A Short Introduction to Hopf Algebras

Lecture 3

Memorial
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Summary

1. The skew-antipode equation

$$S_H^{-1}(h_{(2)})h_{(1)} = \varepsilon_H(h)1_H$$
$$h_{(2)}S_H^{-1}(h_{(1)}) = \varepsilon_H(h)1_H$$

2. Special elements: Grouplikes, primitives, skew-primitives
3. $G(H)$ is a group. Its elements are linearly independent.
4. Dualization in Sweedler notation:

$$(\varphi\psi)(c) = \varphi(c_{(1)})\psi(c_{(2)}) \qquad \varphi(ab) = \varphi_{(1)}(a)\varphi_{(2)}(b)$$

5. Left comodules: Coaction $\delta : V \to C \otimes V$ satisfying

$$(\Delta \otimes \mathrm{id}_V) \circ \delta = (\mathrm{id}_C \otimes \delta) \circ \delta$$

$$(\varepsilon \otimes \mathrm{id}_V)(\delta(v)) = 1_K \otimes v$$

6. (Left) comodule homomorphisms: $f : V \to W$ satisfying

$$\delta_W \circ f = (\mathrm{id}_C \otimes f) \circ \delta_V$$

7. Tensor products of left comodules

$$\delta_{V \otimes W}(v \otimes w) = v^{(1)} w^{(1)} \otimes v^{(2)} \otimes w^{(2)}$$

8. Right comodules: Coaction $\delta : V \to V \otimes C$ satisfying

$$(\mathrm{id}_V \otimes \Delta) \circ \delta = (\delta \otimes \mathrm{id}_C) \circ \delta \qquad (\mathrm{id}_V \otimes \varepsilon)(\delta(v)) = v \otimes 1_K$$

9. Hopf modules

Overview

1. The structure theorem for Hopf modules
2. Integrals
3. Modular functions and modular elements
4. Frobenius algebras

Hopf modules

H: Hopf algebra.

Hopf module: Right H-module and right H-comodule M.

Compatibility condition:

$$\delta(m \leftarrow h) = (m^{(1)} \leftarrow h_{(1)}) \otimes m^{(2)} h_{(2)}$$

Example 1

$M = H$, action=multiplication, coaction=comultiplication,

i.e., $\delta = \Delta$.

Generalization: V vector space, $M := V \otimes H$

Action:

$$(v \otimes h) \leftarrow h' := v \otimes hh'$$

Coaction:

$$\delta : M \to M \otimes H, \; v \otimes h \mapsto v \otimes h_{(1)} \otimes h_{(2)}$$

Example 2

H finite-dimensional.

H^* is a Hopf module:

Left coregular action, plus antipode:

$$\varphi \leftarrow h := \varphi_{(2)}(S_H(h))\, \varphi_{(1)}$$

Right coregular coaction:

$$\delta(\varphi) = \sum_{i=1}^{n} \varphi(h_{i\,(2)})\, h_i^* \otimes h_{i\,(1)}$$

Proof. Have to show:

$$\delta(\varphi \leftarrow h) = (\varphi^{(1)} \leftarrow h_{(1)}) \otimes \varphi^{(2)} h_{(2)}$$

This means:

$$\sum_{i=1}^{n} \varphi_{(2)}(S_H(h))h_i^* \otimes h_{i(1)}\varphi_{(1)}(h_{i(2)})$$
$$= \sum_{i=1}^{n} \varphi(h_{i(2)})(h_i^* \leftarrow h_{(1)}) \otimes h_{i(1)}h_{(2)}$$

Take $h' \in H$ and evaluate the left tensorand on it:

$$\sum_{i=1}^{n} \varphi_{(2)}(S_H(h))h_i^*(h')\ h_{i(1)}\varphi_{(1)}(h_{i(2)})$$
$$= \sum_{i=1}^{n} \varphi(h_{i(2)})h_i^*(h' S_H(h_{(1)}))\ h_{i(1)}h_{(2)}$$

Equivalently:

$$\varphi_{(2)}(S_H(h))h'_{(1)}\varphi_{(1)}(h'_{(2)}) = \varphi(h'_{(2)}S_H(h_{(1)}))h'_{(1)}S_H(h_{(2)})h_{(3)}$$

Structure theorem for Hopf modules

M: Hopf module over H.

Consider the space of coinvariants

$$V := M^{\mathrm{co}\,H} = \{m \in M \mid \delta(m) = m \otimes 1\}$$

Then

$$f : V \otimes H \to M, \; v \otimes h \mapsto (v \leftarrow h)$$

is an isomorphism of Hopf modules.

Proof: Easy steps

f is obviously H-linear.

f is colinear:

$\delta(f(v \otimes h)) = \delta(v \leftarrow h) = (v^{(1)} \leftarrow h_{(1)}) \otimes v^{(2)} h_{(2)}$

$= (v \leftarrow h_{(1)}) \otimes h_{(2)} = f(v \otimes h_{(1)}) \otimes h_{(2)} = (f \otimes \mathrm{id})(\delta(v \otimes h))$

Projecting to the coinvariants

Define

$$p : M \to V,\ \ m \mapsto (m^{(1)} \leftarrow S_H(m^{(2)}))$$

Note: $p(v) = v$ for $v \in V$.

p is well-defined:

$\delta(p(m)) = \delta(m^{(1)} \leftarrow S_H(m^{(2)}))$

$= (m^{(1)} \leftarrow S_H(m^{(4)})) \otimes m^{(2)} S_H(m^{(3)})$

$= (m^{(1)} \leftarrow S_H(m^{(2)})) \otimes 1 = p(m) \otimes 1$

Similarly, we have

$p(m \leftarrow h) = (m^{(1)} \leftarrow h_{(1)}) \leftarrow S_H(m^{(2)} h_{(2)})$

$= \varepsilon_H(h)(m^{(1)} \leftarrow S_H(m^{(2)})) = \varepsilon_H(h) p(m)$

f is bijective

Claim:

$$f^{-1} : M \to V \otimes H,\ \ m \mapsto p(m^{(1)}) \otimes m^{(2)}$$

Obviously:

$$f(p(m^{(1)}) \otimes m^{(2)}) = (m^{(1)} \leftarrow S_H(m^{(2)})) \leftarrow m^{(3)} = m$$

Conversely:

$$p(f(v \otimes h)^{(1)}) \otimes f(v \otimes h)^{(2)} = p(v \leftarrow h_{(1)}) \otimes h_{(2)}$$

$$= p(v)\varepsilon_H(h_{(1)}) \otimes h_{(2)} = v \otimes h$$

Integrals

An element $\Lambda_H \in H$ is called a left integral $:\Leftrightarrow$

$$h\Lambda_H = \varepsilon_H(h)\Lambda_H$$

for all $h \in H$.

An element $\Gamma_H \in H$ is called a right integral $:\Leftrightarrow$

$$\Gamma_H h = \varepsilon_H(h)\Gamma_H$$

for all $h \in H$.

Note: Left integrals (and right integrals) form a subspace of H.

H unimodular $\Leftrightarrow$ Every left integral is also a right integral, and vice versa.

Integrals of H^*

H finite-dimensional. $\lambda_H \in H^*$ is a left integral

$\Leftrightarrow \varphi\lambda_H = \varphi(1_H)\lambda_H$ (for all $\varphi \in H^*$)

$\Leftrightarrow \varphi(h_{(1)})\lambda_H(h_{(2)}) = \varphi(1_H)\lambda_H(h)$

$\Leftrightarrow h_{(1)}\lambda_H(h_{(2)}) = 1_H\lambda_H(h)$

Similarly: $\rho_H \in H^*$ is a right integral $\Leftrightarrow$

$$\rho_H(h_{(1)})h_{(2)} = \rho_H(h)1_H$$

for all $h \in H$.

Example

G finite group, $H = K[G]$.

Then

$$\Lambda_H := \sum_{g \in G} g$$

is a left and right integral.

Theorem

Suppose that H is a finite-dimensional Hopf algebra.

Then the space of left integrals (and the space of right integrals) is one-dimensional.

Proof: The space of left integrals of H^* is exactly the space of coinvariants for the Hopf module structure in Example 2:

$$\delta(\varphi) = \varphi \otimes 1_H \Leftrightarrow \sum_{i=1}^{n} \varphi(h_{i(2)})\, h_i^* \otimes h_{i(1)} = \varphi \otimes 1_H$$

$$\Leftrightarrow \varphi(h_{(2)})h_{(1)} = \varphi(h)1_H$$

Corollary

Suppose that H is a finite-dimensional Hopf algebra.

Then the antipode of H is bijective.

Proof: Suppose $S_H(h) = 0$.

With action from Example 2:

$$\varphi \leftarrow h = \varphi_{(2)}(S_H(h))\varphi_{(1)} = 0$$

By structure theorem: Isomorphism

$$f : H^{*\,\mathrm{co}\,H} \otimes H \to H^*,\ \varphi \otimes h \mapsto (\varphi \leftarrow h)$$

Choose nonzero left integral $\lambda_H \in H^{*\,\mathrm{co}\,H}$.

Then $f(\lambda_H \otimes h) \neq 0$.

Modular functions

H finite-dimensional.

$\Lambda_H \in H$ nonzero left integral.

For $h \in H$, $\Lambda_H h$ is again a left integral.

It is therefore proportional to Λ_H:

$$\Lambda_H h = \alpha^L(h)\Lambda_H$$

We have

$$\alpha^L(hh')\Lambda_H = \Lambda_H hh' = \alpha^L(h)\Lambda_H h' = \alpha^L(h)\alpha^L(h')\Lambda_H$$

The algebra homomorphism

$$\alpha^L : H \to K$$

is called the left modular function.

Modular elements

Γ_H nonzero right integral.

Right modular function: $h\Gamma_H = \alpha^R(h)\Gamma_H$

Antipode maps left integrals to right integrals $\Rightarrow \alpha^R = (\alpha^L)^{-1}$

Modular functions of H^*:

Elements of $H^{**} \cong H$, called modular elements.

Left modular element a^L:

$$\lambda_H \varphi = \varphi(a^L)\lambda_H \Leftrightarrow \lambda_H(h_{(1)})\varphi(h_{(2)}) = \varphi(a^L)\lambda_H(h)$$
$$\Leftrightarrow \lambda_H(h_{(1)})h_{(2)} = a^L\lambda_H(h)$$

Right modular element:

$$h_{(1)}\rho_H(h_{(2)}) = a^R\rho_H(h)$$

a^R is the inverse of a^L.

Remark

Consider the conjugation with a modular element:

$$H \to H,\ h \mapsto a^R h a^L$$

Dually: Coconjugation with a modular function:

$$H \to H,\ h \mapsto \alpha^R(h_{(1)})h_{(2)}\alpha^L(h_{(3)})$$

Both mappings commute:

$$\alpha^R(a^R h_{(1)} a^L)\ a^R h_{(2)} a^L\ \alpha^L(a^R h_{(3)} a^L) = \alpha^R(h_{(1)})\ a^R h_{(2)} a^L\ \alpha^L(h_{(3)})$$

Frobenius algebras

Definition: A finite-dimensional algebra A is called a Frobenius algebra if there is a nondegenerate bilinear form

$$\langle \cdot, \cdot \rangle : A \times A \to K$$

which is associative in the sense that

$$\langle ab, c \rangle = \langle a, bc \rangle$$

for all $a, b, c \in A$.

Elementary properties

Define the Frobenius homomorphism:

$$\phi : A \to K,\ a \mapsto \langle a, 1 \rangle$$

Then we can write

$$\langle a, b \rangle = \phi(ab)$$

The mapping

$$\Phi : A \to A^*,\ a \mapsto \phi(a\_)$$

is an isomorphism.

$b_1, \ldots, b_n$ basis of A with dual basis $b_1^*, \ldots, b_n^*$.

Define $a_i := \Phi^{-1}(b_i^*)$.

Then: $\langle a_i, b_j \rangle = \delta_{ij}$

The Casimir element

Define the Casimir element

$$c := \sum_{i=1}^{n} b_i \otimes a_i$$

Then we have

$$(a \otimes 1)c = c(1 \otimes a)$$

for all $a \in A$.

Proof: $b = \sum_{i=1}^{n} \langle a_i, b \rangle b_i \Rightarrow ab = \sum_{i=1}^{n} \langle a_i, b \rangle ab_i$

But also

$$ab = \sum_{i=1}^{n} \langle a_i, ab \rangle b_i = \sum_{i=1}^{n} \langle a_i a, b \rangle b_i$$

Therefore: $\sum_{i=1}^{n} ab_i \otimes a_i = \sum_{i=1}^{n} b_i \otimes a_i a$

The Nakayama automorphism

Fix $a \in A$ and consider the linear form

$$A \to K, \;\; b \mapsto \langle a, b \rangle$$

By nondegeneracy, there exists $\nu(a) \in A$ such that

$$\langle a, b \rangle = \langle b, \nu(a) \rangle$$

for all $b \in A$. We have

$$\langle b, \nu(aa') \rangle = \langle aa', b \rangle = \langle a, a'b \rangle = \langle a'b, \nu(a) \rangle$$
$$= \langle a', b\nu(a) \rangle = \langle b\nu(a), \nu(a') \rangle = \langle b, \nu(a)\nu(a') \rangle$$

So $\nu(aa') = \nu(a)\nu(a')$ and ν is an algebra automorphism, the Nakayama automorphism.