A Short Introduction to Hopf Algebras

Lecture 6

Memorial
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Nichols-Zoeller theorem

H: finite-dimensional Hopf algebra.

A: Hopf subalgebra.

Then H is free as a (left) A-module.

In particular: $\dim(A)$ divides $\dim(H)$.

Case of groups: Lagrange's theorem

G finite group, $H = K[G]$

U subgroup, $A = K[U]$

Lagrange's theorem:

$|U| = \dim(A)$ divides $|G| = \dim(H)$.

Summary

1. First trace formula:

$$\mathrm{tr}(R_h \circ S_H^{-2} \circ r_\varphi) = \lambda_H(h)\varphi(\Lambda_H)$$

2. In characteristic zero, semisimple Hopf algebras are cosemisimple.
3. Second trace formula:

$$\mathrm{tr}_{H^*}(S_H^{2*}) = \dim(H)\,\mathrm{tr}(S_H^{2*}\mid_{\chi_R H^*})$$

4. In characteristic zero, semisimple Hopf algebras are involutory.

Overview

1. The semisimple case
2. The Krull-Remak-Schmidt theorem
3. Quasi-Frobenius algebras
4. The Nichols-Zoeller theorem
5. Hopf subalgebras of semisimple Hopf algebras

The semisimple case

H: semisimple Hopf algebra over a field of characteristic zero

A: a Hopf subalgebra. Then H is free as a left A-module.

Proof: Lecture 5: H is involutory.

So A is involutory and therefore semisimple.

χ_R: character of the regular representation of H

χ'_R: character of the regular representation of A.

End of Lecture 5: χ_R is a two-sided integral of H^*.

χ'_R is a two-sided integral of A^*.

Clearly, the restriction of χ_R to A is also a two-sided integral of A^*.

The space of integrals is one-dimensional: $\chi_R \mid_A = m\chi'_R$ for $m \in K$.

Evaluating this equation at 1, we get

$$\dim(H) = \chi_R(1) = m\chi'_R(1) = m \dim(A) \Rightarrow m \in \mathbb{Q}$$

Maschke's theorem: There is an integral $\Lambda_A \in A$ with $\varepsilon(\Lambda_A) = 1$.

Proof of the second trace formula, Step 3: Also $\chi'_R(\Lambda_A) = 1$.

So $\Lambda_A^2 = \varepsilon(\Lambda_A)\Lambda_A = \Lambda_A$ is an idempotent.

So it maps to a projection in any representation.

Its trace in any representation is the dimension of the image.

Therefore $\chi_R(\Lambda_A)$ is an integer. But

$$\chi_R(\Lambda_A) = m\chi'_R(\Lambda_A) = m$$

so m is not only a rational number, but rather an integer.

We have $\chi_R \mid_A = m\chi'_R$ by definition of m.

Therefore $H \cong A^m$ as left A-modules, because the characters agree.

Krull-Remak-Schmidt theorem

Suppose that A is an algebra and that M is a finite-dimensional module. Then there exists a decomposition

$$M = \bigoplus_{i=1}^{k} M_i$$

into (nonzero) indecomposable submodules. If

$$M = \bigoplus_{j=1}^{l} M'_j$$

is a second such decomposition, then $k = l$, and there is a permutation $\sigma \in S_k$ such that

$$M_{\sigma(i)} \cong M'_i$$

Corollary 1

Suppose that A is an algebra and that M and N are finite-dimensional modules. If

$$M^n \cong N^n$$

for some $n \in \mathbb{N}$, then $M \cong N$.

Proof:

$$M \cong \bigoplus_{i=1}^{k} P_i^{m_i} \qquad N \cong \bigoplus_{i=1}^{k} P_i^{n_i}$$

where $P_1, \ldots, P_k$ are pairwise nonisomorphic indecomposable modules.

$$M^n \cong N^n \Rightarrow \bigoplus_{i=1}^{k} P_i^{nm_i} \cong \bigoplus_{i=1}^{k} P_i^{nn_i}$$

$$\Rightarrow nm_i = nn_i \Rightarrow m_i = n_i \Rightarrow M \cong N$$

Corollary 2

Suppose that A is an algebra and that M, N, and L are finite-dimensional modules. If

$$M \oplus L \cong N \oplus L$$

then $M \cong N$.

Proof:

$$M \cong \bigoplus_{i=1}^{k} P_i^{m_i} \qquad N \cong \bigoplus_{i=1}^{k} P_i^{n_i} \qquad L \cong \bigoplus_{i=1}^{k} P_i^{l_i}$$

where $P_1, \ldots, P_k$ are pairwise nonisomorphic indecomposable modules.

$$M \oplus L \cong N \oplus L \Rightarrow \textstyle\bigoplus_{i=1}^{k} P_i^{m_i+l_i} \cong \bigoplus_{i=1}^{k} P_i^{n_i+l_i}$$
$$\Rightarrow m_i + l_i = n_i + l_i \Rightarrow m_i = n_i \Rightarrow M \cong N$$

Dualization

If M is a right module over an algebra A, then the dual space $M^* = \mathrm{Hom}_K(M, K)$ is a left A-module via

$$(a\varphi)(m) = \varphi(ma)$$

and in a similar way, the dual space of a left module becomes a right module via $(\varphi a)(m) = \varphi(am)$.

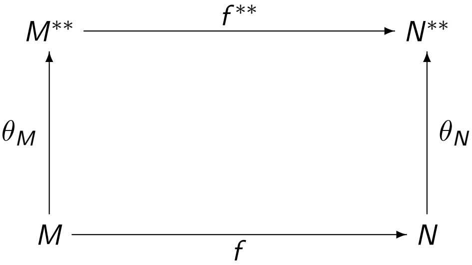
Biduals

Define the canonical injection $\theta_M : M \to M^{**}$ via

$$\theta_M(m)(\varphi) = \varphi(m)$$

for $m \in M$ and $\varphi \in M^*$. θ_M is A-linear on the left or the right, depending on the situation.

If $f : M \to N$ is A-linear, or even only K-linear, then the diagram



is commutative:

$$((f^{**} \circ \theta_M)(m))(\varphi) = \theta_M(m)(f^*(\varphi)) = f^*(\varphi)(m)$$
$$= \varphi(f(m)) = ((\theta_N \circ f)(m))(\varphi)$$

The mappings $\theta_{M^*} : M^* \to M^{***}$ and $\theta_M^* : M^{***} \to M^*$ can be composed.
Their composition $\theta_M^* \circ \theta_{M^*}$ is the identity:
For $m \in M$ and $\varphi \in M^*$, we have that

$$((\theta_M^* \circ \theta_{M^*})(\varphi))(m) = \theta_{M^*}(\varphi)(\theta_M(m)) = \theta_M(m)(\varphi) = \varphi(m)$$

Lemma

If M is a projective right module, then M^* is an injective left module.

Proof:

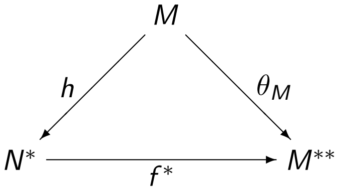
Suppose that $f : M^* \to N$ is an injective left A-linear map.

We have to find an A-linear map $g : N \to M^*$ with $g \circ f = \mathrm{id}_{M^*}$.

Certainly, there is a K-linear map $\tilde{g} : N \to M^*$ with $\tilde{g} \circ f = \mathrm{id}_{M^*}$.

Take the transpose: $f^* \circ \tilde{g}^* = \mathrm{id}_{M^{**}}$, so f^* is surjective.

Since M is projective, there is an A-linear map $h : M \to N^*$ that makes the diagram



commutative. Then $g := h^* \circ \theta_N$ satisfies our requirement:

$$g \circ f = h^* \circ \theta_N \circ f = h^* \circ f^{**} \circ \theta_{M^*} = (f^* \circ h)^* \circ \theta_{M^*} = \theta_M^* \circ \theta_{M^*} = \mathrm{id}_{M^*}$$

Quasi-Frobenius algebras

A finite-dimensional algebra is called a quasi-Frobenius algebra

$\Leftrightarrow$ The left regular module is injective.

Remark: Every Frobenius algebra is a quasi-Frobenius algebra.

Proof: Suppose that $\phi : A \to K$ is the Frobenius homomorphism.

Then

$$A \to A^*, \ a \mapsto \phi(\_ \, a)$$

is bijective and left A-linear:

$$a' \phi(\_ \, a) = \phi(\_ \, a' a)$$

A projective $\Rightarrow$ preceding lemma applies.

Proposition 1

Let A be a quasi-Frobenius algebra and

$$A = \bigoplus_{i=1}^{k} P_i$$

be a decomposition into indecomposable (nonzero) left ideals.

Then every P_i contains exactly one minimal (nonzero) left ideal.

Proof: Existence: By finite-dimensionality.

Uniqueness: $L_1, L_2 \subset P_i$ distinct minimal left ideals.

Note: P_i injective. (Hilton/Stammbach, Prop. I.6.3).

Consider maximal essential extensions E_1 and E_2 of L_1 and L_2.

E_1 and E_2 are injective. (Hilton/Stammbach, Thm. I.9.2).

$L_1 \cap E_2 \neq 0 \Rightarrow L_1 \cap L_2 \neq 0$ (essential extensions)

Contradiction. Similarly: $L_2 \cap E_1 = 0$.

This implies: $E_1 \cap E_2 = 0$

(Again essential: $E_1 \cap E_2 \neq 0 \Rightarrow E_1 \cap L_2 \neq 0$)

Now $E_1 \oplus E_2$ is injective.

Thus: $P_i = E_1 \oplus E_2 \oplus L$ for some left ideal L

(Hilton/Stammbach, Thm. I.8.4(4)).

Contradiction to P_i indecomposable.

Proposition 2

Let A be a quasi-Frobenius algebra and

$$A = \bigoplus_{i=1}^{k} P_i$$

be a decomposition into indecomposable left ideals.

Let M be a (left) A-module.

Then the following assertions are equivalent:

1. P_i is isomorphic to a direct summand of M.
2. There exist $m \in M$ and $a \in \mathrm{Soc}(P_i)$ such that $am \neq 0$.

1. $\Rightarrow$ 2.

Write

$$1 = \sum_{j=1}^{k} e_j, \qquad e_j \in P_j$$

Let $f : P_i \to M$ be A-linear injective.

Put $m := f(e_i)$.

For $a \in \mathrm{Soc}(P_i) \setminus \{0\}$, we have

$$a = \sum_{j=1}^{k} ae_j \Rightarrow a = ae_i$$

Therefore

$$am = af(e_i) = f(ae_i) = f(a) \neq 0$$

$2. \Rightarrow 1.$

Suppose that $m \in M$ and $a \in \mathrm{Soc}(P_i)$ such that $am \neq 0$. Consider

$$f : P_i \to M, \; b \mapsto bm$$

f not injective $\Rightarrow \ker(f)$ nonzero left ideal

$\Rightarrow$ (Prop. 1): $\mathrm{Soc}(P_i) \subset \ker(f) \Rightarrow am = 0$. Contradiction.

Therefore f injective. $f(P_i)$ is a direct summand

(Hilton/Stammbach, Thm. I.8.4(4)).

Remark:

Proof shows: For $f : P_i \to M$, we have

$$f \text{ injective } \Leftrightarrow f\mid_{\mathrm{Soc}(P_i)} \text{ injective } \Leftrightarrow f\mid_{\mathrm{Soc}(P_i)} \neq 0$$

Proposition 3

Let A be a quasi-Frobenius algebra and

$$A = \bigoplus_{i=1}^{k} P_i$$

be a decomposition into indecomposable left ideals.

Enumerate: $P_1, \ldots, P_l$ pairwise nonisomorphic;

every $P_{l+1}, \ldots, P_k$ isomorphic to one of $P_1, \ldots, P_l$.

Let M be a finite-dimensional A-module.

Then the following assertions are equivalent:

1. M is faithful.
2. $\bigoplus_{i=1}^{l} P_i$ is isomorphic to a direct summand of M.
3. Every P_i is isomorphic to a direct summand of M.

1. $\Rightarrow$ 2.

Inductively: $\bigoplus_{i=1}^{j} P_i$ is isomorphic to a direct summand of M.

$j = 0$: Clear.

Induction step: $j \to j+1$.

Choose $a \in \mathrm{Soc}(P_{j+1}) \setminus \{0\}$.

M faithful $\Rightarrow$ Exists $m \in M$ with $am \neq 0$.

Decompose: $M = M_1 \oplus M_2$, $M_1 \cong \bigoplus_{i=1}^{j} P_i$

$m = m_1 + m_2 \Rightarrow am_1 \neq 0$ or $am_2 \neq 0$

Assume: $am_1 \neq 0 \Rightarrow$ (Prop. 2)

P_{j+1} is direct summand in $M_1 \cong \bigoplus_{i=1}^{j} P_i$.

Contradiction to Krull-Remak-Schmidt.

Therefore: $am_1 = 0 \Rightarrow am_2 \neq 0$

$\Rightarrow P_{j+1}$ is a direct summand in M_2.

2. $\Rightarrow$ 3.: Clear

3. $\Rightarrow$ 1.

Suppose that $a \in A$ annihilates M.

Then it annihilates every P_i.

Then it annihilates A.

But then $a = 0$.

Proposition 4

A: quasi-Frobenius algebra.

M: finite-dimensional A-module.

Claim: There exists $n \in \mathbb{N}$ such that M^n can be decomposed in the form

$$M^n = F \oplus N$$

where F is free and N is not faithful.

Proof

Decompose A into indecomposable left ideals: $A = \bigoplus_{i=1}^{k} P_i$

Enumerate: $P_1, \ldots, P_l$ pairwise nonisomorphic;

every $P_{l+1}, \ldots, P_k$ isomorphic to one of $P_1, \ldots, P_l$.

Then: $A \cong \bigoplus\limits_{i=1}^{l} P_i^{n_i}$

Decompose M into indecomposable submodules: $M = \bigoplus\limits_{i=1}^{r} M_i$

P: Sum of those isomorphic to some P_i.

M': Sum of the others $\Rightarrow M = P \oplus M'$

$$P \cong \bigoplus_{i=1}^{l} P_i^{m_i}$$

If some $m_i = 0$: M not faithful, assertion correct.

Determine $j \leq l$ such that

$$\frac{m_j}{n_j} = \min\left\{\frac{m_i}{n_i} \mid i = 1, \ldots, l\right\}$$

$n := n_j$, $m := m_j$. Then

$$nm_i - mn_i \geq 0 \qquad nm_j - mn_j = 0$$

Therefore

$M^n \cong (\bigoplus_{i=1}^{l} P_i^{nm_i}) \oplus M'^n$
$\cong (\bigoplus_{i=1}^{l} P_i^{mn_i}) \oplus (\bigoplus_{i=1}^{l} P_i^{nm_i - mn_i}) \oplus M'^n$
$\cong A^m \oplus (\bigoplus_{i=1}^{l} P_i^{nm_i - mn_i}) \oplus M'^n \cong F \oplus N$

Lemma

Suppose that A is a finite-dimensional augmented algebra and that M is a finite-dimensional A-module.
If M^n is free for some $n \in \mathbb{N}$, then M is free.

Proof: Augmentation: $\varepsilon_A : A \to K$

K is an A-module via $a.\lambda := \varepsilon_A(a)\lambda$.

Tensor product preserves direct sums:

$$M^n \cong A^m \Rightarrow (M \otimes_A K)^n \cong (A \otimes_A K)^m \cong K^m$$

Take dimensions: $m = nl$, $l = \dim_K(M \otimes_A K)$.

Therefore: $M^n \cong (A^l)^n$

Corollary 1 to Krull-Remak-Schmidt: $M \cong A^l$

Repetition

H finite-dimensional Hopf algebra, V finite-dimensional module

Lecture 5, Lemma 2: $V \otimes H \cong H^{\dim(V)}$

W free H-module $\Rightarrow W \cong H^n \Rightarrow$

$$V \otimes W \cong (V \otimes H)^n \cong (H^{\dim(V)})^n \cong W^{\dim(V)}$$

Similarly:

$$W \otimes V \cong W^{\dim(V)}$$

Converse

H finite-dim. Hopf algebra, V, W finite-dim. modules

Suppose that

$$V \otimes W \cong W^{\dim(V)}$$

If V is faithful, then W is free.

Proof: Proposition 4: $W^n \cong F \oplus N$, F free, N not faithful.

$V \otimes W^n \cong (W^n)^{\dim(V)}$.

Preceding lemma: Suffices to show W^n free.

$\Rightarrow$ can replace W^n by $W \Rightarrow W \cong F \oplus N$

Step 2

$V \otimes W \cong W^{\dim(V)} \Rightarrow$

$$(V \otimes F) \oplus (V \otimes N) \cong F^{\dim(V)} \oplus N^{\dim(V)}$$

By repetition: $V \otimes F \cong F^{\dim(V)}$

Corollary 2 to Krull-Remak-Schmidt: $V \otimes N \cong N^{\dim(V)}$

Step 3: $V^m \cong F' \oplus N'$, F' free, N' not faithful.

V faithful $\Rightarrow F' \neq 0$.

$V^m \otimes N \cong N^{m \dim(V)} \Rightarrow$

$$(F' \otimes N) \oplus (N' \otimes N) \cong N^{\dim(F')} \oplus N^{\dim(N')}$$

not faithful.

By repetition: $F' \otimes N \cong F'^{\dim(N)}$

Faithful if $N \neq 0$. Thus $N = 0$, W free.

Relative Hopf modules

Suppose that H is a Hopf algebra and that A is a Hopf subalgebra.

M is a left A-H-Hopf module $:\Leftrightarrow$

1. M is a left A-module.
2. M is a left H-comodule.
3. $\delta(a.m) = a_{(1)} m^{(1)} \otimes a_{(2)}.m^{(2)}$

Example: $M = H$, Coaction=Comultiplication, i.e., $\delta = \Delta$.

Theorem

Suppose that H is a finite-dimensional Hopf algebra
and that A is a Hopf subalgebra.
Suppose that M is a finite-dimensional left A-H-Hopf module.
Then M is a free A-module.

Proof: H is a faithful A-module.
It suffices to show: $H \otimes M \cong M^{\dim(H)}$.
As in repetition:

$$f : H \otimes M \to H \otimes M,\ h \otimes m \mapsto m^{(1)} h \otimes m^{(2)}$$

is A-linear. $f^{-1}(h \otimes m) = S_H^{-1}(m^{(1)}) h \otimes m^{(2)}$

Hopf subalgebras of semisimple Hopf algebras

Suppose that A is a Hopf subalgebra of the semisimple Hopf algebra H. Then A is semisimple.

Proof: H is finite-dimensional. By the Nichols-Zoeller theorem:

Choose a basis $h_1, \dots, h_n$ of the left A-module H.

Λ: nonzero left integral of H.

Expand it: $\Lambda = \sum_{i=1}^{n} a_i h_i$, with $a_i \in A$.

For $a \in A$, the equation $a\Lambda = \varepsilon(a)\Lambda$ reads

$$\sum_{i=1}^{n} a a_i h_i = \sum_{i=1}^{n} \varepsilon(a) a_i h_i$$

Compare coefficients: $aa_i = \varepsilon(a)a_i$

So every a_i is a left integral of A.

If Λ' is a nonzero left integral of A, we have $a_i = \mu_i \Lambda'$ with $\mu_i \in K$.

Insert this back into original equation:

$$\Lambda = \sum_{i=1}^{n} \mu_i \Lambda' h_i = \Lambda' \left(\sum_{i=1}^{n} \mu_i h_i \right)$$

Therefore $\varepsilon(\Lambda) = \varepsilon(\Lambda')\varepsilon\left(\sum_{i=1}^{n} \mu_i h_i\right)$.

By Maschke's theorem: $\varepsilon(\Lambda) \neq 0$.

So $\varepsilon(\Lambda') \neq 0$.

Again by Maschke's theorem: A is semisimple.