1. Let A be a square matrix of odd size. Prove that A is singular if and only if it can be carried to $-A$ by elementary operations of adding a multiple of one row to another row.

2. Let ξ be a primitive complex n-th root of unity, $n \geq 2$. Determine the n-th power of the following $n \times n$ matrix:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \xi & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \xi^2 & -1 & 0 & \dots & 0 & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & \xi^{n-2} & -1 \\ -1 & 0 & 0 & 0 & 0 & \dots & 0 & \xi^{n-1} \end{pmatrix}.$$

3. Harry Potter and Voldemort are playing a game by filling the squares of a chess board with integers as follows. First Voldemort fills the dark squares and then Harry must fill the light squares so that the resulting 8×8 matrix has rank r. For what values of r does Harry have a winning strategy?

4. Let $f(z)$ be a polynomial function with complex coefficients. Prove that if f maps the set of all complex roots of unity to itself, then f is a monomial.

5. Let p be a prime. For any integers $a_0, a_1, \dots, a_p$, prove that

$$\sum a_{i_1} \cdots a_{i_p} \equiv a_1^p \pmod{p},$$

where the summation is over all $i_1, \dots, i_p \geq 0$ such that $i_1 + \cdots + i_p = p$.

6. Let G be a finite simple group. Suppose G contains a subgroup of prime index p. Prove that p is the largest prime divisor of $|G|$ and, moreover, p^2 does not divide $|G|$.