A Short Introduction to Hopf Algebras

Lecture 1

Memorial
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Overview

1. Representations of groups and algebras
2. The group algebra
3. Hopf algebras
4. Coalgebras and bialgebras
5. Dualization
6. Sweedler notation
7. Properties of the antipode

Group representations

G: Group.

$\rho : G \to \mathrm{GL}(V)$ group homomorphism.

Notation: $\rho(g)(v) = g.v$

Tensor product of representations:

$\rho' : G \to \mathrm{GL}(W)$ second representation.

Then

$$G \to \mathrm{GL}(V \otimes W), \; g \mapsto \rho(g) \otimes \rho'(g)$$

is again a representation.

In other words:

$$g.(v \otimes w) = g.v \otimes g.w$$

Modules for algebras

A: Algebra over a field K.

(In the following, K-vector space structures are understood.)

Representation of A:

Algebra homomorphism $\rho : A \to \mathrm{End}(V)$

Alternatively: V is an A-module via

$$A \times V \to V,\ (a, v) \mapsto a.v := \rho(a)(v)$$

For a second representation

$\rho' : A \to \mathrm{End}(W)$, the map

$$A \to \mathrm{End}(V \otimes W),\ a \mapsto \rho(a) \otimes \rho'(a)$$

is not an algebra homomorphism.

Coproduct

Additional structure element:

Algebra homomorphism

$$\Delta : A \to A \otimes A$$

Tensor product of representations:

$$A \to \mathrm{End}(V \otimes W),\ a \mapsto (\rho \otimes \rho')\Delta(a)$$

Canonical isomorphism

$$(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$$

should be A-linear $\Rightarrow$ Coproduct should be coassociative:

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

Group algebra

G Group. Group algebra $K[G]$:

Vector space with basis G

Multiplication: Multiplication on basis elements already defined, extend as a bilinear function:

$$(\sum_{g \in G} \lambda_g g)(\sum_{h \in G} \mu_h h) := \sum_{g,h \in G} \lambda_g \mu_h \,(gh)$$

Representations of G = Modules of $K[G]$:

A group homomorphism $\rho : G \to \mathrm{GL}(V)$ extends linearly to an algebra homomorphism $K[G] \to \mathrm{End}(V)$

Coproduct of the group algebra

$$\Delta : K[G] \to K[G] \otimes K[G]$$

Define on basis elements $g \in G$:

$$\Delta(g) := g \otimes g$$

Extend linearly:

$$\Delta(\sum_{g \in G} \lambda_g \, g) := \sum_{g \in G} \lambda_g \, g \otimes g$$

Tensor product of representations:

$$(\rho \otimes \rho')\Delta(g) = \rho(g) \otimes \rho'(g)$$

for $g \in G$.

Special representations

G group. Trivial representation:

$$\varepsilon : G \to \mathrm{End}(K) \cong K,\ g \mapsto \mathrm{id}_K \cong 1$$

It satisfies $K \otimes V \cong V \cong V \otimes K$

For any representation V, we have the dual representation:

$$\rho^* : G \to \mathrm{End}(V^*),\ g \mapsto \rho(g^{-1})^T$$

It makes the evaluation map

$V^* \otimes V \to K$ $K[G]$-linear:

$$\rho(g)^*(\varphi) \otimes \rho(g)(v) \mapsto \varphi(\rho(g^{-1})\rho(g)(v)) = \varphi(v)$$

Counit and antipode

Linear extension: $\varepsilon : K[G] \to K$. Then:

$$(\varepsilon \otimes \mathrm{id})\Delta(g) = 1 \otimes g \qquad (\mathrm{id} \otimes \varepsilon)\Delta(g) = g \otimes 1$$

ε: counit

Antipode: $S : K[G] \to K[G]$, defined on basis elements by

$$S(g) = g^{-1}$$

and extended linearly. It satisfies

$$m(S \otimes \mathrm{id})\Delta(g) = \varepsilon(g)1 = m(\mathrm{id} \otimes S)\Delta(g)$$

Hopf algebras

A Hopf algebra is an algebra H together with a coassociative algebra homomorphism (called the coproduct)

$$\Delta : H \to H \otimes H$$

an algebra homomorphism

$$\varepsilon : H \to K$$

called the counit and satisfying

$$(\varepsilon \otimes \mathrm{id})\Delta(h) = 1 \otimes h \qquad (\mathrm{id} \otimes \varepsilon)\Delta(h) = h \otimes 1$$

and a linear map

$$S : H \to H$$

called the antipode and satisfying

$$m(S \otimes \mathrm{id})\Delta(h) = \varepsilon(h)1 = m(\mathrm{id} \otimes S)\Delta(h)$$

Coalgebras

A coalgebra is a vector space C together with a coassociative linear map (called the coproduct)

$$\Delta : C \to C \otimes C$$

an a linear map

$$\varepsilon : C \to K$$

called the counit and satisfying

$$(\varepsilon \otimes \mathrm{id})\Delta(c) = 1 \otimes c \qquad (\mathrm{id} \otimes \varepsilon)\Delta(c) = c \otimes 1$$

Bialgebras

A bialgebra is an algebra that is simultaneously a coalgebra such that the coproduct and the counit are algebra homomorphisms.

A Hopf algebra is a bialgebra with an antipode.

Dualization

C: Coalgebra with coproduct Δ.

Transpose Δ^T maps $(C \otimes C)^*$ to C^*.

There is a canonical injective map from $C^* \otimes C^*$ to $(C \otimes C)^*$.

So we can define a multiplication on C^* as the composition

$$C^* \otimes C^* \longrightarrow (C \otimes C)^* \xrightarrow{\Delta^T} C^*$$

The coassociativity of the coproduct implies comparatively directly that this product is associative.

Counit = unit element of the dual

Lemma: If $\varphi \in C^*$, we have $\varepsilon\varphi = \varphi = \varphi\varepsilon$.

Proof: By the definition of the product, we have

$$(\varepsilon\varphi)(c) = (\Delta^T(\varepsilon \otimes \varphi))(c) = (\varepsilon \otimes \varphi)(\Delta(c))$$

We have $(\varepsilon \otimes \mathrm{id}_C)\Delta(c) = 1_K \otimes c$. Apply φ to this equation to get the first assertion.

For the second equation, use the counit equation on the other side.

Converse

Suppose that A is an algebra.

Multiplication induces a linear map $m : A \otimes A \to A$.

The transpose m^T is a map from A^* to $(A \otimes A)^*$.

Problem: The canonical map $A^* \otimes A^* \to (A \otimes A)^*$ is not invertible if A is infinite-dimensional.

If $\dim(A) < \infty \Rightarrow$ map is bijective.

Get coproduct as the composition

$$A^* \stackrel{m^T}{\to} (A \otimes A)^* \to A^* \otimes A^*$$

Coassociativity follows from associativity.

Counit of the dual

Counit of $A^* =$ Evaluation at the unit element:

Proposition The map

$$\varepsilon : A^* \to K,\ \varphi \mapsto \varphi(1_A)$$

is a counit.

Proof

For $\varphi \in A^*$, write $m^T(\varphi)$ as a sum

$$m^T(\varphi) = \sum_{i=1}^{n} \psi_i \otimes \psi_i'$$

of decomposable tensors (not unique). For $a, b \in A$, we then have

$$\sum_{i=1}^{n} \psi_i(a)\psi_i'(b) = m^T(\varphi)(a \otimes b) = \varphi(m(a \otimes b)) = \varphi(ab)$$

For $b = 1_A$, this means $\sum_{i=1}^{n} \psi_i(a)\psi_i'(1_A) = \varphi(a)$, or alternatively

$$\varphi = \sum_{i=1}^{n} \psi_i'(1_A)\ \psi_i = \sum_{i=1}^{n} \varepsilon(\psi_i')\ \psi_i = (\mathrm{id}_{A^*} \otimes \varepsilon)\left(\sum_{i=1}^{n} \psi_i \otimes \psi_i'\right)$$

$$= (\mathrm{id}_{A^*} \otimes \varepsilon)(m^T(\varphi))$$

which is one of the two conditions for a counit. The other condition can be shown in a similar way.

Sweedler notation

Suppose that $\beta : C^n = C \times C \times \ldots \times C \to V$ is a multilinear map. Universal property of the tensor product: β induces a linear map $f : C^{\otimes n} \to V$ with

$$f(c_1 \otimes c_2 \otimes \ldots \otimes c_n) = \beta(c_1, c_2, \ldots, c_n)$$

Sweedler notation:

$$\beta(c_{(1)}, c_{(2)}, \ldots, c_{(n)}) := f(\Delta_n(c))$$

where $\Delta_n : C \longrightarrow C^{\otimes n} = C \otimes C \otimes \ldots \otimes C$ is the iterated coproduct.

Simple example

$\beta = \otimes$, the ordinary tensor product.

Then $n = 2$ and $f = \mathrm{id}$, so the preceding equation becomes

$$\Delta(c) = c_{(1)} \otimes c_{(2)}$$

Not asserted: $\Delta(c)$ is a decomposable tensor.

There is no well-defined 'first component' $c_{(1)}$ of the tensor $\Delta(c)$.

To remind the reader of the fact that this expression does not constitute a decomposable tensor, but rather is a sum of decomposable tensors, many textbooks introduce a sigma in this notation and write the preceding equation as

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$$

Note also

$$\Delta_n(c) = c_{(1)} \otimes c_{(2)} \otimes \ldots \otimes c_{(n)}$$

Identities in Sweedler notation

Counit: $(\varepsilon \otimes \mathrm{id}_C)\Delta(c) = 1_K \otimes c$ and $(\mathrm{id}_C \otimes \varepsilon)\Delta(c) = c \otimes 1_K$

Sweedler notation:

$$\varepsilon(c_{(1)})c_{(2)} = c = c_{(1)}\varepsilon(c_{(2)})$$

Multiplicativity of the coproduct:

$$(hh')_{(1)} \otimes (hh')_{(2)} = h_{(1)}h'_{(1)} \otimes h_{(2)}h'_{(2)}$$

Antipode equation $m(S \otimes \mathrm{id})\Delta(h) = \varepsilon(h)1 = m(\mathrm{id} \otimes S)\Delta(h)$ reads

$$S(h_{(1)})h_{(2)} = \varepsilon(h)1 = h_{(1)}S(h_{(2)})$$

Properties of the antipode

1. The antipode is an algebra antihomomorphism:

$$S_H(hh') = S_H(h')S_H(h)$$

and also $S_H(1_H) = 1_H$.

2. The antipode is a coalgebra antihomomorphism:

$$\Delta_H(S_H(h)) = S_H(h_{(2)}) \otimes S_H(h_{(1)})$$

and also $\varepsilon_H(S_H(h)) = \varepsilon_H(h)$.

Proof

Multiplicativity:

$$S_H(hh') = S_H(h_{(1)}h'_{(1)})h_{(2)}h'_{(2)}S_H(h'_{(3)})S_H(h_{(3)}) = S_H(h')S_H(h)$$

Preservation of the unit:

$$\Delta_H(1_H) = 1_H \otimes 1_H \Rightarrow 1_H S_H(1_H) = \varepsilon_H(1_H)1_H = 1_H$$

Comultiplicativity:

$$\Delta_H(S_H(h)) = S_H(h)_{(1)} \otimes S_H(h)_{(2)} =$$

$$S_H(h_{(1)})_{(1)} h_{(2)} S_H(h_{(5)}) \otimes S_H(h_{(1)})_{(2)} h_{(3)} S_H(h_{(4)})$$

$$= S_H(h_{(2)}) \otimes S_H(h_{(1)})$$

Preservation of the counit:

$$h_{(1)} S_H(h_{(2)}) = \varepsilon_H(h) 1_H$$

$$\Rightarrow \varepsilon_H(S_H(h)) = \varepsilon_H(h_{(1)}) \varepsilon_H(S_H(h_{(2)})) = \varepsilon_H(h)$$

Summary

1. Coalgebras: Coproduct, counit
2. Bialgebras: Coproduct and counit are algebra homomorphisms
3. Hopf algebras: Antipode
4. Example: Group rings
5. Coproduct: Can form tensor products of modules
6. Dualization leads from coalgebras to algebras.

7. The antipode is an algebra and a coalgebra antihomomorphism:

$$S_H(hh') = S_H(h')S_H(h)$$

$$\Delta_H(S_H(h)) = S_H(h_{(2)}) \otimes S_H(h_{(1)})$$