A Short Introduction to Hopf Algebras

Lecture 2

Memorial
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Summary

1. Coalgebras: Coproduct, counit
2. Bialgebras: Coproduct and counit are algebra homomorphisms
3. Hopf algebras: Antipode
4. Example: Group rings
5. Coproduct: Can form tensor products of modules
6. Dualization leads from coalgebras to algebras.

7. The antipode is an algebra and a coalgebra antihomomorphism:

$$S_H(hh') = S_H(h')S_H(h)$$

$$\Delta_H(S_H(h)) = S_H(h_{(2)}) \otimes S_H(h_{(1)})$$

Overview

The skew-antipode equation

Suppose that S_H is bijective.

Then we have

$$S_H^{-1}(h_{(2)})h_{(1)} = \varepsilon_H(h)1_H$$
$$h_{(2)}S_H^{-1}(h_{(1)}) = \varepsilon_H(h)1_H$$

Proof: (of first equation)

Apply the inverse antipode to:

$$S_H(h_{(1)})h_{(2)} = \varepsilon_H(h)1_H$$

The opposite Hopf algebra

H: Hopf algebra with bijective antipode.

Opposite Hopf algebra:

Change multiplication to

$$h \cdot_{\mathrm{op}} h' := h' h$$

Change antipode to its inverse.

Coopposite Hopf algebra:

Change comultiplication to

$$\Delta^{\mathrm{cop}}(h) := h_{(2)} \otimes h_{(1)}$$

Change antipode to its inverse.

Special elements

H a Hopf algebra.

$g \in H$ is called grouplike if $g \neq 0$ and

$$\Delta_H(g) = g \otimes g$$

$x \in H$ is called primitive if

$$\Delta_H(x) = x \otimes 1_H + 1_H \otimes x$$

$x \in H$ is called skew-primitive if

$$\Delta_H(x) = x \otimes g + h \otimes x$$

for grouplike elements g and h.

Values of counit and antipode

g grouplike:

$$\varepsilon_H(g) = 1 \qquad S_H(g) = g^{-1}$$

$x \in H$ skew-primitive:

$$\varepsilon_H(x) = 0 \qquad S_H(x) = -h^{-1}xg^{-1}$$

Proof: g grouplike $\Rightarrow$

$$g = \varepsilon_H(g)g \qquad gS_H(g) = \varepsilon_H(g)1_H$$

$x \in H$ skew-primitive $\Rightarrow$

$$x = \varepsilon_H(x)g + \varepsilon_H(h)x$$

$$xg^{-1} + hS_H(x) = \varepsilon_H(x)1_H = 0$$

Proposition

The set $G(H)$ of grouplike elements of H is a group under multiplication.

Distinct grouplike elements are linearly independent.

Proof: g, h grouplike $\Rightarrow$

$$\Delta_H(gh) = gh \otimes gh$$

The unit element 1_H is grouplike, and invertibility was just proved.

Suppose: $g_1, \ldots, g_n \in G(H)$ linearly independent,

$g \in \mathrm{Span}(g_1, \ldots, g_n)$ also grouplike. Then

$$g = \sum_{i=1}^{n} \lambda_i g_i$$

Apply coproduct:

$$\sum_{i=1}^{n} \lambda_i \; g_i \otimes g_i = g \otimes g = \sum_{i,j=1}^{n} \lambda_i \lambda_j \; g_i \otimes g_j$$

Compare coefficients: $\lambda_i \lambda_j = \delta_{ij} \lambda_i$

$\lambda_i \neq 0 \Rightarrow \lambda_j = 0$ for $j \neq i \Rightarrow g = \lambda_i g_i \Rightarrow g = g_i$

Now the assertion follows by induction on the number of grouplikes.

Dualization in Sweedler notation: Coalgebras

Recall: If C is a coalgebra with coproduct Δ, then the dual space C^* is an algebra with product

$$C^* \otimes C^* \longrightarrow (C \otimes C)^* \xrightarrow{\Delta^T} C^*$$

In Sweedler notation, this means

$$(\varphi\psi)(c) = \varphi(c_{(1)})\psi(c_{(2)})$$

Proof:

$$(\varphi\psi)(c) = (\Delta^T(\varphi \otimes \psi))(c) = (\varphi \otimes \psi)(\Delta(c))$$

$$= (\varphi \otimes \psi)(c_{(1)} \otimes c_{(2)}) = \varphi(c_{(1)})\psi(c_{(2)})$$

Dualization in Sweedler notation: Algebras

Recall: If A is a finite-dimensional algebra whose multiplication is described by $m : A \otimes A \to A$, then the dual space A^* is a coalgebra with coproduct

$$A^* \xrightarrow{m^*} (A \otimes A)^* \to A^* \otimes A^*$$

In Sweedler notation, this means

$$\varphi(ab) = \varphi_{(1)}(a)\varphi_{(2)}(b)$$

Proof:

$$\varphi_{(1)}(a)\varphi_{(2)}(b) = (\varphi_{(1)} \otimes \varphi_{(2)})(a \otimes b) = m^T(\varphi)(a \otimes b)$$

$$= \varphi(m(a \otimes b)) = \varphi(ab)$$

Modules

Like the product on an algebra A can be described via a linear map $m : A \otimes A \to A$, the action of A on a left module M can be described via a linear map

$$\alpha : A \otimes M \to M$$

Since $(ab).m = a.(b.m)$, we have $\alpha \circ (m \otimes \mathrm{id}_M) = \alpha \circ (\mathrm{id}_A \otimes \alpha)$. Expressed as a diagram, this equation becomes

$$\begin{array}{ccc}
A \otimes A \otimes M & \xrightarrow{\;m \otimes \mathrm{id}_M\;} & A \otimes M \\
\Big\downarrow {\scriptstyle \mathrm{id}_A \otimes \alpha} & & \Big\downarrow {\scriptstyle \alpha} \\
A \otimes M & \xrightarrow[\;\alpha\;]{} & M
\end{array}$$

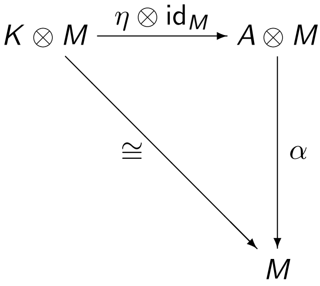
Unital modules

We encode the unit element as a map

$$\eta : K \to A,\ \lambda \mapsto \lambda 1_A$$

The axiom $1_A.m = m$ then reads $(\alpha \circ (\eta \otimes \mathrm{id}_M))(\lambda \otimes m) = \lambda m$.

Expressed as a diagram, this equation becomes



Comodules

Like an algebra can have modules, a coalgebra C can have comodules. They arise from modules by reversing the arrows. For a comodule V over C, we have a coaction

$$\delta : V \to C \otimes V$$

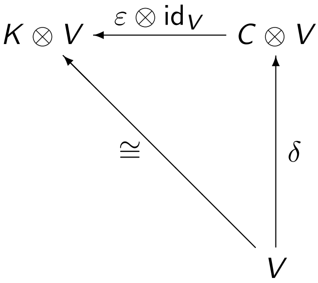
that replaces $\alpha : A \otimes M \to M$. The first diagram above becomes

$$\begin{array}{ccc} C \otimes C \otimes V & \xleftarrow{\Delta \otimes \mathrm{id}_V} & C \otimes V \\ \Big\uparrow{\scriptstyle \mathrm{id}_C \otimes \delta} & & \Big\uparrow{\scriptstyle \delta} \\ C \otimes V & \xleftarrow[\delta]{} & V \end{array}$$

In other words, we have $(\Delta \otimes \mathrm{id}_V) \circ \delta = (\mathrm{id}_C \otimes \delta) \circ \delta$.

Counitality

If we reverse the arrows in the diagram that expresses unitality, we get



Expressed in terms of equations, this reads

$$(\varepsilon \otimes \mathrm{id}_V)(\delta(v)) = 1_K \otimes v$$

Definition

A left comodule over a coalgebra C is a vector space V together with a linear map

$$\delta : V \to C \otimes V$$

satisfying

$$(\Delta \otimes \mathrm{id}_V) \circ \delta = (\mathrm{id}_C \otimes \delta) \circ \delta$$

and

$$(\varepsilon \otimes \mathrm{id}_V)(\delta(v)) = 1_K \otimes v$$

for all $v \in V$.

Sweedler notation for coactions

We use a version of Sweedler's notation with upper indices for coactions: Instead of $\Delta(c) = c_{(1)} \otimes c_{(2)}$, we write

$$\delta(v) = v^{(1)} \otimes v^{(2)} \in C \otimes V$$

Iterated cooperations are represented in the form

$$(\Delta \otimes \mathrm{id}_V)(\delta(v)) = (\mathrm{id}_C \otimes \delta)(\delta(v)) = v^{(1)} \otimes v^{(2)} \otimes v^{(3)} \in C \otimes C \otimes V$$

Counitality becomes

$$\varepsilon(v^{(1)})v^{(2)} = v$$

Our convention deviates from the standard: Most authors write

$$\delta(v) = v^{(-1)} \otimes v^{(0)} \in C \otimes V$$

Module homomorphisms

If A is an algebra (over a field K) and M and N are A-modules, then a K-linear map $f : M \to N$ is a module homomorphism if $f(a.m) = a.f(m)$. In terms of the action maps, this means

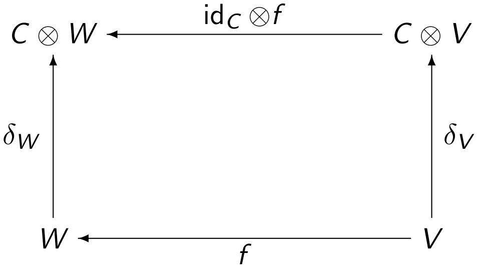
$$f \circ \alpha = \alpha \circ (\mathrm{id}_A \otimes f)$$

or in diagrammatic form

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\mathrm{id}_A \otimes f} & A \otimes N \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ M & \xrightarrow[f]{} & N \end{array}$$

Comodule homomorphisms

Dually, if C is a coalgebra and V and W are left comodules, then a K-linear map $f : V \to W$ is a comodule homomorphism if the diagram



is commutative, in other words, if

$$\delta_W \circ f = (\mathrm{id}_C \otimes f) \circ \delta_V$$

Tensor products of modules and comodules

If H is a Hopf algebra and V and W are H-modules, we have seen in the first lecture that $V \otimes W$ is an H-module via

$$h.(v \otimes w) := \Delta(h).(v \otimes w) = h_{(1)}.v \otimes h_{(2)}.w$$

The action map for $V \otimes W$ is therefore the composition

$$H \otimes V \otimes W \xrightarrow{\Delta \otimes \mathrm{id}_{V \otimes W}} H \otimes H \otimes V \otimes W \xrightarrow{\mathrm{id}_V \otimes \tau \otimes \mathrm{id}_W} H \otimes V \otimes H \otimes W \xrightarrow{\alpha_V \otimes \alpha_W} V \otimes W$$

Dually, for H-comodules V and W, we define the coaction on $V \otimes W$ as

$$H \otimes V \otimes W \xleftarrow{m \otimes \mathrm{id}_{V \otimes W}} H \otimes H \otimes V \otimes W \xleftarrow{\mathrm{id}_V \otimes \tau \otimes \mathrm{id}_W} H \otimes V \otimes H \otimes W \xleftarrow{\delta_V \otimes \delta_W} V \otimes W$$

In formulas, this says that

$$\delta_{V \otimes W}(v \otimes w) = v^{(1)} w^{(1)} \otimes v^{(2)} \otimes w^{(2)}$$

Right comodules

So far, we have worked with left comodules.

There are also right comodules.

Left comodules: Coaction $\delta : V \to C \otimes V$

Axioms:

$$(\Delta \otimes \mathrm{id}_V) \circ \delta = (\mathrm{id}_C \otimes \delta) \circ \delta$$

$$(\varepsilon \otimes \mathrm{id}_V)(\delta(v)) = 1_K \otimes v$$

Right comodules: Coaction $\delta : V \to V \otimes C$

Axioms:

$$(\mathrm{id}_V \otimes \Delta) \circ \delta = (\delta \otimes \mathrm{id}_C) \circ \delta$$

$$(\mathrm{id}_V \otimes \varepsilon)(\delta(v)) = v \otimes 1_K$$

Sweedler notation: $\delta(v) = v^{(1)} \otimes v^{(2)} \in V \otimes C$

Hopf modules

H: Hopf algebra.

Hopf module: Right H-module and right H-comodule M.

Compatibility condition:

$$\delta(m \leftarrow h) = (m^{(1)} \leftarrow h_{(1)}) \otimes m^{(2)} h_{(2)}$$

Example 1

$M = H$, action=multiplication, coaction=comultiplication,

i.e., $\delta = \Delta$.

Generalization: V vector space, $M := V \otimes H$

Action:

$$(v \otimes h) \leftarrow h' := v \otimes hh'$$

Coaction:

$$\delta : M \to M \otimes H, \; v \otimes h \mapsto v \otimes h_{(1)} \otimes h_{(2)}$$

Example 2

H finite-dimensional.

H^* is a Hopf module:

Left coregular action, plus antipode:

$$\varphi \leftarrow h := \varphi_{(2)}(S_H(h))\, \varphi_{(1)}$$

Right coregular coaction:

$$\delta(\varphi) = \sum_{i=1}^{n} \varphi(h_{i(2)})\, h_i^* \otimes h_{i(1)}$$

Proof. Have to show:

$$\delta(\varphi \leftarrow h) = (\varphi^{(1)} \leftarrow h_{(1)}) \otimes \varphi^{(2)} h_{(2)}$$

This means:

$$\sum_{i=1}^{n} \varphi_{(2)}(S_H(h))h_i^* \otimes h_{i(1)}\varphi_{(1)}(h_{i(2)})$$
$$= \sum_{i=1}^{n} \varphi(h_{i(2)})(h_i^* \leftarrow h_{(1)}) \otimes h_{i(1)}h_{(2)}$$

Take $h' \in H$ and evaluate the left tensorand on it:

$$\sum_{i=1}^{n} \varphi_{(2)}(S_H(h))h_i^*(h')\ h_{i(1)}\varphi_{(1)}(h_{i(2)})$$
$$= \sum_{i=1}^{n} \varphi(h_{i(2)})h_i^*(h' S_H(h_{(1)}))\ h_{i(1)}h_{(2)}$$

Equivalently:

$$\varphi_{(2)}(S_H(h))h'_{(1)}\varphi_{(1)}(h'_{(2)}) = \varphi(h'_{(2)}S_H(h_{(1)}))h'_{(1)}S_H(h_{(2)})h_{(3)}$$