbbk$-algebra and B a $\Bbbk$-bialgebra.

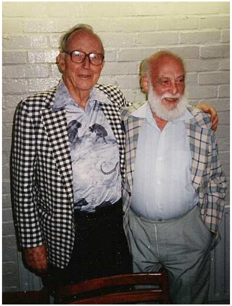
View $\Bbbk$ as an A-bimodule via the augmentation,
and the same for B via the counit.

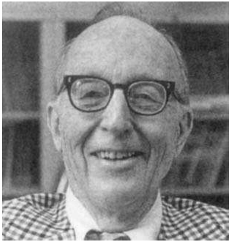
Facts.

- The cup product on the Hochschild cohomology $\mathrm{H}(A, \Bbbk)$ is not always commutative.
- But the cup product on the Hochschild cohomology $\mathrm{H}(\mathrm{B}, \Bbbk)$ is commutative.

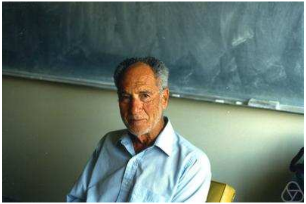
Why?

Samuel Eilenberg
(1913-1998)





Saunders Maclane
(1909-2005)



Gerhard Hochschild
(1915-2010)

Simplicial sets

Let sSet be the category of simplicial sets $X = (X_n,\ d_i^{(n)},\ s_j^{(n)})$.

For each $n \geq 0$, X_n is a set, and

$$d_i^{(n)} : X_{n+1} \to X_n \text{ for } 0 \leq i \leq n+1,$$
$$s_j^{(n)} : X_n \to X_{n+1} \text{ for } 0 \leq j \leq n$$

are collections of maps satisfying the simplicial relations.
These maps are called faces and degeneracies.

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are collections of maps satisfying the simplicial relations.
These maps are called faces and degeneracies.

The category sSet is symmetric monoidal (in fact, cartesian) under

$$(X \times Y)_n = X_n \times Y_n.$$

The faces and degeneracies on $X \times Y$ are also defined diagonally.

Chain complexes

Let $cC_{\mathbb{k}}$ be the category of chain complexes of $\mathbb{k}$-modules $C = (C_n, \partial_n)$.
For each $n \geq 0$, C_n is a $\mathbb{k}$-module, and

$$\partial_n : C_{n+1} \to C_n$$

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is a collection of morphisms such that $\partial_n \partial_{n+1} = 0$.

The category $cC_{\mathbb{k}}$ is symmetric monoidal under

$$(C \bullet D)_n = \bigoplus_{i=0}^{n} C_i \otimes D_{n-i},$$

$$\partial_n(x \otimes y) = \partial_i(x) \otimes y + (-1)^i x \otimes \partial_{n+1-i}(y)$$

for $x \in C_i$, $y \in D_{n+1-i}$.

The chain complex functor

Define a functor $\mathscr{F} : \mathsf{sSet} \longrightarrow \mathsf{cC}_{\mathbb{k}}$ by

$$(\mathscr{F}X)_n = \mathbb{k}X_n \quad \text{and} \quad \partial_n = \sum_{i=0}^{n+1} (-1)^i d_i^{(n)}.$$

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Define a functor $\mathscr{F} : \mathsf{sSet} \to \mathsf{cC}_{\Bbbk}$ by

$$(\mathscr{F}X)_n = \Bbbk X_n \quad \text{and} \quad \partial_n = \sum_{i=0}^{n+1} (-1)^i d_i^{(n)}.$$

Theorem. There are transformations

$$\varphi_{X,Y} : \mathscr{F}X \bullet \mathscr{F}Y \to \mathscr{F}(X \times Y) \quad \text{and} \quad \psi_{X,Y} : \mathscr{F}(X \times Y) \to \mathscr{F}X \bullet \mathscr{F}Y$$

that make $(\mathscr{F}, \varphi, \psi)$ a bimonoidal functor.

φ is Eilenberg-Zilber, ψ is Alexander-Whitney.

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that make $(\mathscr{F}, \varphi, \psi)$ a bimonoidal functor.

φ is Eilenberg-Zilber, ψ is Alexander-Whitney.

$$\psi_{X,Y} : X_n \otimes Y_n \to \bigoplus_{i=0}^{n} X_i \otimes Y_{n-i},$$

$$\psi_{X,Y}(x \otimes y) = \sum_{i=0}^{n} (d_{i+1} \cdots d_n)(x) \otimes (d_0 \cdots d_{i-1})(y).$$

Fact. The comonoidal functor $(\mathscr{F}, \psi)$ is <u>not</u> braided.

Up to homotopy

Let $\overline{\mathsf{cC}_{\Bbbk}}$ be the category whose objects are chain complexes and whose morphisms are homotopy classes of chain maps.

Fact (Dold). Up to homotopy, $(\mathscr{F}, \psi)$ is braided.

In other words, the functor $\mathsf{sSet} \xrightarrow{\mathscr{F}} \mathsf{cC}_{\Bbbk} \twoheadrightarrow \overline{\mathsf{cC}_{\Bbbk}}$ is braided comonoidal.

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Consequences.

- For any simplicial set X, the chain complex $\mathscr{F}X$ is a cocommutative comonoid.
- Hence, singular cohomology and group cohomology are commutative graded rings.

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Consequences.

- For any simplicial set X, the chain complex $\mathscr{F}X$ is a cocommutative comonoid.
- Hence, singular cohomology and group cohomology are commutative graded rings.

However, the Hochschild chain complex of a $\Bbbk$-bialgebra B (with coefficients in $\Bbbk$) arises from a simplicial $\Bbbk$-module, not from a simplicial set.

Why is $H(B, \Bbbk)$ commutative?

DUOIDAL CATEGORIES AND FUNCTORS

Latin	duo
Greek	δύο
Sanskrit	dvá
Old English	twā

Duoidal categories

Let $\mathscr{C}$ be a category. A duoidal structure on $\mathscr{C}$ consists of:

- Two monoidal structures on $\mathscr{C}$: two tensor products

$$\mathscr{C} \times \mathscr{C} \xrightarrow{\diamond} \mathscr{C} \quad \text{and} \quad \mathscr{C} \times \mathscr{C} \xrightarrow{\star} \mathscr{C}$$

  with respective unit objects J and K.
- A natural transformation

$$\zeta_{A,B,C,D} \colon (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

  called the interchange law.
- Three morphisms

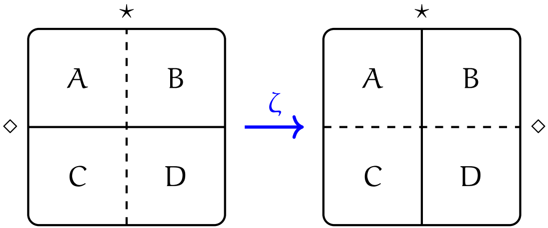
$$\zeta_{J} \colon J \to J \star J, \qquad \zeta_0 \colon J \to K, \qquad \zeta_K \colon K \diamond K \to K.$$

All of these are subject to various axioms.

Complete definition by Garner and (independently) A.-Mahajan.
Precedents in work of Balteanu-Fiedorowicz and others.

The interchange law

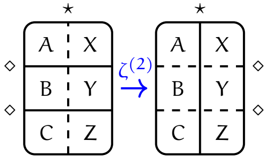
$$(A \star B) \diamond (C \star D) \xrightarrow{\zeta} (A \diamond C) \star (B \diamond D)$$



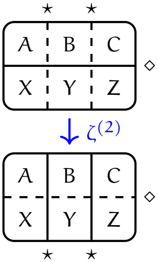
- The interchange law is not required to be invertible.
- The order of the operations $(\diamond, \star)$ matters.

Duoidal categories: two of the axioms

$$
\begin{array}{ccccc}
 & & (A \star X) \diamond (B \star Y) \diamond (C \star Z) & & \\
 & \swarrow^{\zeta \diamond \mathrm{id}} & & \searrow^{\mathrm{id} \diamond \zeta} & \\
\big((A \diamond B) \star (X \diamond Y)\big) \diamond (C \star Z) & & & & (A \star X) \diamond \big((B \diamond C) \star (Y \diamond Z)\big) \\
 & \searrow_{\zeta} & & \swarrow_{\zeta} & \\
 & & (A \diamond B \diamond C) \star (X \diamond Y \diamond Z) & &
\end{array}
$$



$$
\begin{array}{ccccc}
 & & (A \star B \star C) \diamond (X \star Y \star Z) & & \\
 & \swarrow^{\zeta} & & \searrow^{\zeta} & \\
\big((A \star B) \diamond (X \star Y)\big) \star (C \diamond Z) & & & & (A \diamond X) \star \big((B \star C) \diamond (Y \star Z)\big) \\
 & \searrow_{\zeta \star \mathrm{id}} & & \swarrow_{\mathrm{id} \star \zeta} & \\
 & & (A \diamond X) \star (B \diamond Y) \star (C \diamond Z) & &
\end{array}
$$



The remaining axioms involve the unit objects and unit maps.

Duoidal categories: the axioms

Recall the structure maps:

$$(A \star B) \diamond (C \star D) \xrightarrow{\zeta_{A,B,C,D}} (A \diamond C) \star (B \diamond D),$$

$$J \xrightarrow{\zeta_J} J \star J, \qquad J \xrightarrow{\zeta_0} K, \qquad K \diamond K \xrightarrow{\zeta_K} K.$$

Let $\mathscr{C} \times \mathscr{C} \underset{\mathscr{G}}{\overset{\mathscr{F}}{\rightrightarrows}} \mathscr{C}$ be $\mathscr{F}(A, B) = A \star B$ and $\mathscr{G}(A, B) = A \diamond B$.

Duoidal categories: the axioms

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The axioms are:

- The functor $\mathscr{F}$ is monoidal with respect to $\diamond$ with

  $$\mathscr{F}(A, B) \diamond \mathscr{F}(C, D) \xrightarrow{\zeta_{A,B,C,D}} \mathscr{F}\big((A, B) \diamond (C, D)\big) \quad \text{and} \quad J \xrightarrow{\zeta_J} \mathscr{F}(J, J).$$

- (K, ζ_K, ζ_0) is a monoid in $(\mathscr{C}, \diamond, J)$.
- The functor $\mathscr{G}$ is comonoidal with respect to $\star$ with

  $$\mathscr{G}\big((A, C) \star (B, D)\big) \xrightarrow{\zeta_{A,B,C,D}} \mathscr{G}(A, C) \diamond \mathscr{G}(B, D) \quad \text{and} \quad \mathscr{G}(K, K) \xrightarrow{\zeta_K} K.$$

- (J, ζ_J, ζ_0) is a comonoid in $(\mathscr{C}, \star, K)$.

Example: braided monoidal categories

Let $(\mathscr{C}, \otimes, I, \beta)$ be a braided monoidal category with braiding

$$\beta_{A,B} : A \otimes B \to B \otimes A.$$

Then $\mathscr{C}$ is duoidal with

$$\diamond = \star = \otimes \qquad \text{and} \qquad J = K = I.$$

The interchange law is

$$A \otimes B \otimes C \otimes D \xrightarrow{\mathrm{id} \otimes \beta \otimes \mathrm{id}} A \otimes C \otimes B \otimes D.$$

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Proposition (Joyal-Street).
Let $\mathscr{C}$ be a duoidal category in which all structure maps are invertible. Then the duoidal structure arises from a braided monoidal structure as above.

Example: M-graded modules

Let M be a monoid and $\Bbbk$ a commutative ring.
Let $\mathscr{C}$ be the category of M-graded $\Bbbk$-modules:

$$X = (X_m)_{m \in M}, \text{ each } X_m \text{ is a } \Bbbk\text{-module.}$$

Then $\mathscr{C}$ is duoidal with $\diamond$ and $\star$ defined by

$$(X \diamond Y)_m = \bigoplus_{p \cdot q = m} X_p \otimes Y_q \qquad \text{and} \qquad (X \star Y)_m = X_m \otimes Y_m.$$

The unit objects J and K are defined by

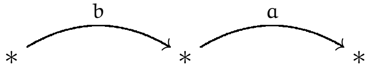
$$J_m = \begin{cases} \Bbbk & \text{if } m = 1 \\ 0 & \text{if not} \end{cases} \qquad \text{and} \qquad K_m = \Bbbk \text{ for all } m.$$

Exercise. Define the interchange law.

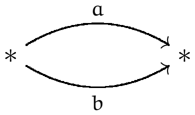
Example: spans

Let X be a set. The category Span_X is duoidal with

$$A \diamond B = \{(a, b) \in A \times B : s(a) = t(b)\},$$
$$A \star B = \{(a, b) \in A \times B : s(a) = s(b) \text{ and } t(a) = t(b)\}.$$



in series



in parallel

Exercise. Describe the unit objects and define the interchange law.

Example

Let $(\mathscr{C}, \otimes)$ be an arbitrary monoidal category.

Suppose that in $\mathscr{C}$ all finite products exist and consider the corresponding cartesian monoidal category $(\mathscr{C}, \times)$.

Then $(\mathscr{C}, \otimes, \times)$ is a duoidal category: the interchange law

$$\zeta_{A,B,C,D} \colon (A \times B) \otimes (C \times D) \to (A \diamond C) \times (B \otimes D)$$

is the unique map with components

$$\pi_A^{A \times B} \otimes \pi_C^{C \times D} \quad \text{and} \quad \pi_B^{A \times B} \otimes \pi_D^{C \times D},$$

where π denotes the canonical projections.

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where π denotes the canonical projections.

The duoidal category Span_X is of this form.

Example: bimodules

Let K be a commutative $\mathbb{k}$-algebra.

Let $\mathscr{C} = \mathsf{Bimod}_K = \mathsf{Mod}_{K \otimes K}$.

Then $\mathscr{C}$ is duoidal with $\diamond$ and $\star$ defined by

$$X \diamond Y = X \otimes_{K \otimes K} Y \qquad \text{and} \qquad X \star Y = X \otimes_K Y.$$

The unit objects are $K \otimes K$ and K.

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Let K be a commutative $\Bbbk$-algebra.

Let $\mathscr{C} = \mathsf{Bimod}_K = \mathsf{Mod}_{K \otimes K}$.

Then $\mathscr{C}$ is duoidal with $\diamond$ and $\star$ defined by

$$X \diamond Y = X \otimes_{K \otimes K} Y \qquad \text{and} \qquad X \star Y = X \otimes_K Y.$$

The unit objects are $K \otimes K$ and K.

The interchange law is induced from the braiding on $\mathsf{Mod}_{\Bbbk}$:

$$\begin{array}{ccc}
(A \otimes B) \otimes (C \otimes D) & \xrightarrow{\mathrm{id} \otimes \beta \otimes \mathrm{id}} & (A \otimes C) \otimes (B \otimes D) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
(A \star B) \otimes (C \star D) & & (A \diamond C) \otimes (B \diamond D) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
(A \star B) \diamond (C \star D) & \overset{}{\underset{\zeta}{\dashrightarrow}} & (A \diamond C) \star (B \diamond D)
\end{array}$$

What goes wrong if K is not commutative?
For noncommutative K, see Gabi's lectures.

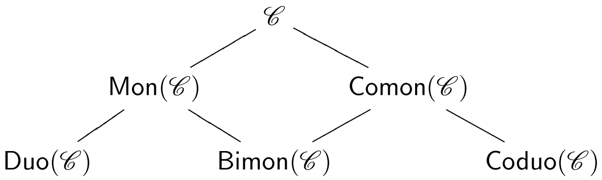
Algebraic structures in duoidal categories

Proposition. Let $(\mathscr{C}, \diamond, \star, \zeta)$ be a duoidal category. Then $\mathrm{Mon}(\mathscr{C}, \diamond)$ is monoidal under $\star$ and $\mathrm{Comon}(\mathscr{C}, \star)$ is monoidal under $\diamond$. In addition,

$$\mathrm{Comon}(\mathrm{Mon}(\mathscr{C}, \diamond), \star) = \mathrm{Mon}(\mathrm{Comon}(\mathscr{C}, \star), \diamond).$$

Define duoids, bimonoids, and coduoids in $\mathscr{C}$ by

- $\mathrm{Bimon}(\mathscr{C}, \diamond, \star) = \mathrm{Comon}(\mathrm{Mon}(\mathscr{C}, \diamond), \star) = \mathrm{Mon}(\mathrm{Comon}(\mathscr{C}, \star), \diamond)$,
- $\mathrm{Duo}(\mathscr{C}, \diamond, \star) = \mathrm{Mon}(\mathrm{Mon}(\mathscr{C}, \diamond), \star)$,
- $\mathrm{Coduo}(\mathscr{C}, \diamond, \star) = \mathrm{Comon}(\mathrm{Comon}(\mathscr{C}, \star), \diamond)$.



Bimonoids

A bimonoid B in a duoidal category $\mathscr{C}$ has

$$B \diamond B \xrightarrow{\mu} B \quad \text{and} \quad B \xrightarrow{\delta} B \star B$$

subject (among other axioms) to

$$\begin{array}{ccc}
(B \star B) \diamond (B \star B) & \xrightarrow{\quad\zeta\quad} & (B \diamond B) \star (B \diamond B) \\
\uparrow{\scriptstyle \delta\diamond\delta} & & \downarrow{\scriptstyle \mu\star\mu} \\
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\delta\diamond\delta \uparrow & & & & \downarrow \mu\star\mu \\
B \diamond B & \xrightarrow[\mu]{} & B & \xrightarrow[\delta]{} & B \star B.
\end{array}$$

This is equivalent to either of:

- $\mu : B \diamond B \to B$ is a morphism of $\star$-comonoids,
- $\delta : B \to B \star B$ is a morphism of $\diamond$-monoids.

Beyond bimonoids

What about Hopf monoids?

Things are more subtle: see Gabi's lectures.

Bimonoids: examples

- In a braided monoidal category, the notion of bimonoid acquires its usual meaning:

$$B \otimes B \xrightarrow{\mu} B \quad \text{and} \quad B \xrightarrow{\delta} B \otimes B.$$

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- A bimonoid B in M-graded modules is a semi-Hopf M-algebra in the sense of Turaev. It carries operations

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- A bimonoid in Bimod_K is a bialgebroid with commutative base K.

Duoids

A duoid D in a duoidal category $\mathscr{C}$ has

$$\mathrm{D} \diamond \mathrm{D} \xrightarrow{\mu} \mathrm{D} \quad \text{and} \quad \mathrm{D} \star \mathrm{D} \xrightarrow{\nu} \mathrm{D}$$

subject (among other axioms) to

$$\begin{array}{ccccc}
(\mathrm{D} \star \mathrm{D}) \diamond (\mathrm{D} \star \mathrm{D}) & & \xrightarrow{\zeta} & & (\mathrm{D} \diamond \mathrm{D}) \star (\mathrm{D} \diamond \mathrm{D}) \\
\downarrow{\scriptstyle \nu \diamond \nu} & & & & \downarrow{\scriptstyle \mu \star \mu} \\
\mathrm{D} \diamond \mathrm{D} & \xrightarrow[\mu]{} & \mathrm{D} & \xleftarrow[\nu]{} & \mathrm{D} \star \mathrm{D}.
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\big\downarrow{\scriptstyle \nu \diamond \nu} & & & & \big\downarrow{\scriptstyle \mu \star \mu} \\
D \diamond D & \xrightarrow[\mu]{} & D & \xleftarrow[\nu]{} & D \star D.
\end{array}$$

This is equivalent to:

- $\nu : D \star D \to D$ is a morphism of $\diamond$-monoids.

Note: $\mu : D \diamond D \to D$ is not a morphism of $\star$-monoids. Why?

Duoids in braided monoidal categories

Proposition (Eckmann-Hilton argument).
Let $\mathscr{C}$ be a braided monoidal category.
Let M be a duoid in the associated duoidal category.
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Question.
Does this hold (in some form) in more general duoidal categories?
Answer: next time.

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- A duoid D in M-graded $\mathbb{k}$-modules is an M-graded $\mathbb{k}$-algebra

$$D_p \otimes D_q \to D_{p \cdot q}, \quad a \otimes b \mapsto a \circ b,$$

for which each component D_m is itself a $\mathbb{k}$-algebra

$$D_m \otimes D_m \to D_m, \quad a \otimes a' \mapsto a \bullet b,$$

in such a way that

$$(a \bullet a') \circ (b \bullet b') = (a \circ b) \bullet (a' \circ b').$$

This is the Eckmann-Hilton axiom, but only when $\deg(a) = \deg(a')$ and $\deg(b) = \deg(b')$.

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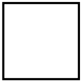
An example when $M = (\mathbb{N}, +)$:
let A be a $\Bbbk$-algebra, define $D_n = A^{\otimes n}$, and

$$(a_1 \otimes \cdots \otimes a_p) \circ (b_1 \otimes \cdots \otimes b_q) = a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q,$$
$$(a_1 \otimes \cdots \otimes a_m) \bullet (b_1 \otimes \cdots \otimes b_m) = a_1 b_1 \otimes \cdots \otimes a_m b_m.$$

Then D is a duoid.

The 2-sphere

Let $S^2 = \mathrm{D}^2/\partial\mathrm{D}^2$. We represent it as a solid square with the boundary identified to a point. This point is the base.



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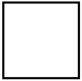

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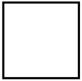

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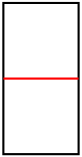
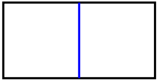
We may represent $S^2 \vee S^2$ either as



or



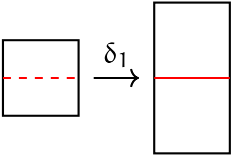
The numbers label the factors in the wedge.

The middle line is identified, along with the boundary, to the base point.

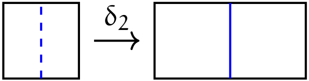
The most famous coduoid

We turn S^2 into a coduoid in the symmetric monoidal (cocartesian) category $(\mathscr{C}, \vee)$.
The counits are the unique maps to the base point.
The coproducts are as follows:

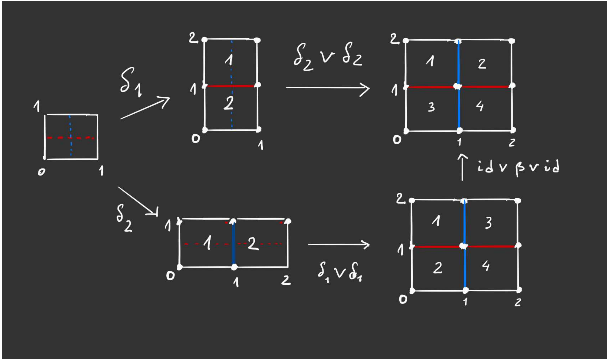
$$\delta_1 : S^2 \to S^2 \vee S^2$$
$$\delta_1(x, y) = (x, 2y)$$



$$\delta_2 : S^2 \to S^2 \vee S^2$$
$$\delta_2(x, y) = (2x, y)$$



The coduoid axiom



Thus, S^2 is a coduoid.
But $\mathscr{C}$ is a symmetric monoidal category.
By Eckmann-Hilton, S^2 is a cocommutative comonoid.

Therefore, $\pi_2(X) = \mathrm{Hom}_{\mathscr{C}}(S^2, X)$ is commutative.

Functors between duoidal categories

Recall: there are 2 kinds of functors between monoidal categories:

monoidal and comonoidal.

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- Each of these kinds is closed under composition.
- Each kind preserves the corresponding class of objects: bimonoidal functors preserve bimonoids, etc.

Duoidal functors

Let $\mathscr{C}$ and $\mathscr{D}$ be duoidal categories.

A functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is duoidal if it is equipped with two monoidal structures

$$\varphi : \mathscr{F}X \diamond \mathscr{F}Y \to \mathscr{F}(X \diamond Y) \qquad \text{and} \qquad \gamma : \mathscr{F}X \star \mathscr{F}Y \to \mathscr{F}(X \star Y).$$

These are subject to various axioms, including:

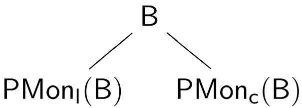
$$\begin{array}{ccc}
\big(\mathscr{F}A \star \mathscr{F}B\big) \diamond \big(\mathscr{F}C \star \mathscr{F}D\big) & \xrightarrow{\zeta} & \big(\mathscr{F}A \diamond \mathscr{F}C\big) \star \big(\mathscr{F}B \diamond \mathscr{F}D\big) \\
\downarrow{\scriptstyle \gamma_{A,B} \diamond \gamma_{C,D}} & & \downarrow{\scriptstyle \varphi_{A,C} \star \varphi_{B,D}} \\
\mathscr{F}(A \star B) \diamond \mathscr{F}(C \star D) & & \mathscr{F}(A \diamond C) \star \mathscr{F}(B \diamond D) \\
\downarrow{\scriptstyle \varphi_{A \star B, C \star D}} & & \downarrow{\scriptstyle \gamma_{A \diamond C, B \diamond D}} \\
\mathscr{F}\big((A \star B) \diamond (C \star D)\big) & \xrightarrow[\mathscr{F}(\zeta)]{} & \mathscr{F}\big((A \diamond C) \star (B \diamond D)\big)
\end{array}$$

The return of pseudomonoids

Recall: given a bicategory B, we may consider pseudomonoids therein.

Let $\mathsf{PMon}_{\mathsf{l}}(\mathsf{B})$ be the category of pseudomonoids and lax morphisms.

Let $\mathsf{PMon}_{\mathsf{c}}(\mathsf{B})$ be the category of pseudomonoids and colax morphisms.



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$$\begin{array}{ccc} & \mathsf{B} & \\ \swarrow & & \searrow \\ \mathsf{PMon}_\mathsf{l}(\mathsf{B}) & & \mathsf{PMon}_\mathsf{c}(\mathsf{B}) \end{array}$$

These are in fact <u>bicategories</u> themselves.

But they are not always <u>monoidal</u> bicategories.

However, if $\mathsf{B} = \mathsf{Cat}$, then they <u>are</u>. So in this case, we can iterate.

Duoidal categories as pseudomonoids

Recall:

- $\mathsf{PMon_l}(\mathsf{Cat})$ consists of monoidal categories and monoidal functors.
- $\mathsf{PMon_c}(\mathsf{Cat})$ consists of monoidal categories and comonoidal functors.

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- $\mathsf{PMon_l}(\mathsf{PMon_l}(\mathsf{Cat}))$
  consists of duoidal categories and duoidal functors,
- $\mathsf{PMon_c}(\mathsf{PMon_l}(\mathsf{Cat})) = \mathsf{PMon_l}(\mathsf{PMon_c}(\mathsf{Cat}))$
  consists of duoidal categories and bimonoidal functors,
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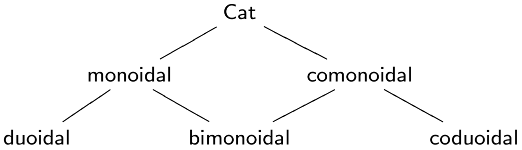

THE HOCHSCHILD COCHAIN COMPLEX
AS A DUOIDAL FUNCTOR

Goal

Let $\Bbbk$ be a commutative ring. Let H be a $\Bbbk$-bialgebra.
View $\Bbbk$ as a trivial H-bimodule.

Fact.
The cup product on the Hochschild cohomology ring $H(H, \Bbbk)$ is commutative:

$$f \smile g = (-1)^{|f|\,|g|} g \smile f.$$

We will explain this fact from the perspective of the Eckmann-Hilton argument.

We will generalize it:

- By extending it to the setting of duoidal categories.
- By (suitably) extending it from the cohomological level to the cosimplicial and cochain levels.
- By extending it to a statement about the Hochschild cochain functor.

Preliminary: bimodules

Let $(\mathscr{C}, \diamond)$ be a monoidal category.

Let H be a monoid in $(\mathscr{C}, \diamond)$.
Let $\mathscr{C}_{\mathrm{H}}$ denote the category of H-bimodules M in $\mathscr{C}$:

$$\mathrm{H} \diamond \mathrm{M} \xrightarrow{\lambda} \mathrm{M}, \quad \mathrm{M} \diamond \mathrm{H} \xrightarrow{\rho} \mathrm{M}.$$

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Suppose all reflexive coequalizers exist in $\mathscr{C}$ and are preserved by $\mathrm{X} \diamond (-)$ and $(-) \diamond \mathrm{X}$ for each X in $\mathscr{C}$. Then:

- $(\mathscr{C}_{\mathrm{H}}, \diamond_{\mathrm{H}})$ is monoidal with $\diamond_{\mathrm{H}}$ defined by

$$\mathrm{M} \diamond \mathrm{H} \diamond \mathrm{N} \underset{\mathrm{id} \diamond \lambda}{\overset{\rho \diamond \mathrm{id}}{\rightrightarrows}} \mathrm{M} \diamond \mathrm{N} \longrightarrow \mathrm{M} \diamond_{\mathrm{H}} \mathrm{N}.$$

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Let H be a monoid in $(\mathscr{C}, \diamond)$.
Let $\mathscr{C}_H$ denote the category of H-bimodules M in $\mathscr{C}$:

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Suppose all reflexive coequalizers exist in $\mathscr{C}$ and are preserved by $X \diamond (-)$ and $(-) \diamond X$ for each X in $\mathscr{C}$. Then:

- $(\mathscr{C}_H, \diamond_H)$ is monoidal with $\diamond_H$ defined by

$$M \diamond H \diamond N \overset{\rho \diamond \mathrm{id}}{\underset{\mathrm{id} \diamond \lambda}{\rightrightarrows}} M \diamond N \longrightarrow M \diamond_H N.$$

Suppose now that $(\mathscr{C}, \diamond, \star)$ and H is a bimonoid therein. Then:

- If M and N are H-bimodules in $(\mathscr{C}, \diamond)$, so is $M \star N$ under

$$H \diamond (M \star N) \xrightarrow{\delta \diamond \mathrm{id}} (H \star H) \diamond (M \star N) \xrightarrow{\zeta} (H \diamond M) \star (H \diamond N) \to M \star N.$$

- $(\mathscr{C}_H, \diamond_H, \star)$ is duoidal.

Two canonical maps

Let $(\mathscr{C}, \diamond, J, \star, K, \zeta)$ be a duoidal category.

Define (after Garner and López Franco) two transformations:

$$\sigma : X \diamond Y \xrightarrow{\cong} (X \star K) \diamond (K \star Y) \xrightarrow{\zeta} (X \diamond K) \star (K \diamond Y),$$
$$\tau : X \diamond Y \xrightarrow{\cong} (K \star X) \diamond (Y \star K) \xrightarrow{\zeta} (K \diamond Y) \star (X \diamond K).$$

Note: when $(\mathscr{C}, \otimes, \beta)$ is braided monoidal,

$$\sigma = \mathrm{id} : X \otimes Y \to X \otimes Y \qquad \text{and} \qquad \tau = \beta : X \otimes Y \to Y \otimes X.$$

Normal duoidal categories

A duoidal category $\mathscr{C}$ is normal if $J = K$.
(More precisely, if $\zeta_0 : J \to K$ is invertible.)

In this case,

$$\sigma : X \diamond Y \to X \star Y \qquad \text{and} \qquad \tau : X \diamond Y \to Y \star X.$$

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Examples of normal duoidal categories.

- Any braided monoidal category.
- The category of pointed sets under coproduct and product:

$$(X, x_0) \bigvee (Y, y_0) = \big((X \sqcup Y)/(x_0 \equiv y_0),\ x_0 \equiv y_0\big),$$
$$(X, x_0) \times (Y, y_0) = \big(X \times Y,\ (x_0, y_0)\big).$$

The common unit object is a singleton.

The duoidal category of cosimplicial modules

Let $\mathscr{D}$ be the category of cosimplicial $\Bbbk$-modules:

$$X = \left(X^n,\, d^i,\, s^j\right), \text{ each } X^n \text{ is a } \Bbbk\text{-module.}$$

The diagonal product is

$$(X \times Y)^n = X^n \otimes Y^n.$$

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There is another monoidal structure on $\mathscr{D}$ defined by Batanin:

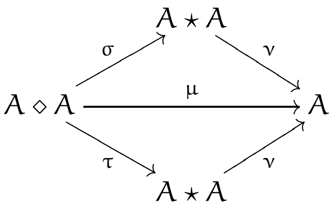
$$(X \diamond Y)^n = \bigoplus_{i+j=n} X^i \otimes Y^j \Big/ \Big\langle d^i x \otimes y - x \otimes d^0 y : x \in X^{i-1},\ y \in Y^j \Big\rangle.$$

Proposition. $(\mathscr{C}, \diamond, \times)$ is duoidal and normal.
Remark. $J = K =$ the constant cosimplicial module $\mathbb{k}$.

Duoids in normal duoidal categories

Proposition (Garner and López Franco).

Let (A, μ, ν) be a duoid in a normal duoidal category. Then:



When $(\mathscr{C}, \otimes, \beta)$ is braided monoidal, this is Eckmann-Hilton.

Exercise.

- What is a duoid in the category of pointed sets?
- What are σ and τ?
- What does the proposition say?

Let $\mathscr{C}$ and $\mathscr{D}$ be duoidal categories.

Recall that $(\mathscr{F}, \varphi, \gamma) : \mathscr{C} \to \mathscr{D}$ is duoidal if both $(\mathscr{F}, \varphi)$ and $(\mathscr{F}, \gamma)$ are monoidal and

$$\begin{array}{ccc}
\big(\mathscr{F}A \star \mathscr{F}B\big) \diamond \big(\mathscr{F}C \star \mathscr{F}D\big) & \xrightarrow{\ \zeta\ } & \big(\mathscr{F}A \diamond \mathscr{F}C\big) \star \big(\mathscr{F}B \diamond \mathscr{F}D\big) \\
{\scriptstyle \gamma_{A,B} \diamond \gamma_{C,D}} \big\downarrow & & \big\downarrow {\scriptstyle \varphi_{A,C} \star \varphi_{B,D}} \\
\mathscr{F}(A \star B) \diamond \mathscr{F}(C \star D) & & \mathscr{F}(A \diamond C) \star \mathscr{F}(B \diamond D) \\
{\scriptstyle \varphi_{A\star B, C\star D}} \big\downarrow & & \big\downarrow {\scriptstyle \gamma_{A\diamond C, B\diamond D}} \\
\mathscr{F}\big((A \star B) \diamond (C \star D)\big) & \xrightarrow[\mathscr{F}(\zeta)]{} & \mathscr{F}\big((A \diamond C) \star (B \diamond D)\big)
\end{array}$$

Eckmann-Hilton for duoidal functors

Proposition (A.-Cóppola).
Let $\mathscr{C}$ and $\mathscr{D}$ be normal duoidal categories.
Let $(\mathscr{F}, \varphi, \gamma) : \mathscr{C} \to \mathscr{D}$ be a duoidal functor. Then:

$$
\begin{array}{ccc}
\mathscr{F}X \diamond \mathscr{F}Y & \xrightarrow{\sigma} & \mathscr{F}X \star \mathscr{F}Y \\
\varphi \downarrow & \text{(A)} & \downarrow \gamma \\
\mathscr{F}(X \diamond Y) & \xrightarrow[\mathscr{F}(\sigma)]{} & \mathscr{F}(X \star Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{F}X \diamond \mathscr{F}Y & \xrightarrow{\tau} & \mathscr{F}Y \star \mathscr{F}X \\
\varphi \downarrow & \text{(B)} & \downarrow \gamma \\
\mathscr{F}(X \diamond Y) & \xrightarrow[\mathscr{F}(\tau)]{} & \mathscr{F}(Y \star X)
\end{array}
$$

When $\mathscr{C}$ is the unit duoidal category, this recovers the result of Garner and López Franco.

Eckmann-Hilton converse

Theorem (A.-Cóppola).
Let $\mathscr{C}$ and $\mathscr{D}$ be normal duoidal categories.
Let $(\mathscr{F}, \varphi, \gamma) : \mathscr{C} \to \mathscr{D}$ be such that:

- both $(\mathscr{F}, \varphi)$ and $(\mathscr{F}, \varphi, \gamma)$ are monoidal,
- the transformation $\mathscr{F}(\sigma \star \sigma)$ is monic,
- axioms (A) and (B) hold.

Then $(\mathscr{F}, \varphi, \gamma)$ is duoidal.

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Back to the source:
an operation • that is both commutative and associative satisfies

$$(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d).$$

The Hochschild complex

Let $(\mathscr{C}, \diamond)$ be a $\mathbb{k}$-linear monoidal category.
Let H be a monoid and A an H-bimodule in $(\mathscr{C}, \diamond)$.

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Suppose A is in addition a monoid in $(\mathscr{C}, \diamond)$.
The cup product of cochains $f \in \mathscr{H}^p(H, A)$ and $g \in \mathscr{H}^q(H, A)$ is

$$f \smile g : H^{\diamond(p+q)} = H^{\diamond p} \diamond H^{\diamond q} \xrightarrow{f \diamond g} A \diamond A \to A.$$

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Instead of focusing on $\mathscr{H}(H, A)$, we consider the functor $\mathscr{H}(H, -)$.

The Hochschild complex of a bimonoid

Let $(\mathscr{C}, \diamond, \star)$ be a duoidal category.
Let H be a bimonoid in $(\mathscr{C}, \diamond, \star)$.
We consider the functors

$$\mathscr{C}_{\mathrm{H}} \xrightarrow{\mathscr{H}(\mathrm{H},-)} \mathscr{E} \twoheadrightarrow \overline{\mathscr{E}},$$

where

- $\mathscr{C}_{\mathrm{H}}$ is the category of H-bimodules in $(\mathscr{C}, \diamond)$.
- $\mathscr{E}$ is the category of cochain complexes of $\mathbb{k}$-modules.
- $\overline{\mathscr{E}}$ is the homotopy category of such complexes.

We analyze the monoidal properties of these functors.

Cochain complexes

Let $\mathscr{E}$ be the category of cochain $\mathbb{k}$-complexes:

$$C = (C^n, d^n).$$

For each $n \geq 0$, C^n is a $\mathbb{k}$-module, and

$$d^n : C_n \to C_{n+1}$$

is a collection of $\mathbb{k}$-module morphisms such that $d^{n+1}d^n = 0$.

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The category $\mathscr{E}$ is symmetric monoidal under

$$(C \bullet D)^n = \bigoplus_{i=0}^{n} C^i \otimes D^{n-i},$$

$$d^n(x \otimes y) = d^i(x) \otimes y + (-1)^i x \otimes d^{n-i}(y)$$

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Recall the operations $\diamond$ and $\star$ for cosimplicial modules.

Which one resembles $\bullet$?

The Hochschild complex of a bimonoid

Recall:

- $\mathscr{C}_{\mathrm{H}}$, the category of H-bimodules in $(\mathscr{C}, \diamond)$.
- $\mathscr{D}$, the category of cosimplicial $\mathbb{k}$-modules.
- $\mathscr{E}$, the category of cochain complexes of $\mathbb{k}$-modules.

They are all duoidal. $\mathscr{D}$ is normal, while $\mathscr{E}$ is symmetric monoidal.

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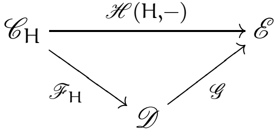
The Hochschild complex arises from a cosimplicial $\mathbb{k}$-module.
In other words:

The Hochschild functor factors:



We focus on these two functors.

The Hochschild complex of a bimonoid

We focus on $\mathscr{F}_{\mathrm{H}} : \mathscr{C}_{\mathrm{H}} \to \mathscr{D}$.

The degree n component of $\mathscr{F}_{\mathrm{H}}(M)$ is

$$\mathrm{Hom}_{\mathscr{C}}(H^{\diamond n}, M).$$

Proposition (A.-Cóppola).
$\mathscr{F}_{\mathrm{H}}$ carries two monoidal structures:

$$\mathscr{F}_{\mathrm{H}}M \diamond \mathscr{F}_{\mathrm{H}}N \xrightarrow{\varphi} \mathscr{F}_{\mathrm{H}}(M \diamond_{\mathrm{H}} N), \quad \mathscr{F}_{\mathrm{H}}M \times \mathscr{F}_{\mathrm{H}}N \xrightarrow{\gamma} \mathscr{F}_{\mathrm{H}}(M \star N).$$

- φ is concatenation of cochains.
- γ is convolution of cochains using $\delta : H \to H \star H$.

With these structures, $(\mathscr{F}_{\mathrm{H}}, \varphi, \gamma)$ is duoidal.

The cochain complex of a cosimplicial module

We focus on $\mathscr{G} : \mathscr{D} \to \mathscr{E}$.

This functor simply builds the differential from the coface maps:

$$(X^n, d^i, s^j) \overset{\mathscr{G}}{\longmapsto} (X^n, \sum_{i=0}^{n+1} (-1)^i d^i).$$

Proposition (A.-Cóppola).

- $\mathscr{G}$ carries 3 monoidal structures:

$$\mathscr{G}X \bullet \mathscr{G}Y \xrightarrow{\varphi} \mathscr{G}(X \diamond Y), \quad \mathscr{G}X \bullet \mathscr{G}Y \xrightarrow{\mathrm{AW}} \mathscr{G}(X \times Y), \quad \mathscr{G}X \bullet \mathscr{G}Y \xrightarrow{\widetilde{\mathrm{AW}}} \mathscr{G}(X \times Y).$$

  - φ is strong.
  - AW and $\widetilde{\mathrm{AW}}$ are the two versions of Alexander-Whitney (front-back and back-front).

- $(\mathscr{G}, \varphi, \mathrm{AW})$ satisfies axiom (A), while $(\mathscr{G}, \varphi, \widetilde{\mathrm{AW}})$ satisfies axiom (B).

This last statement was the crux of the matter.

Up to homotopy

Classical facts. Dold: AW and $\widetilde{\mathrm{AW}}$ coincide up to homotopy.
Eilenberg-Zilber: AW is invertible up to homotopy.

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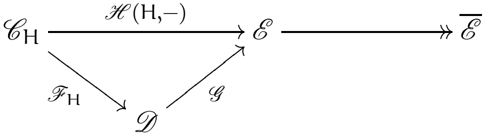

Corollary. (A.-Cóppola).

- The functor $\mathscr{D} \xrightarrow{\mathscr{G}} \mathscr{E} \twoheadrightarrow \overline{\mathscr{E}}$ is duoidal.
- The functor $\mathscr{C}_{\mathrm{H}} \xrightarrow{\mathscr{H}(\mathrm{H},-)} \mathscr{E} \twoheadrightarrow \overline{\mathscr{E}}$ is duoidal.

Proof. When we pass to $\overline{\mathscr{E}}$, axioms (A) and (B) hold for AW by Dold. Then we apply the EH converse for duoidal functors. (We use EZ for the hypothesis on monomorphisms.)

Commutativity of the cup product

Lemma. Let H be a bimonoid and D a duoid in $(\mathscr{C}, \diamond, \star)$.
View D as a trivial H-bimodule via

$$\mathrm{H} \xrightarrow{\delta_0} \mathrm{K} \xrightarrow{\nu_0} \mathrm{D}.$$

Then D is a duoid in $\mathscr{C}_{\mathrm{H}}$. Moreover, every duoid in $\mathscr{C}_{\mathrm{H}}$ is of this form.

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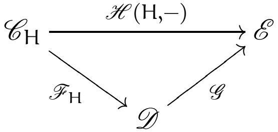
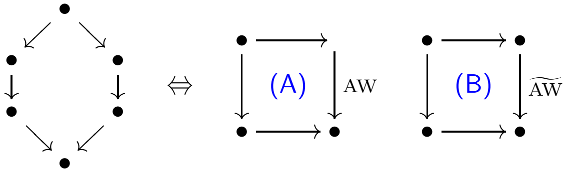


Proposition (A.-Cóppola)
Let H and D be as above.
Then the Hochschild cochain complex $\mathscr{H}(\mathrm{H}, \mathrm{D})$ carries two products which, up to homotopy, coincide and are commutative.

Proof. The duoidal functor $\mathscr{C}_{\mathrm{H}} \xrightarrow{\mathscr{H}(\mathrm{H},-)} \mathscr{E} \to \overline{\mathscr{E}}$ sends the duoid D in $\mathscr{C}_{\mathrm{H}}$ to a duoid $\mathscr{H}(\mathrm{H}, \mathrm{D})$ in $\overline{\mathscr{E}}$. Since the latter category is braided monoidal, Eckmann-Hilton applies.

Comments

- The classical proofs of the commutativity of the cup product (at the level of cohomology) rely on Dold's fact about AW. Our proof of commutativity (up to homotopy of cochains) does too.
- We obtain the more general property that the Hochschild functor $\mathscr{H}(\mathrm{H}, -)$ is duoidal, up to homotopy.
- The initial obstacle for proving this is: how to prove that a noncommutative diagram actually commutes up to homotopy? Constructing explicit chain-homotopies is challenging.
- We circumvent this by applying the EH converse for duoidal functors. Diagrams (A) and (B) do commute on the nose. They break the lack of commutativity into two different pieces that agree up to homotopy, by Dold.



References

- Monoidal functors, species and Hopf algebras,
  Marcelo Aguiar and Swapneel Mahajan,
  CRM-AMS Monograph Series, Vol. 29 (2010).
- The Eckmann-Hilton argument in duoidal categories,
  Marcelo Aguiar and Javier Cóppola,
  in preparation.
- Understanding the small object argument,
  Richard Garner,
  Appl. Categ. Structures 17 (2009) n3, 247–285.
- Commutativity,
  Richard Garner and Ignacio López Franco,
  J. Pure Appl. Algebra 220 (2016) n5, 1707–1751.
- Monoidal categories in, and linking, geometry and algebra,
  Ross Street,
  Bull. Belg. Math. Soc. Simon Stevin 19 (2012), 769–821.