A Short Introduction to Hopf Algebras

Lecture 5

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Summary

1. Radford's formula for the fourth power of the antipode.
2. The antipode of a finite-dimensional Hopf algebra has finite order.
3. Maschke's theorem: H semisimple $\Leftrightarrow \varepsilon_H(\Lambda_H) \neq 0$
4. Semisimple Hopf algebras are unimodular.

Overview

1. The first trace formula
2. Semisimple Hopf algebras in characteristic zero are cosemisimple
3. The second trace formula
4. Semisimple Hopf algebras in characteristic zero are involutory

Traces in Frobenius algebras

Suppose that A is a Frobenius algebra with Frobenius homomorphism ϕ and Casimir element $c = \sum_{i=1}^{n} b_i \otimes a_i$.
If $f : A \to A$ is a linear map, then we have

$$\mathrm{tr}(f) = \sum_{i=1}^{n} \phi(a_i f(b_i)) = \sum_{i=1}^{n} \phi(f(a_i) b_i)$$

Proof: The first formula holds because the linear forms $\phi(a_i\_)$ form the dual basis of $b_1, \ldots, b_n$. The second formula holds for similar reasons.

The first trace formula

$\Lambda_H \in H$, $\lambda_H \in H^*$ left integrals with $\lambda_H(\Lambda_H) = 1$.

For $h \in H$, consider right multiplication

$$R_h : H \to H,\ h' \mapsto h'h$$

For $\varphi \in H^*$, consider transpose of right multiplication

$$r_\varphi : H \to H,\ h \mapsto \varphi(h_{(2)})h_{(1)}$$

Then

$$\mathrm{tr}(R_h \circ S_H^{-2} \circ r_\varphi) = \lambda_H(h)\varphi(\Lambda_H)$$

Proof

We have seen that $\Lambda_{H(2)} \otimes S_H^{-1}(\Lambda_{H(1)})$ is the Casimir element corresponding to λ_H. Therefore

$$\begin{gathered} \operatorname{tr}(R_h \circ S_H^{-2} \circ r_\varphi) = \\ \lambda_H(S_H^{-1}(\Lambda_{H(1)})(R_h \circ S_H^{-2} \circ r_\varphi)(\Lambda_{H(2)})) = \\ \lambda_H(S_H^{-1}(\Lambda_{H(1)})S_H^{-2}(\Lambda_{H(2)})h)\varphi(\Lambda_{H(3)}) = \\ \lambda_H(h)\varphi(\Lambda_H) \end{gathered}$$

Reformulation

$\Gamma_H \in H$ right integral, $\lambda_H \in H^*$ left integral with $\lambda_H(\Gamma_H) = 1$.

For $h \in H$, consider left multiplication

$$L_h : H \to H,\ h' \mapsto hh'$$

For $\varphi \in H^*$, as before

$$r_\varphi : H \to H,\ h \mapsto \varphi(h_{(2)})h_{(1)}$$

Then: $\mathrm{tr}(L_h \circ S_H^2 \circ r_\varphi) = \lambda_H(h)\varphi(\Gamma_H)$.

In particular: $\mathrm{tr}(S_H^2) = \lambda_H(1_H)\varepsilon_H(\Gamma_H)$.

Proof: Replace H by H^{op}.

Proposition

K algebraically closed, $\mathrm{char}(K) = 0$.

Suppose that H is a semisimple Hopf algebra.

Then S_H^2 is an inner automorphism.

Proof: λ_H nonzero left integral. Since H is unimodular:

S_H^2 is the Nakayama automorphism corresponding to the Frobenius homomorphism λ_H.

Choose $c \in H$ such that

$$\chi_R(h) = \lambda_H(hc)$$

χ_R is also a Frobenius homomorphism $\Rightarrow$ c is invertible.

$$\chi_R(hh') = \chi_R(h'h) \Rightarrow \lambda_H(hh'c) = \lambda_H(h'hc)$$
$$\Rightarrow \lambda_H(h'cS_H^2(h)) = \lambda_H(h'hc)$$
$$\Rightarrow cS_H^2(h) = hc$$
$$\Rightarrow S_H^2(h) = c^{-1}hc$$

Lemma 1

Suppose that

$$f : M(n \times n, \mathbb{C}) \to M(n \times n, \mathbb{C})$$

is an algebra automorphism of finite order.

Then $\mathrm{tr}(f)$ is a nonnegative real number.

Proof: Skolem-Noether theorem: f is inner, i.e.,

$$f(A) = CAC^{-1}$$

$$f^m = \mathrm{id} \Rightarrow C^m \text{ central} \Rightarrow C^m = \lambda E_n$$

Write $\lambda = \mu^m$. Then $(\frac{1}{\mu} C)^m = E_n$.

Replace C by $\frac{1}{\mu} C$: Can assume $C^m = E_n$.

C is diagonalizable. Eigenvalues $\zeta_1, \ldots, \zeta_n$ are m-th roots of unity.

$$\mathrm{tr}(f) = \mathrm{tr}(C)\,\mathrm{tr}(C^{-1}) =$$
$$(\zeta_1 + \ldots + \zeta_n)(\frac{1}{\zeta_1} + \ldots + \frac{1}{\zeta_n}) =$$
$$(\zeta_1 + \ldots + \zeta_n)\overline{(\zeta_1 + \ldots + \zeta_n)} \geq 0$$

Theorem 1

Suppose that H is a semisimple Hopf algebra

and that the base field has characteristic zero.

Then H^* is a semisimple Hopf algebra.

Proof: Question is whether a number is nonzero

$\Rightarrow$ Can assume that the base field is algebraically closed.

Wedderburn decomposition:

$$H = \bigoplus_{i=1}^{k} I_i \qquad I_i \cong M(n_i \times n_i, K)$$

$n_1 = 1$: Counit corresponds to the projection to the first summand.

$$\mathrm{tr}(S_H^2) = \sum_{i=1}^{k} \mathrm{tr}(S_H^2 \mid_{I_i})$$

$$S_H^2(\Lambda_H) = \Lambda_H \Rightarrow \mathrm{tr}(S_H^2 \mid_{I_1}) = 1$$

All these traces are sums of roots of unity, in particular algebraic numbers, i.e., contained in the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in K.

Uniqueness of algebraic closures: This is isomorphic to the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Under this isomorphism:

$\mathrm{tr}(S_H^2 \mid_{I_i})$ becomes a nonnegative real number.

In particular: $\mathrm{tr}(S_H^2) \neq 0$.

First trace formula: $\lambda_H(1_H)\varepsilon_H(\Lambda_H) \neq 0$

Maschke's theorem: H^* semisimple.

Lemma 2

For every finite-dimensional H-module V:

$$V \otimes H \cong H^{\dim V}$$

Proof: Consider

$$f : V \otimes H \to V \otimes H, \ v \otimes h \mapsto h_{(1)}.v \otimes h_{(2)}$$

f is H-linear for the following module structures:

Left-hand side: $h.(v \otimes h') = v \otimes hh'$

Right-hand side: Usual structure, i.e.,

$$h.(v \otimes h') = h_{(1)}v \otimes h_{(2)}h'$$

f bijective: $f^{-1}(v \otimes h) = S_H^{-1}(h_{(1)}).v \otimes h_{(2)}$

Left-hand side: $V \otimes H \cong H^{\dim V}$

The second trace formula

$$\mathrm{tr}_{H^*}(S_H^{2*}) = \dim(H)\,\mathrm{tr}(S_H^{2*}\,|_{\chi_R H^*})$$

Proof: Suppose that $\rho_H \in H^*$ is a right integral and $\Lambda_H \in H$ is a left integral satisfying $\rho_H(\Lambda_H) = 1$.

H^* is a Frobenius algebra with Frobenius homomorphism

$$H^* \to K,\ \varphi \mapsto \varphi(\Lambda_H)$$

and Casimir element $S_H^*(\rho_{H(1)}) \otimes \rho_{H(2)}$.

(Apply the theorem to $H^{*\,\mathrm{op}}$.)

Step 2

Consider

$$f : H^* \to H^*,\ \varphi \mapsto \chi_R S_H^{2*}(\varphi)$$

Formulas for traces in Frobenius algebras:

$$\mathrm{tr}(f) = \big(\chi_R S_H^{2*}(\rho_{H(2)}) S_H^*(\rho_{H(1)})\big)(\Lambda_H) = \chi_R(\Lambda_H)\rho_H(1_H)$$

Step 3: $\chi_R(\Lambda_H) = \varepsilon_H(\Lambda_H)$

If $\varepsilon_H(\Lambda_H) = 0$, then ${\Lambda_H}^2 = 0$, then $\chi_R(\Lambda_H) = 0$.

If $\varepsilon_H(\Lambda_H) \neq 0$, then H semisimple, then multiplication by Λ_H is (essentially) projection to the first Wedderburn component, so $\chi_R(\Lambda_H) = \varepsilon_H(\Lambda_H)$.

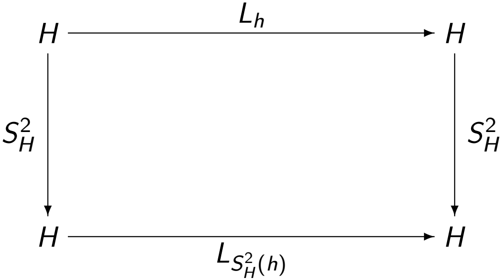
By first trace formula:

$$\mathrm{tr}(f) = \varepsilon_H(\Lambda_H)\rho_H(1_H) = \mathrm{tr}(S_H^2)$$

Step 4: $S_H^{2*}(\chi_R) = \chi_R$

For $h \in H$: L_h left multiplication by h.

The following diagram commutes:



Therefore:

$$\chi_R(h) = \mathrm{tr}(L_h) = \mathrm{tr}(L_{S_H^2(h)}) = \chi_R(S_H^2(h)) = S_H^{2*}(\chi_R)(h)$$

Step 5

To show: $\mathrm{tr}(f) = \dim(H)\,\mathrm{tr}(S_H^{2*}\mid_{\chi_R H^*})$

By Lemma 2:

$$H \otimes H \cong H^{\dim(H)} \Rightarrow \chi_R^2 = (\dim H)\chi_R$$

First case: $\dim(H) = 0 \in K$

Then:

$$f^2(\varphi) = \chi_R S_H^{2*}(\chi_R S_H^{2*}(\varphi)) = \chi_R^2 S_H^{4*}(\varphi) = 0$$

Second case: $\dim(H) \neq 0 \in K$

Then $e := \frac{1}{\dim(H)}\chi_R$ is an idempotent.

$$H^* = \chi_R H^* \oplus (1-e)H^*$$

$$\mathrm{tr}(f) = \mathrm{tr}(f\mid_{\chi_R H^*}) = \dim(H)\,\mathrm{tr}(S_H^{2*}\mid_{\chi_R H^*})$$

Kaplansky's seventh conjecture

Suppose that $\mathrm{char}(K)$ divides $\dim(H)$.

Then H or H^* are not semisimple.

Proof: Assume H and H^* semisimple.

Maschke's theorem + first trace formula:

$$\mathrm{tr}(S_H^2) = \lambda_H(1_H)\varepsilon_H(\Gamma_H) \neq 0$$

Second trace formula:

$$\mathrm{tr}_{H^*}(S_H^{2*}) = \dim(H)\,\mathrm{tr}(S_H^{2*}\mid_{\chi_R H^*}) = 0$$

Theorem 2

Suppose H semisimple and cosemisimple.

Assume: $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > \dim(H)^2$.

Then $S_H^2 = \mathrm{id}$.

Proof:

H unimodular and counimodular $\Rightarrow S_H^4 = \mathrm{id}$.

$$n_{\pm 1} := \dim(\{\varphi \in H^* \mid S_H^{2*}(\varphi) = \pm\varphi\})$$

$$m_{\pm 1} := \dim(\{\varphi \in \chi_R H^* \mid S_H^{2*}(\varphi) = \pm\varphi\})$$

$V_i \otimes H \cong H^{n_i} \Rightarrow \chi_i \chi_R = n_i \chi_R \Rightarrow$

χ_R not invertible $\Rightarrow \chi_R H^* \neq H^*$.

$d := n_1 - n_{-1} - \dim(H)(m_1 - m_{-1})$. Then

$$d1_K = \mathrm{tr}(S_H^{2*}) - \dim(H)\,\mathrm{tr}(S_H^{2*}\mid_{\chi_R H^*}) = 0$$

In positive characteristic:

$$\begin{aligned} |d| &\leq n_1 + n_{-1} + \dim(H)(m_1 + m_{-1}) \\ &\leq \dim(H) + \dim(H)(\dim(H) - 1) \\ &= \dim(H)^2 < \mathrm{char}(K) \end{aligned}$$

Therefore $d = 0$, i.e.,

$$n_1 - n_{-1} = \dim(H)(m_1 - m_{-1})$$

$$-\dim(H) \leq n_1 - n_{-1} \leq \dim(H) \Rightarrow$$
$$n_1 - n_{-1} \in \{-\dim(H), 0, \dim(H)\}$$

Maschke's theorem:

$$\mathrm{tr}(S_H^{2*}) \neq 0 \Rightarrow n_1 - n_{-1} \neq 0$$

$S_H^{2*}(\varepsilon_H) = \varepsilon_H \Rightarrow n_1 \neq 0.$
Therefore $n_{-1} = 0.$

Equivalences

Suppose that H is a finite-dimensional Hopf algebra over a field of characteristic zero.

Then the following assertions are equivalent:

1. H is semisimple.
2. H is cosemisimple.
3. H is involutory. (I.e., $S_H^2 = \mathrm{id}$.)

Proof

1\. $\Leftrightarrow$ 2. by Theorem 1.

1.,2. $\Rightarrow$ 3. by Theorem 2.

3\. $\Rightarrow$ 1.,2. by the first trace formula.

Corollary

Suppose that H is a semisimple Hopf algebra over a field K of characteristic zero.
Then $\chi_R \in H^*$ is a two-sided integral.

Proof: Second trace formula:

$$\mathrm{tr}_{H^*}(S_H^{2*}) = \dim(H)\,\mathrm{tr}(S_H^{2*}\mid_{\chi_R H^*})$$

H involutory:

$$\dim(H) = \dim(H)\dim(\chi_R H^*)$$

$\Rightarrow \dim(\chi_R H^*) = 1$

$$\chi_R \varphi = \mu \chi_R \Rightarrow \dim(H)\varphi(1_H) = \mu \dim(H) \Rightarrow \varphi(1_H) = \mu$$