Representation theory of Lie algebra of vector fields on a torus.

We begin in a general setting of a real C^∞-manifold M, $\dim M = n$.

Consider the group $\mathrm{Diff}(M)$.

1-parametric subgroup in $\mathrm{Diff}(M)$ is a flow on M.

Elements of the Lie algebra of $\mathrm{Diff}(M)$ are vector fields on M. A vector field acts on functions via the directional derivative. Thus we can write vector fields as the first order differential operators.

Let $\Phi \in \mathrm{Diff}(M)$. Φ acts on functions:

$$(\Phi f)(p) = f(\Phi^{-1}(p)), \quad p \in M$$

In local coordinates,

$$\Phi(x_i) = g_i(x_1, \dots, x_n), \quad i = 1, \dots, n$$

$$\mathrm{Vect}(M): \quad \eta = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}$$

Diff(M) acts not only on functions, but also on tensor fields on M.

(0,1) - tensors : $\Omega^1(M)$ - differential 1-forms

$$\Phi\left(h(x_1,\dots,x_n)\,dx_j\right) = h(g_1(x),\dots,g_n(x))\,d(g_j(x))$$

$$\eta\left(h(x_1,\dots,x_n)\,dx_j\right) = (\eta h)\,dx_j + h(x)\,d(\eta x_j)$$

Let $\eta = f(x)\frac{\partial}{\partial x_a}$, then

$$d(\eta x_j) = \delta_{aj}\, d(\eta) = \sum_{p=1}^{n} \frac{\partial f}{\partial x_p}\,\delta_{aj}\,dx_p$$

$$= \sum_{p=1}^{n} \frac{\partial f}{\partial x_p}\,\rho_V(E_{pa})\,dx_j\,,$$

where ρ_V is the natural representation of gl_n on $V = \mathrm{Span}\{dx_1,\dots,dx_n\}$.

(1,0) - tensors : vector fields on M :

$$\eta\left(h(x)\frac{\partial}{\partial x_j}\right) = \left(f\frac{\partial h}{\partial x_a}\right)\frac{\partial}{\partial x_j} - \left(h\frac{\partial f}{\partial x_j}\right)\frac{\partial}{\partial x_a}$$

$$= (\eta h)\frac{\partial}{\partial x_j} + \sum_{p=1}^{n} h\frac{\partial f}{\partial x_p}\,\rho_{V^*}(E_{pa})\frac{\partial}{\partial x_j}\,,$$

where ρ_{V^*} is the dual of the natural representation of gl_n:

$$\rho_{V^*}(E_{pa})\frac{\partial}{\partial x_j} = -\delta_{pj}\frac{\partial}{\partial x_a}.$$

Likewise, the action of vector fields on (m,k)-tensors is given in terms of the gl_n-module

$$\underbrace{V\otimes\ldots\otimes V}_{k}\otimes\underbrace{V^*\otimes\ldots\otimes V^*}_{m}$$

$$f\frac{\partial}{\partial x_a}\left(h(x)w\right) = f\frac{\partial h}{\partial x_a}w + h\sum_{p=1}^{n}\frac{\partial f}{\partial x_p}\rho(E_{pa})w$$

Now let us focus on the case $M=\mathbb{T}^n$

Functions on $\mathbb{T}^n$: 2π-periodic functions in n variables and may be represented by Fourier polynomials. Even though $\mathbb{T}^n$ is a real manifold, we complexify all vector bundles and consider complex-valued functions, 1-forms, vector fields, etc.

We take $t_j = e^{ix_j}$ as generators. Then

$$\frac{\partial}{\partial x_j} = \frac{\partial t_j}{\partial x_j}\frac{\partial}{\partial t_j} = i\, t_j \frac{\partial}{\partial t_j} = i d_j$$

We can ~~simplify~~ consider an algebraic version of these objects by taking Fourier polynomials $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ instead of C^∞-periodic functions; polynomial vector fields,

$$\mathrm{Vect}(\mathbb{T}^n) = \mathrm{Der}\, \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

$$= \bigoplus_{p=1}^{n} \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]\, d_p \text{ , etc.}$$

$$[t^m d_a, t^s d_b] = s_a t^{m+s} d_b - m_b t^{m+s} d_a.$$

Note that in the infinite-dimensional case there are difficulties with the Lie correspondence between Lie groups and Lie algebras. In particular, $\mathrm{Diff}(\mathbb{T}^n)$ does not have an algebraic version.

<u>Goal</u>: Study the representation theory of the Lie algebra $W_n = \mathrm{Vect}(\mathbb{T}^n)$ of polynomial vector fields on a torus.

We need to impose restrictions on the class of representations we consider, otherwise this is not attainable.

The Lie algebra $\mathrm{Vect}(\mathbb{T}^n)$ has a Cartan subalgebra $\langle d_1, \dots, d_n \rangle$. The action is diagonalizable:

$$[d_a, t^m d_b] = m_a t^m d_b.$$

The actiom of the Cartan subalgebra induces a $\mathbb{Z}^n$-grading on $\mathrm{Vect}(\mathbb{T}^n)$ with n-dim. components

$$\mathrm{Vect}(\mathbb{T}^n)_m = \bigoplus_{p=1}^{n} t^m d_p \, , \quad m \in \mathbb{Z}^n.$$

<u>Definition</u> A $\mathrm{Vect}(\mathbb{T}^n)$-module N is called a weight module if

$$N = \bigoplus_{m \in \mathbb{C}^n} N_m \, , \text{ where}$$

$$N_m = \{ v \in N \mid \rho(d_j) v = m_j \cdot v \, , \, j = 1 \dots n \}.$$

<u>Problem</u>: Classify irreducible weight modules for $\mathrm{Vect}(\mathbb{T}^n)$ with finite-dimensional weight spaces.

Define the support of a weight module N as $\{m \in \mathbb{C}^n \mid \dim N_m > 0\}$.

<u>Remark</u>: If N is an irreducible $\mathrm{Vect}(\mathbb{T}^n)$-module then $\mathrm{supp}(N) \subset \alpha + \mathbb{Z}^n$ for some $\alpha \in \mathbb{C}^n$.

Examples of weight modules with f.dim. weight spaces: modules of tensor fields

Let W be a fin. dim. gl_n-module, $\alpha \in \mathbb{C}$.

Consider $T(W, \alpha) = t^{\alpha} \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes W$

$$(t^m d_a)(t^r \otimes w) = r_a t^{m+r} \otimes w + \sum_{p=1}^{n} m_p t^{r+m} \otimes E_{pa} w$$

An obvious necessary condition for $T(W, \alpha)$ to be irreducible as a $\mathrm{Vect}(\mathbb{T}^n)$-mod is that W is an irreducible gl_n-module.

<u>Theorem</u> (Rao'96 / Rudakov'74)

Let W be a finite-dimensional irreducible gl_n-module. Then $T(W, \alpha)$ is irreducible as a $\mathrm{Vect}(\mathbb{T}^n)$-module, except when it appears in de Rham complex:

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n$$

Here $\Omega^k = T(\Lambda^k V, \alpha)$

The differential $d : \Omega^k \longrightarrow \Omega^{k+1}$ is given by

$$d\left(e^{imx}\, dx_{j_1} \wedge \ldots \wedge dx_{j_k}\right) = \sum_{p=1}^{n} m_p\, e^{imx}\, dx_p \wedge dx_{j_1} \ldots$$

The map d is a homomorphism of $\mathrm{Vect}(\mathbb{T}^n)$-modules, hence $\mathrm{Ker}\, d$ and $\mathrm{Im}\, d$ are submodules. As a result, all modules in the middle, Ω^k, $k = 1, \ldots, n-1$, are reducible, while the modules at the ends are reducible when $\alpha \in \mathbb{Z}^n$.

<u>Example</u> $\Omega^0 = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ has a submodule of constant functions and $\Omega^0 / \mathbb{C}1$ is irreducible. This quotient has a hole at $m = 0$.

Definition A module M is an AW_n-module if it is a module for the Lie algebra W_n and for the commutative unital algebra A, with two actions being compatible:

$$\eta(fv) = (\eta f)v + f(\eta v),$$
$$\eta \in W_n,\ f \in A,\ v \in M.$$

Let M be an AW_n-module, which is a weight module with $\operatorname{supp} M \subset \alpha + \mathbb{Z}^n$.

Then M is frelly generated by M_α as an A-module:

$$M \cong A \otimes M_\alpha.$$

If $\dim M_\alpha < \infty$ then all weight spaces have the same dimension.

Consider the operators $D_a(s) : M_\alpha \to M_\alpha$,

$$D_a(s) = t^{-s} \circ (t^s d_a).$$

We can recover AW_n-action on M from these operators

Let us compute the Lie bracket for the operators $D_a(s)$:

$$[D_a(s), D_b(m)] = t^{-s} \circ t^s d_a \circ t^{-m} \circ t^m d_b$$

$$- t^{-m} \circ t^m d_b \circ t^{-s} \circ t^s d_a$$

$$= -m_a\, t^{-s} \circ t^{s-m} \circ t^m d_b + s_b\, t^{-m} \circ t^{m-s} \circ t^s d_a$$

$$+ t^{-s-m} \circ [t^s d_a, t^m d_b]$$

$$= m_a\,(D_b(s+m) - D_b(m)) - s_b\,(D_a(s+m) - D_a(s))$$

<u>Theorem</u>. The family of operators $\{D_a(s) \mid s \in \mathbb{Z}\}$ has a polynomial dependence on s.

<u>Proof</u> We consider the case $n=1$

Let $F(k) = \sum_{i=0}^{k+1} (-1)^{k+1-i} \binom{k+1}{i} D(-1+i)$.

Then $[F(k), F(m)] = (m-k)\,F(m+k)$.

In particular, $[F(0), F(m)] = m\,F(m)$.

We can view $\operatorname{ad} F(0)$ as an operator on a finite-dimensional space $\operatorname{End}(M_\alpha)$.

An operator on a finite-dimensional vector space may have only a finite number of eigenvalues. Thus

$$F(k) = \sum_{i=0}^{k+1} (-1)^{k+1-i} \binom{k+1}{i} D(-1+i) = 0$$

for $k \gg 0$.

These expressions are discrete derivatives of $D(s)$. Hence $D(s)$ is a polynomial in s.

<u>Corollary</u> Let M be an AW_n-module with finite-dimensional weight spaces, $\operatorname{supp} M \subset \alpha + \mathbb{Z}$. Then M_α admits the action of the Lie alg. $\operatorname{Der}_+ \mathbb{C}[z_1, \dots, z_n]$, $M \cong A \otimes M_\alpha$ and the action of W_n is given by

$$(t^s d_p)(t^m \otimes w) = (m_p + \alpha_p)\, t^{m+s} \otimes w$$

$$+ \; t^{m+s} \otimes \sum_{k \in \mathbb{Z}_+^n \setminus \{0\}} \frac{s^k}{k!} \rho\left(z^s \frac{\partial}{\partial z_p}\right) w$$

<u>Corollary</u> Irreducible AW_n-modules with finite-dim weight spaces are parametrized by simple fin. dim. $gl_n(\mathbb{C})$-modules.

Mazorchuk - Zhao (2011) classified supports of irreducible W_n-modules with finite-dim weight spaces:

supp $M =$ (1) $\alpha + \mathbb{Z}^n$

(2) $\mathbb{Z}^n \setminus \{0\}$

(3) $\alpha + \{ s \in \mathbb{Z}^n \mid s \cdot r \geq 0 \}$ for some $r \in \mathbb{Z}^n$

(4) $\{ s \in \mathbb{Z}^n \setminus \{0\} \mid s \cdot r \geq 0 \}$

(5) $\{0\}$

Moreover they showed that in cases (1) and (2) dimensions of weight spaces have a global bound.

In cases (3) and (4) there is a global bound on dimensions of weight spaces with $s \cdot r = 0$.

Def. We call a weight W_n-module cuspidal if there is a global bound on dimensions of weight spaces.

Example: AW_n-modules studied above

We want to classify simple cuspidal W_n-modules. A technical obstruction is the existence of modules with "holes".

Theorem (B.-Futorny, 2013) Let M be a cuspidal simple W_n-module. Then there exists a cuspidal AW_n-module $\hat{M}$ and a surjective homomorphism of W_n-modules

$$\pi : \hat{M} \longrightarrow M.$$

Corollary Classification of cuspidal simple W_n-modules:

(1) $t^{\alpha} \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes W$, where W is a simple f.dim. $gl_n(\mathbb{C})$-module, different from $\Lambda^k V$.

(2) $d\,\Omega^k(\alpha)$, $k=0,\dots,n-1$

(3) Trivial 1-dimensional module

Construction of the cover $\hat{M}$.

We need to pass from the category of W_n-modules to the category of AW_n-modules. We use the coinduction functor.

Definition / Theorem The coinduced module is the space $Hom_{\mathbb{C}}(A,M)$, which is an AW_n-module with the action given by

$$(\eta\varphi)(f) = \eta(\varphi(f)) - \varphi(\eta(f)),$$
$$(g\varphi)(f) = \varphi(gf),$$

$\varphi \in Hom(A,M)$, $\eta \in W_n$, $g, f \in A$.

There is a projection $\pi: Hom(A,M) \to M$ by $\varphi \mapsto \varphi(1)$.

The problem with the coinduced module is that it is too big. To define the cover $\hat{M}$ we exploit the fact that A is W_n-module, but W_n is A-module.

Definition An A-cover $\hat{M}$ of a W_n-module M is the following submodule in $Hom(A, M)$:

$$\hat{M} = Span\{\psi(\eta, v)\}_{\eta \in W_n, v \in M}, \text{ where}$$

$$\psi(\eta, v) \in Hom(A, M)$$

$$\psi(\eta, v)(f) = (f\eta)v.$$

Proposition (1) AW_n-action on $\hat{M}$ is given by

$$\tau\psi(\eta, v) = \psi([\tau, \eta], v) + \psi(\eta, \tau v)$$

$$g\psi(\eta, v) = \psi(g\eta, v)$$

(2) If M is a weight module, ~~so~~ then so is $\hat{M}$

(3) $\pi(\hat{M}) = W_n \cdot M$

Theorem If M is cuspidal then $\hat{M}$ is also cuspidal.

Let us sketch the proof of this theorem in the case $n=1$.

Mathieu (1992) classified simple W_1-modules with finite-dimensional weight spaces:

(1) Tensor modules

(2) $\mathbb{C}[t,t^{-1}]/\mathbb{C}1$

(3) Highest weight modules

(4) Lowest weight modules.

In particular, simple cuspidal W_1 modules are tensor modules or their quotients.

Tensor modules $T(\alpha,\beta)$, $\alpha,\beta \in \mathbb{C}$ have bases $\{v_k \mid k \in \alpha + \mathbb{Z}\}$ and W_1-action:

$$e_m \cdot v_k = (k+\beta m)\, v_{k+m}$$

Here $e_m = t^m d$.

Define the following operators in $U(W_1)$

$$\Omega^{(m)}_{k,s} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} e_{k-i}\, e_{s+i}\, .$$

Lemma $\Omega^{(3)}_{k,s}$ annihilates $T(\alpha,\beta)$.

Theorem Let M be a cuspidal W_1-module. Then there exists $m\in\mathbb{N}$ such that $\Omega^{(m)}_{k,s}$ annihilate M for all $k,s\in\mathbb{Z}$.

The proof of this is based on the following key algebraic identity

$$\sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j}\left(\left\{\Omega^{(m)}_{k-i,s-j},\Omega^{(m)}_{q+i,p+j}\right\}\right.$$
$$\left.-\left\{\Omega^{(m)}_{k-i,q-j},\Omega^{(m)}_{s+i,p+j}\right\}\right)=(q-s)(p-k+2m)\times$$
$$\Omega^{(4m)}_{k+p+2m,\,s+q-2m}$$

for $m\geq 2$

Here $\{A,B\}=AB+BA$.

Proof: $\displaystyle LHS=\sum_{i,j,a,b=0}^{m}(-1)^{i+j+a+b}\binom{m}{i}\binom{m}{j}\binom{m}{a}\binom{m}{b}$

$$\Big(e_{k-i-a}e_{s-j+a}e_{q+i-b}e_{p+j+b}+e_{q+i-b}e_{p+j+b}e_{k-i-a}e_{s-j+a}$$

$$-e_{k-i-a}e_{q-j+a}e_{s+i-b}e_{p+j+b}-e_{s+i-b}e_{p+j+b}e_{k-i-a}e_{q-j+a}\Big)$$

Switch $a\leftrightarrow i$, $b\leftrightarrow j$ in the last two terms:

$$= \sum_{i,j,a,b=0}^{m} (-1)^{i+j+a+b} \binom{m}{i}\binom{m}{j}\binom{m}{a}\binom{m}{b} \times$$

$$\Big(e_{k-i-a}\, e_{s-j+a}\, e_{q+i-b}\, e_{p+j+b} + e_{q+i-b}\, e_{p+j+b}\, e_{k-i-a}\, e_{s-j+a}$$

$$- e_{k-i-a}\, e_{q-b+i}\, e_{s+a-j}\, e_{p+b+j} - e_{s+a-j}\, e_{p+b+j}\, e_{q-b+i}\, e_{k-i-a} \Big)$$

$$= \sum_{i,j,a,b} (-1)^{i+j+a+b} \binom{m}{i}\binom{m}{j}\binom{m}{a}\binom{m}{b} \times$$

$$\Big(e_{k-i-a} \left[e_{s-j+a}, e_{q+i-b} \right] e_{p+j+b}$$

$$+ e_{p+j+b}\, e_{k-i-a} \left[e_{q+i-b}, e_{s-j+a} \right]$$

$$+ \left[e_{q+i-b}, e_{p+j+b} \right] e_{k-i-a}\, e_{s-j+a}$$

$$+ e_{p+j+b} \left[e_{q+i-b}, e_{k-i-a} \right] e_{s-j+a}$$

$$+ e_{p+j+b} \left[e_{k-i-a}, e_{s-j+a} \right] e_{q+i-b}$$

$$+ \left[e_{p+j+b}, e_{s-j+a} \right] e_{k-i-a}\, e_{q+i-b} \Big)$$

all vanish

For the last term:

$$\sum_{ijab=0}^{m} (-1)^{i+j+a+b} \binom{m}{i}\binom{m}{j}\binom{m}{a}\binom{m}{b} (s-p-2j+a-b) \times$$

$$e_{s+p+a+b}\, e_{k-i-a}\, e_{q+i-b}$$

The last monomial is independent of j, so we can carry out summation in j.

But $$\sum_{j=0}^{m} (-1)^j \binom{m}{j} = 0 \quad \text{and} \quad \sum_{j=0}^{m} (-1)^j \cdot j \binom{m}{j} = 0$$

The first two terms can be written as:

$$\sum_{i,j,a,b=0}^{m} (-1)^{i+j+a+b} \binom{m}{i}\binom{m}{j}\binom{m}{a}\binom{m}{b} \times$$

$$\Big([e_{k-i-a}, e_{p+j+b}][e_{s-j+a}, e_{q+i-b}]$$

$$+ e_{k-i-a} [[e_{s-j+a}, e_{q+i-b}], e_{p+j+b}] \Big)$$

The last summand again vanishes and we get

$$\sum_{i,j,a,b=0}^{m} (-1)^{i+j+a+b} \binom{m}{a}\binom{m}{b}\binom{m}{i}\binom{m}{j} \times$$

$$\times (q-s+i+j-a-b)(p-k+i+j+a+b) \times$$

$$\times e_{k+p-i+j-a+b}\, e_{s+q+i-j+a-b}.$$

We can symmetrize in $\{i,a\}$ and $\{b,j\}$, the first factor will become $(q-s)$.

Make a change of variables: $j \mapsto m-j$, $b \mapsto m-b$

$$(q-s) \sum_{i,j,a,b=0}^{m} (-1)^{i+j+a+b} \binom{m}{i}\binom{m}{j}\binom{m}{a}\binom{m}{b}$$

$$(p-k+2m+\underbrace{(i+a)-(j+b)}_{\text{cancel by symmetrizing } i\leftrightarrow j,\ a\leftrightarrow b}) \times$$

$$e_{k+p+2m-i-j-a-b}\, e_{s+q-2m+i+j+a+b}$$

Set $u = i+j+a+b$ - runs from 0 to $4m$

The coefficient at $e_{k+p+2m-u}\, e_{k+p+2m+u}$

is the same at t^u in

$$(q-s) \sum_{i,j,a,b=0}^{m} (-1)^{i+j+a+b} \binom{m}{i}\binom{m}{j}\binom{m}{a}\binom{m}{b}$$

$$\times\ (p-k+2m+\cancel{(i+a)-(j+b)})\, t^{i+j+a+b}$$

$$= (q-s)(p-k+2m)(1-t)^m \times (1-t)^m \times (1-t)^m \times (1-t)^m$$

$\Rightarrow$ Coefficient at t^u is $(q-s)(p-k+2m)(-1)^u \binom{4m}{u}$

$\blacksquare$

As an immediate Corollary we get the Theorem

$\Omega^{(m)}_{k,s} \in \mathrm{Ann}(M)$ for $m \gg 0$.

We can use the key identity to prove cuspidality of $\hat{M}$ in case of W_1. Indeed, we have

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \psi(e_{k-i}, e_{s+i} v) = 0 \quad \forall k, s$$

This gives enough linear dependencies between ψ's of the same weight to get cuspidality of $\hat{M}$.

To go from W_1 to W_n we use solenoidal Lie algebras $W(\mu) \subset W_n$:

Fix $\mu \in \mathbb{C}^n$ such that for $r \in \mathbb{Z}^n$

$\mu \cdot r = 0 \Rightarrow r = 0$. Set $d_\mu = \mu_1 d_1 + \dots + \mu_n d_n$

Let $W(\mu) = A d_\mu$.

$W(\mu)$ has properties very similar to W_1, yet in size it is commeasurable to W_n:

$$W_n = W(\mu_1) \oplus \dots \oplus W(\mu_n)$$

as a vector space.

It then suffices to prove cuspidality of AW_μ-covers, which is essentially the same as for W_1.

This establishes the classification of the simple cuspidal W_n-modules as the tensor modules or their quotients.

Bounded W_n-modules.

This class of modules was constructed by Berman-B. (1999) and B.-Zhao (2004).
It will be convenient to switch to W_{n+1} with variables $t_0, t_1, \dots, t_n$.
Consider a $\mathbb{Z}$-grading on W_{n+1} by degrees in t_0:

$$W_{n+1} = W_{n+1}^- \oplus W_{n+1}^0 \oplus W_{n+1}^+.$$

Here $W_{n+1}^0 = W_n \oplus A_n d_0$

Consider a representation $T(U, \alpha, \gamma)$, $\alpha \in \mathbb{C}^n$, $\gamma \in \mathbb{C}$ for W_n^0:

$$T(U, \alpha, \gamma) = t^{\alpha}\, \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes U$$

where U is a fin. dim. simple $gl_n(\mathbb{C})$-module

$$t^m d_a (t^s \otimes u) = s_a t^{s+m} \otimes u + \sum_{p=1}^{n} m_p\, t^{s+m} \otimes E_{pa} u,$$

$$t^m d_0 (t^s \otimes u) = \gamma\, t^{s+m} \otimes u.$$

Postulate $\quad W_{n+1}^{+} \cdot T(U, \alpha, \gamma) = 0$

and define the generalized Verma module

$$M(U, \alpha, \gamma) = \operatorname{Ind}_{W_{n+1}^{0}}^{W_{n+1}} T(U, \alpha, \gamma)$$

$$\cong \mathcal{U}(W_{n+1}^{-}) \otimes T(U, \alpha, \gamma).$$

The generalized Verma module is a weight module, but its weight spaces below T are infinite-dimensional.

<u>Theorem</u> (Berman - B. 1999)

(i) $M(U, \alpha, \gamma)$ has a unique maximal submodule

(ii) The simple quotient $L(U, \alpha, \gamma)$ has finite-dimensional weight spaces

Now we get the classification of simple W_n-modules with finite-dimensional weight spaces:

(1) Tensor modules with $U \not\cong \Lambda^k V$

(2) $d\Omega^k(\alpha)$

(3) $L(U, \alpha, \gamma)$ twisted by an automorphism from G

We would like to understand the structure of the modules $L(U, \alpha, \gamma)$ and in particular compute the dimensions of its weight spaces. Methods used in the proof of the Theorem do not provide such explicit information.

To answer these questions, it turns out to be helpful to consider a bigger Lie algebra.

Consider an abelian extension of W_n:

$$W_n \ltimes \Omega^1 / d\Omega^0$$

Motivation: toroidal Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and let $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes \mathfrak{g}$ be the multiloop Lie algebra. Then

$$\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes \mathfrak{g} \oplus \Omega^1 / d\Omega^0$$

is the universal central extension, with Lie bracket $[f \otimes x, g \otimes y] =$

$$= fg \otimes [x, y] + (x|y) \overline{g\, df}$$

This is a generalization of affine Lie algebras

Now we can add outer derivation and get $\left(\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes \mathfrak{g} \oplus \Omega^1 / d\Omega^0 \right) \rtimes W_n$.

Finally setting $\mathfrak{g} = (0)$ we get $W_n \ltimes \Omega^1 / d\Omega^0$.

Let us write down the Lie bracket

Set notations $\quad k_a = t_a^{-1} dt_a = i\, dx_a$

$$\left(e^{imx} \frac{\partial}{\partial x_a}\right)\left(e^{isx} dx_b\right) =$$

$$= s_a\, e^{i(m+s)x} dx_b + \delta_{ab}\, e^{isx}\, d e^{imx}$$

$$= i s_a\, e^{i(m+s)x} dx_b + \delta_{ab} \sum_{p=1}^{n} i m_p\, e^{i(m+s)x} dx_p .$$

In t-coordinates:

$$(t^m d_a)(t^s k_b) = s_a\, t^{m+s} k_b + \delta_{ab} \sum_{p=1}^{n} m_p\, t^{m+s} k_p$$

We have also a relation $\displaystyle\sum_{p=1}^{n} s_p\, t^s k_p = 0$

The elements $\{k_1, \dots, k_n\}$ are central.

Now let us go back to rank $n+1$ and variables $t_0, t_1, \dots, t_n$.

<u>Proposition</u> $\operatorname{Span}\{ t_0^s d_a,\ t_0^s k_b,\ k_0 \mid s \in \mathbb{Z},\ a, b = 1, \dots, n \}$ forms an infinite-dimensional Heisenberg subalgebra.

Proof $\quad [t_0^s d_a, t_0^m d_b] = 0, \quad [t_0^s k_a, t_0^m k_b] = 0$

$$[t_0^s d_a, t_0^m k_b] = \delta_{ab}\, t_0^m\, d(t_0^s) = s \cdot \delta_{ab}\, t_0^{m+s-1}\, dt_0$$

Note that $t_0^{k-1} dt_0 = \frac{1}{k} dt_0^k$, so it is exact

when $k \neq 0$. We get

$$[t_0^s d_a, t_0^m k_b] = \begin{cases} s k_0 & \text{if } a=b,\ m=-s \\ 0, & \text{otherwise.} \end{cases}$$

Highest weight representation of the Heisenberg Lie algebra: $V = \mathbb{C}[x_{pj}, y_{pj}]_{\substack{p=1,\dots,n \\ j=1,2,\dots}}$

$$t_0^{-\cancel{s}j} k_a \mapsto j x_{aj}$$

$$t_0^{-j} d_a \mapsto j y_{aj}$$

$$t_0^{j} k_a \mapsto \frac{\partial}{\partial y_{aj}} \qquad t_0^j d_a \mapsto \frac{\partial}{\partial x_{aj}}$$

$$k_0 \mapsto \mathrm{Id}, \quad k_1, \dots, k_n \mapsto 0$$

The action of $d_1, \dots, d_n$ is to be specified later.

Next we would like to determine the action of $t_0^j t^m k_0$, $m \in \mathbb{Z}^n$.

Form a generating series $k_0(m,z) = \sum_{j \in \mathbb{Z}} t_0^j t^m k_0 \cdot z^{-j}$

$$[t_0^s k_a, k_0(m,z)] = 0$$

$$[t_0^s d_a, k_0(m,z)] = \sum_{j \in \mathbb{Z}} [t_0^s d_a, t_0^j t^m k_0]\, z^{-j}$$

$$= \sum_{j \in \mathbb{Z}} m_a\, t_0^{s+j} t^m k_0 \cdot z^{-j} = m_a z^s k_0(m,z).$$

Suppose $k_0(m,z)$ is represented by an operator $K_0(m,z)$, which is a differential operator on V.

Then $K_0(m,z)$ does not involve multiplications by y_{aj} and differentiations $\frac{\partial}{\partial x_{aj}}$.

$$\left[\frac{\partial}{\partial x_{aj}}, K_0(m,z)\right] = m_a z^j K_0(m,z)$$

$$\left[\frac{\partial}{\partial x}, f(x)\right] = f'(x) = \lambda f(x) \Rightarrow f(x) = Ce^{\lambda x}$$

$$\Rightarrow K_0(m,z) \sim \exp(m_a z^j x_{aj}) \sim \exp\left(\sum_{p=1}^{n} m_p \sum_{j=1}^{\infty} x_{pj} z^j\right)$$

$$[j y_{aj}, K_0(m,z)] = m_a z^{-j} K_0(m,z)$$

$$\Rightarrow K_0(m,z) \sim \exp\left(-\sum_{p=1}^{n} m_p \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial y_{pj}}\right)$$

Note also $[d_a, K_0(m,z)] = m_a K_0(m,z)$

We set $d_a = t_a \frac{\partial}{\partial t_a}$ and $V = t^\alpha \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes \mathbb{C}[x_{pj} y_{pj}$

$$K_0(m,z) = t^m \exp\left(\sum_{p=1}^{n} m_p \sum_{j=1}^{\infty} x_{pj} z^j\right) \exp\left(-\sum_{p=1}^{n} m_p \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial y_{pj}}\right)$$

Form also $k_a(m,z) = \sum_{j \in \mathbb{Z}} t_0^j t^m k_a z^{-j-1}$

$$d_a(m,z) = \sum_{j \in \mathbb{Z}} t_0^j t^m d_a z^{-j-1}$$

We need to determine $K_a(m,z)$, $D_a(m,z)$

and we know $K_a(z)$, $D_a(z)$

Using a computation similar to the previous one,

we see $\quad K_a(m,z) = K_a(z)\cdot K_0(m,z)$

Note: $\quad D_a(z) = \sum_{j=1}^{\infty} j\, y_{aj}\, z^{j-1} + t_a \frac{\partial}{\partial t_a} z^{-1} + \sum_{j=1}^{\infty} \frac{\partial}{\partial x_{aj}} z^{-j-1}$

does not commute with $K_0(m,z)$

In this case we need to use normally ordered

product:

$$D_a(z)_+ = \sum_{j=1}^{\infty} j\, y_{aj}\, z^{j-1}$$

$$D_a(z)_- = \sum_{j=1}^{\infty} \frac{\partial}{\partial x_{aj}} z^{-j-1} + t_a \frac{\partial}{\partial t_a} z^{-1}$$

$$:D_a(z)\, K_0(m,z): \; = D_a(z)_+ K_0(m,z) + K_0(m,z) D_a(z)_-$$

~~This will define a representation~~

There is still one ingredient missing:

Recall the structure of the top space:

$$T(U,\alpha,\gamma) = t^{\alpha}\, \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}] \otimes U$$

where U is a $gl_n(\mathbb{C})$-module with action

$$(t^m d_a)(t^s \otimes u) = s_a t^{m+s} \otimes u + \sum_{p=1}^{n} m_p t^{m+s} \otimes E_{pa} u$$

$D_a(z)$ is the "affinization" of the first term

with $t^m \left(t_a \frac{\partial}{\partial t_a}\right)$, but we also need to

affinize E_{pa}-term.

Let $L_{\hat{gl}}(U)$ be the irred highest weight $\hat{gl}_n$-module with a central charge 1 and the gl_n-submodule generated by the highest weight isomorphic to U.

On the space

$$t^\alpha \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes \mathbb{C}[x_{pj}, y_{pj}] \otimes L_{\hat{gl}_n}(U)$$

we define the action of $d_a(m,z)$ by

$$D_a(m,z) = :D_a(z) K_0(m,z): + \sum_{p=1}^{n} m_p E_{pa}(z) K_0(m,z)$$

where $E_{pa}(z)$ acts on $L(U)$ via $\sum_{j \in \mathbb{Z}} t_0^j E_{pa} \cdot z^{-j-1}$

We still need to define the action of $D_0(m,z)$

In order to do so, and to account for ~~a~~ parameter γ, we need an extra tensor factor $L_{Vir}(d)$ - a highest weight module for the Virasoro algebra with $L(0)$ acting on the highest weight by d and central charge 0.

Theorem (B.

The space $t^\alpha \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes \mathbb{C}[x_{pj}, y_{pj}] \otimes L_{\hat{gl}_n}(U) \otimes L_{Vir}(d)$

admits the structure of an irreducible module for the Lie algebra $W_{n+1} \ltimes \Omega^1 / d\Omega^0$

- The formula for $D_0(m, z)$ is complicated

What happens with this module when we reduce to subalgebra W_n ?

Usually when we restrict to a subalgebra the module becomes reducible. Remarkably, the situation is much better here.

<u>Theorem</u> (B. - Futorny

The restriction of the module

$$t^{\alpha} \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes \mathbb{C}[x_{pj}, y_{pj}] \otimes L_{\widehat{gl}_n}(U) \otimes L_{vir}(d)$$

to the subalgebra W_{n+1} remains irreducible except when this module appears in the chiral de Rham complex.

In case when it is irreducible, it is isomorphic to $L(U, \alpha, \gamma)$, so we get a realization and a character of $L(U, \alpha, \gamma)$.

Chiral de Rham complex was introduced by Malikov - Schechtman - Vaintrob (1999)

Consider a Clifford Lie superalgebra with basis $\{\varphi^p_{(j)}, \psi^p_{(j)} \mid p=1,\dots,n, j\in\mathbb{Z}\}$ of odd part and $\{K\}$ - basis of even part and Lie super bracket $[\varphi^a_{(j)}, \psi^b_{(i)}] = \delta_{ab}\,\delta_{i+j,-1}\cdot K$

Consider a highest weight module L_{cl} with $K\cdot\mathbb{1} = \mathbb{1}$, $\psi^p_{(j)}\mathbb{1} = \varphi^p_{(j)}\mathbb{1} = 0$ for $j\geq 0$.

Define $\mathbb{Z}\times\mathbb{Z}$ grading on L_{cl}:

Fermionic $\deg_{fer}(\varphi^p_{(j)}) = 1$, $\deg_{fer}(\psi^p_{(j)}) = -1$

Bosonic $\deg_{bos}(\varphi^p_{(j)}) = -j-1$, $\deg_{bos}(\psi^p_{(j)}) = -j$

Let L^k_{cl} be the subspace of fermionic deg k:

$$L_{cl} = \bigoplus_{k\in\mathbb{Z}} L^k_{cl}$$

<u>Lemma</u> L^k_{cl} is a $\widehat{gl}_\infty$ -module

Set $\varphi^p(z) = \sum_j \varphi^p_{(j)} z^{-j-1}$, $\psi^p(z) = \sum_{j\in\mathbb{Z}} \psi^p_{(j)} z^{-j-1}$

Then $\widehat{gl}_\infty$ -action is given by $E^{ab}(z) = :\varphi^a(z)\psi^b(z):$

Set $\Omega^k_{ch}(\alpha) = t^\alpha \mathbb{C}[t_1^{\pm1}, \dots, t_n^{\pm1}] \otimes \mathbb{C}[x_{pj}, y_{pj}] \otimes L^k_{cl}$

Note that $\Omega^k_{ch}(\alpha)$ is one of the modules $L(\bar{U}, \alpha, 0)$. for $W_n \ltimes \Omega^1/d\Omega^0$.

<u>Theorem</u> The modules $\Omega^k_{ch}(\alpha)$ form a complex

$$\xrightarrow{d} \Omega^{-1}_{ch}(\alpha) \xrightarrow{d} \Omega^0_{ch}(\alpha) \xrightarrow{d} \Omega^1_{ch}(\alpha) \xrightarrow{d} \cdots$$

where $d = z^{-1}$ coefficient in $\sum_{p=1}^{n} D_p(z)\varphi^p(z)$ is a homomorphism of W_n-modules.

Picture for $n=2$: