1. We will say that a matrix is *sensitive* if its rank changes upon any change of any of its entries. What are the possible ranks of sensitive $n \times n$ matrices

   (a) over the field of complex numbers?

   (b) over an arbitrary field?

2. Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix whose entries satisfy (i) $a_{ii} = 1$ and (ii) $\sum_{j=1}^{n} |a_{ij}| \leq 2$ for all i. Prove that $0 \leq \det A \leq 1$.

3. Let $R = \mathbb{Z}/m\mathbb{Z}$, the ring of residues modulo m ($m > 1$). If $a \in \mathbb{Z}$ is coprime to m, then the map $f_a(x) = ax$ is a bijection $R \to R$, so f_a can be regarded as a permutation of m symbols. Let $\sigma(m, a)$ be the sign of this permutation.

   (a) Show that if $m = 2^\alpha k$ where k is odd and $\alpha \geq 1$, then $\sigma(m, a) = \sigma(2^\alpha, a)$ for all a coprime to m.

   (b) Determine $\sigma(2^\alpha, a)$ as a function of α and a.

4. Show that if a field $\mathbb{K}$ is not algebraically closed, then the solution set in $\mathbb{K}^n$ of any system of equations

$$f_1(x_1, \ldots, x_n) = \ldots = f_m(x_1, \ldots, x_n) = 0,$$

   where $f_1, \ldots, f_m$ are polynomials in n variables over $\mathbb{K}$, coincides with the solution set of one equation $F(x_1, \ldots, x_n) = 0$, for some polynomial F in n variables over $\mathbb{K}$. [For example, if $\mathbb{K} = \mathbb{R}$, then we can take $F = f_1^2 + \cdots + f_m^2$.]

5. We will say that a finite nonzero associative commutative ring (possibly without identity element) is *magical* if the product of all its nonzero elements is not equal to 0 or -1 (if the identity element exists). Find all magical rings.

6. Let G be a group and let e be its identity element. We will say that an element $a \in G$ is *engaged* if a commutes with exactly three elements: e, a and some element b (distinct from e and a). If this is the case, we will also say that a is engaged to b.

   (a) Prove that the relation *engaged to* is symmetric: if a is engaged to b, then b is engaged to a.

   (b) Prove that if G is a finite group, then one of the following three possibilities takes place: (i) there are no engaged elements, (ii) exactly one third of the elements are engaged, (iii) exactly two thirds of the elements are engaged.

   (c) Give examples of groups that realize each possibility in part (b).