A Short Introduction to Hopf Algebras

Lecture 4

Memorial
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Summary

1. Hopf modules: Modules and comodules with compatibility condition
2. Structure theorem: $M \cong V \otimes H$
3. Integrals: $h\Lambda_H = \varepsilon_H(h)\Lambda_H$
4. Integrals exist and are unique up to scalar multiples.
5. Modular functions and elements: $\Lambda_H h = \alpha^L(h)\Lambda_H$
6. Frobenius algebras: Frobenius homomorphism, Casimir element, Nakayama automorphism.

Overview

1. Radford's formula
2. Maschke's theorem

Theorem

Suppose that H is a finite-dimensional Hopf algebra and that $\Lambda_H \in H$ and $\lambda_H \in H^*$ are nonzero left integrals.

Then H is a Frobenius algebra with Frobenius homomorphism λ_H.

We have $\lambda_H(\Lambda_H) \neq 0$.

If $\lambda_H(\Lambda_H) = 1$, then

$$\Lambda_{H(2)} \otimes S_H^{-1}(\Lambda_{H(1)})$$

is the corresponding Casimir element.

The corresponding Nakayama automorphism is

$$\nu(h) = S_H^2(h_{(1)})\, \alpha^R(h_{(2)})$$

Proof: Frobenius algebra

By the structure theorem for Hopf modules:

$$H \to H^*, h \mapsto (\lambda_H \leftarrow h) = \lambda_{H(2)}(S_H(h))\ \lambda_{H(1)}$$

is bijective.

This means that the bilinear form

$$H \times H \to K,\ (h, h') \mapsto \lambda_H(h' S_H(h))$$

is nondegenerate.

Claim: $\lambda_H(\Lambda_H) \neq 0$.

Assume $\lambda_H(\Lambda_H) = 0$. Then

$$\lambda_H(h\Lambda_H) = \varepsilon_H(h)\lambda_H(\Lambda_H) = 0$$

Contradiction to nondegeneracy.

Proof: Casimir element

Now $\lambda_H(\Lambda_H) = 1$. Then we have

$$S_H^{-1}(\Lambda_{H(1)})\lambda_H(h\Lambda_{H(2)}) = h_{(1)}\Lambda_{H(2)}S_H^{-1}(\Lambda_{H(1)})\lambda_H(h_{(2)}\Lambda_{H(3)})$$
$$= h_{(1)}\lambda_H(h_{(2)}\Lambda_H) = h\lambda_H(\Lambda_H) = h$$

This means: If

$$\Lambda_{H(2)} \otimes S_H^{-1}(\Lambda_{H(1)}) = \sum_{i=1}^{n} b_i \otimes a_i$$

then

$$\sum_{i=1}^{n} \lambda_H(hb_i)a_i = h$$

Proof: Nakayama automorphism

The preceding implies:

$$\lambda_H(S_H^{-1}(\Lambda_{H(1)})h)\Lambda_{H(2)} = h$$

In particular: $\lambda_H(S_H^{-1}(\Lambda_H)) = 1$. And

$$\nu(h) = \lambda_H(S_H^{-1}(\Lambda_{H(1)})\nu(h))\Lambda_{H(2)} = \lambda_H(hS_H^{-1}(\Lambda_{H(1)}))\Lambda_{H(2)}$$

so that

$$S_H^{-2}(\nu(h)) = \lambda_H(hS_H^{-1}(\Lambda_{H(1)}))S_H^{-2}(\Lambda_{H(2)})$$
$$= \lambda_H(h_{(2)}S_H^{-1}(\Lambda_{H(1)}))h_{(1)}S_H^{-1}(\Lambda_{H(2)})S_H^{-2}(\Lambda_{H(3)})$$
$$= \lambda_H(h_{(2)}S_H^{-1}(\Lambda_H))h_{(1)}$$
$$= \lambda_H(S_H^{-1}(\Lambda_H))\alpha^R(h_{(2)})h_{(1)} = \alpha^R(h_{(2)})h_{(1)}$$

Nakayama automorphism: Second formula

Proposition: $\nu(h) = \alpha^R(h_{(1)})a^R S_H^{-2}(h_{(2)})a^L$

Proof: We have

$\lambda_H(S_H^{-1}(\Lambda_{H(2)})h)S_H^{-2}(\Lambda_{H(1)})a^L =$
$\lambda_H(S_H^{-1}(\Lambda_{H(3)})h_{(1)})S_H^{-2}(\Lambda_{H(1)})S_H^{-1}(\Lambda_{H(2)})h_{(2)}$
$= \lambda_H(S_H^{-1}(\Lambda_H)h_{(1)})h_{(2)} = \lambda_H(S_H^{-1}(\Lambda_H))h = h$

Therefore

$$\nu(h) = \lambda_H(hS_H^{-1}(\Lambda_{H(2)}))S_H^{-2}(\Lambda_{H(1)})a^L$$

This yields $a^L S_H^2(\nu(h))a^R = \lambda_H(hS_H^{-1}(\Lambda_{H(2)}))a^L\Lambda_{H(1)}$
$= \lambda_H(h_{(1)}S_H^{-1}(\Lambda_{H(3)}))h_{(2)}S_H^{-1}(\Lambda_{H(2)})\Lambda_{H(1)}$
$= \lambda_H(h_{(1)}S_H^{-1}(\Lambda_H))h_{(2)} = \lambda_H(S_H^{-1}(\Lambda_H))\alpha^R(h_{(1)})h_{(2)}$
$= \alpha^R(h_{(1)})h_{(2)}$

Radford's formula

$$S_H^4(h) = \alpha^R(h_{(1)})a^R h_{(2)} a^L \alpha^L(h_{(3)})$$

Proof: From theorem:

$$\nu(h) = S_H^2(h_{(1)})\,\alpha^R(h_{(2)})$$

Just proved:

$$\nu(h) = \alpha^R(h_{(1)})a^R S_H^{-2}(h_{(2)})a^L$$

Therefore

$$S_H^4(h_{(1)})\,\alpha^R(h_{(2)}) = \alpha^R(h_{(1)})a^R h_{(2)} a^L$$

Corollary

The antipode of a finite-dimensional Hopf algebra has finite order.

Proof: By Radford's formula, the fourth power of the antipode is the conjugation with a modular element composed with coconjugation with a modular function. Both mappings have finite order and commute.

Semisimple algebras

A algebra, V module.

V semisimple $:\Leftrightarrow$ for every submodule U there exists a submodule W such that

$$V = U \oplus W$$

A semisimple $:\Leftrightarrow$ the left regular representation is semisimple.

In this case, every module is semisimple.

Wedderburn structure theorem

A semisimple $\Leftrightarrow$

$$A \cong \bigoplus_{i=1}^{k} M(n_i \times n_i, D_i)$$

D_i: Division algebras over K.

K algebraically closed $\Rightarrow D_i \cong K$

In this case:

A has k simple modules $V_1, \ldots, V_k$ of dimension $n_1, \ldots, n_k$.

Character: $\chi_i(h) := \mathrm{tr}(h \,|_{V_i})$

Character of the (left) regular representation: $\chi_R = \sum_{i=1}^{k} n_i \chi_i$

Maschke's theorem

H: Finite-dimensional Hopf algebra.

Λ_H: Nonzero left integral of H.

Then: H semisimple $\Leftrightarrow \varepsilon_H(\Lambda_H) \neq 0$

Example: G finite group, $H = K[G]$

$$\Lambda_H = \sum_{g \in G} g$$

Maschke's theorem:

H semisimple $\Leftrightarrow \mathrm{char}(K) \nmid |G|$

Proof

$\Rightarrow$: Suppose H semisimple.

$H^+ := \ker(\varepsilon_H)$ is a two-sided ideal of codimension 1.

H semisimple $\Rightarrow$ there exists a left ideal L such that

$$H = L \oplus H^+$$

$\Lambda_H \in L$ nonzero $\Rightarrow \varepsilon_H(\Lambda_H) \neq 0$.

$$h\Lambda_H \in L \Rightarrow h\Lambda_H = \lambda\Lambda_H \Rightarrow \varepsilon_H(h)\varepsilon_H(\Lambda_H) = \lambda\varepsilon_H(\Lambda_H)$$
$$\Rightarrow h\Lambda_H = \varepsilon_H(h)\Lambda_H$$

This shows: Λ_H integral.

$\Leftarrow$

Suppose $\varepsilon_H(\Lambda_H) \neq 0$. Can assume $\varepsilon_H(\Lambda_H) = 1$.

$\Lambda_{H(1)} \otimes S_H(\Lambda_{H(2)})$ is a Casimir element:

$$\begin{aligned} h\Lambda_{H(1)} \otimes S_H(\Lambda_{H(2)}) &= h_{(1)}\Lambda_{H(1)} \otimes S_H(\Lambda_{H(2)})S_H(h_{(2)})h_{(3)} \\ &= h_{(1)}\Lambda_{H(1)} \otimes S_H(h_{(2)}\Lambda_{H(2)})h_{(3)} \\ &= \Lambda_{H(1)} \otimes S_H(\Lambda_{H(2)})h \end{aligned}$$

Suppose: V is H-module with submodule W.

Have to show: There exists an H-linear projection $p : V \to W$.

Choose K-linear projection $q : V \to W$. Define:

$$p : V \to W,\ v \mapsto \Lambda_{H(1)} q(S_H(\Lambda_{H(2)})v)$$

p is H-linear:

$p(hv) = \Lambda_{H(1)} q(S_H(\Lambda_{H(2)})hv) = h\Lambda_{H(1)} q(S_H(\Lambda_{H(2)})v) = hp(v)$

p restricts to the identity on W:

$p(w) = \Lambda_{H(1)} q(S_H(\Lambda_{H(2)})w) = \Lambda_{H(1)} S_H(\Lambda_{H(2)})w$

$= \varepsilon_H(\Lambda_H)w = w$

Corollary

A finite-dimensional semisimple Hopf algebra is unimodular.

Proof: Choose a nonzero left integral $\Lambda_H \in H$.

Then we have

$$\Lambda_H h = \alpha^L(h)\Lambda_H$$

Therefore

$$\varepsilon_H(\Lambda_H)\varepsilon_H(h) = \alpha^L(h)\varepsilon_H(\Lambda_H)$$

By Maschke's theorem: $\varepsilon_H(h) = \alpha^L(h)$

Remark

A Hopf algebra that contains a nonzero integral is always finite-dimensional.
(M. Sweedler, Hopf algebras, Exercises to Chap. 5).

In particular, this happens for semisimple Hopf algebras by the above proof, so semisimple Hopf algebras are finite-dimensional.

Alternatively, semisimple Hopf algebras are separable, and separable algebras are finite-dimensional, which is a special case of the Villamayor-Zelinsky theorem.