Second Undergraduate Algebra Competition

Atlantic Algebra Centre invites all undergraduate students from universities of Atlantic Canada to participate in the second Undergraduate Algebra Competition. Solutions are due on March 20, 2008. The solutions can be sent by e-mail to `aac@math.mun.ca` or by regular mail, postmarked no later than March 20, to the following address:

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St. Johns, NL, A1C 5S7.

The three winners of the competition will be determined by April 10, 2008, and they will be awarded book prizes.

1. Let P be a square matrix with complex entries. Prove that P has the property $P^2 = P$ if and only if $\mathrm{rk}P = \mathrm{tr}P$ and $\mathrm{rk}(I-P) = \mathrm{tr}(I-P)$.
2. Student X. decided to compute the 100-th powers of all 17×17 matrices over the field of 17 elements and see what their sum would be, but at that moment his computer broke. Help the student.
3. Let K be a field.
   (a) Prove that any subalgebra of $K[x]$ is finitely generated.
   (b) Is the same statement true for $K[x, y]$?
4. Let R be a commutative ring with 1. As usual, for $a, b \in R$, $a|b$ means $b = ax$ for some $x \in R$. We will write $a \sim b$ if $b = au$ for some invertible element $u \in R$. Let $\mathcal{S}$ be the statement: "If $a|b$ and $b|a$, then $a \sim b$". It is easy to see that $\mathcal{S}$ holds if R is an integral domain.
   (a) Prove that $\mathcal{S}$ holds in the ring $\mathbb{Z}_m$ of integers modulo m, for any $m \geq 2$.
   (b) Does $\mathcal{S}$ hold in $\mathbb{Z}_m[x]$?
5. Let G be a group. Suppose m and n are relatively prime integers such that $x^n y^n = y^n x^n$ and $x^m y^m = y^m x^m$ for all $x, y \in G$. Prove that G is abelian.
6. A group G acts on a set such that any non-identity element has a unique fixed point.
   (a) Suppose G is finite. Show that the fixed point is the same for all non-identity elements of the group.
   (b) Is the same statment true for infinite groups?