# Surface braids and mapping class group IV
# Surface braids vs knot theory in 3-manifolds

Paolo Bellingeri

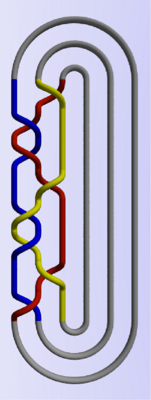
Laboratoire de Mathématiques Nicolas Oresme, Université de Caen

Atlantic Algebra Centre

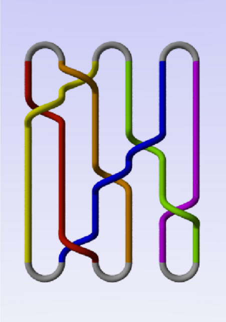
# Braids and knots

**Closure(s) of classical braids :**

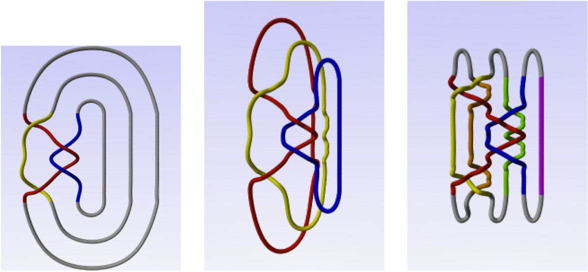
Alexander closure :



Plat closure :



# From Alexander to plat closure.



- Markov Theorem : characterization braids having isotopic Alexander closure ;
- Hilden - Birman Theorem : characterization braids having isotopic plat closure.

# From braids to surface braids

Alexander and plat closure give rise to two different notions of closures for surface braids as links in generic 3-manifolds.

- Alexander closure corresponds to relate surface braids on surfaces with boundary to links in closed 3 manifolds via *open book decomposition*. There exist analogous of Alexander and Markov Theorems (Skora 1992, Sundheim 1993). Similar theorems relating surface braids on a closed surface $\Sigma_g$ to links in $\Sigma_g \times S^1$ have been proved using Birman exact sequence (Grant-Sienicka 2019).
- Plat closure can be generalized to a relation between braids on closed surfaces and links in 3-manifolds via *Heegard splitting* (B-Cattabriga 2012).

A third approach to relate surface braids and links in 3-manifolds is through mixed braids (Lambropoulou, Lambropolou-Holdenburg, Diamantis-Lambropolou-Przytycki...)

# An application to knot theory : Hilden groups

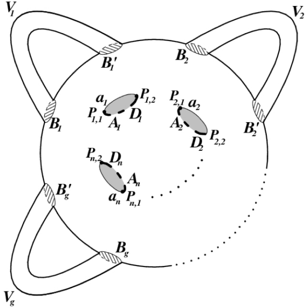
Let $\mathrm{H}_g$ be an oriented handlebody of genus $g \geq 0$ ($\Sigma_g := \partial \mathrm{H}_g$).

**Trivial system of *n* arcs :** $\mathcal{A}_n = \{A_1, \ldots, A_n\}$ set of disjoint unknotted arcs properly embedded in $\mathrm{H}_g$ with endpoints in $\Sigma_g := \partial \mathrm{H}_g$.

# An application to knot theory : Hilden groups

Let $\mathrm{H}_g$ be an oriented handlebody of genus $g \geq 0$ ($\Sigma_g := \partial \mathrm{H}_g$).

**Trivial system of $n$ arcs :** $\mathcal{A}_n = \{A_1, \ldots, A_n\}$ set of disjoint unknotted arcs properly embedded in $\mathrm{H}_g$ with endpoints in $\Sigma_g := \partial \mathrm{H}_g$.




- A trivial system of $n$ arcs can be seen as a trivial *n-bottom tangle* in $\Sigma_g \times I$.

# Hilden groups of genus $g$

**Extending homeomorphisms from $\Sigma_g$ to $\mathrm{H}_g$ :**
$\mathcal{E}_{g,n} := \{\phi \in \mathcal{M}_{2n}(\Sigma_g) \,|\, \phi$ extends to $\tilde{\phi} : \mathrm{H}_g \to \mathrm{H}_g$ s.t. $\tilde{\phi}(\mathcal{A}_n) = \mathcal{A}_n\}$
$\mathcal{E}_g := \{\phi \in \mathcal{M}(\Sigma_g) \,|\, \phi$ extends to $\tilde{\phi} : \mathrm{H}_g \to \mathrm{H}_g\}$

- The forgetting map $\psi_{g,2n} : \mathcal{M}_{2n}(\Sigma_g) \to \mathcal{M}(\Sigma_g)$ restricts to $\psi'_{g,n} : \mathcal{E}_{g,n} \to \mathcal{E}_g$.

# Hilden groups of genus $g$

**Extending homeomorphisms from $\Sigma_g$ to $\mathrm{H}_g$ :**
$\mathcal{E}_{g,n} := \{\phi \in \mathcal{M}_{2n}(\Sigma_g) \,|\, \phi$ extends to $\tilde{\phi} : \mathrm{H}_g \to \mathrm{H}_g$ s.t. $\tilde{\phi}(\mathcal{A}_n) = \mathcal{A}_n \}$
$\mathcal{E}_g := \{\phi \in \mathcal{M}(\Sigma_g) \,|\, \phi$ extends to $\tilde{\phi} : \mathrm{H}_g \to \mathrm{H}_g \}$

- The forgetting map $\psi_{g,2n} : \mathcal{M}_{2n}(\Sigma_g) \to \mathcal{M}(\Sigma_g)$ restricts to $\psi'_{g,n} : \mathcal{E}_{g,n} \to \mathcal{E}_g$.

**Definition (B.-Cattabriga).**
***n*th Hilden group of $\Sigma_g$ :** $Hil_n(\Sigma_g) := ker \psi'_{g,n}$

# Hilden groups of genus $g$

**Extending homeomorphisms from $\Sigma_g$ to $H_g$ :**
$\mathcal{E}_{g,n} := \{\phi \in \mathcal{M}_{2n}(\Sigma_g) \,|\, \phi$ extends to $\tilde{\phi} : H_g \to H_g$ s.t. $\tilde{\phi}(\mathcal{A}_n) = \mathcal{A}_n \}$
$\mathcal{E}_g := \{\phi \in \mathcal{M}(\Sigma_g) \,|\, \phi$ extends to $\tilde{\phi} : H_g \to H_g \}$

- The forgetting map $\psi_{g,2n} : \mathcal{M}_{2n}(\Sigma_g) \to \mathcal{M}(\Sigma_g)$ restricts to $\psi'_{g,n} : \mathcal{E}_{g,n} \to \mathcal{E}_g$.

**Definition (B.-Cattabriga).**
***n*th Hilden group of $\Sigma_g$ :** $Hil_n(\Sigma_g) := ker\psi'_{g,n}$

**Remark.** $Hil_n(\Sigma_g)$ is related to links in closed manifolds via Heegard splitting (extension of plate closures of links in $S^3$ to $(g, n)$-*decompositions* of links in closed manifolds)

$\ker \psi_{g,n} \simeq B_n(\Sigma_g)$ if $g > 1$ ;
$\ker \psi_{g,n} \simeq B_n(\Sigma_g)/ZB_n(\Sigma_g)$ if $\Sigma_g = \mathbb{S}^2$ or $\mathbb{T}^2$ ,
where $\psi_{g,n} : \mathcal{M}_n(\Sigma_g) \to \mathcal{M}(\Sigma_g)$ is the forgetting map.
$Hil_n(\Sigma_g) := \{\phi \in \frac{B_{2n}(\Sigma_g)}{ZB_{2n}(\Sigma_g)} \mid \phi$ extends to $\tilde{\phi} : \mathrm{H}_g \to \mathrm{H}_g$ s.t. $\tilde{\phi}(\mathcal{A}_n) = \mathcal{A}_n \}$

- Hilden (1976) : Finite set of generators for $Hil_n(S^2)$.
- Tawn (2008) Brendle-Hatcher (2011) : Finite type group presentation for $Hil_n(S^2)$.

**Proposition (B.-Cattabriga)** Let $g > 0$. The framed braid group $FB_n(\Sigma_g)$ embeds into $Hil_n(\Sigma_g)$.

$\ker \psi_{g,n} \simeq B_n(\Sigma_g)$ if $g > 1$ ;
$\ker \psi_{g,n} \simeq B_n(\Sigma_g)/ZB_n(\Sigma_g)$ if $\Sigma_g = \mathbb{S}^2$ or $\mathbb{T}^2$ ,
where $\psi_{g,n} : \mathcal{M}_n(\Sigma_g) \to \mathcal{M}(\Sigma_g)$ is the forgetting map.
$Hil_n(\Sigma_g) := \{\phi \in \frac{B_{2n}(\Sigma_g)}{ZB_{2n}(\Sigma_g)} \mid \phi$ extends to $\tilde{\phi} : \mathrm{H}_g \to \mathrm{H}_g$ s.t. $\tilde{\phi}(\mathcal{A}_n) = \mathcal{A}_n \}$

- Hilden (1976) : Finite set of generators for $Hil_n(S^2)$.
- Tawn (2008) Brendle-Hatcher (2011) : Finite type group presentation for $Hil_n(S^2)$.

**Proposition (B.-Cattabriga)** Let $g > 0$. The framed braid group $FB_n(\Sigma_g)$ embeds into $Hil_n(\Sigma_g)$.

$\ker \psi_{g,n} \simeq B_n(\Sigma_g)$ if $g > 1$ ;
$\ker \psi_{g,n} \simeq B_n(\Sigma_g)/ZB_n(\Sigma_g)$ if $\Sigma_g = \mathbb{S}^2$ or $\mathbb{T}^2$ ,
where $\psi_{g,n} : \mathcal{M}_n(\Sigma_g) \to \mathcal{M}(\Sigma_g)$ is the forgetting map.
$Hil_n(\Sigma_g) := \{\phi \in \frac{B_{2n}(\Sigma_g)}{ZB_{2n}(\Sigma_g)} \mid \phi$ extends to $\tilde{\phi} : \mathrm{H}_g \to \mathrm{H}_g$ s.t. $\tilde{\phi}(\mathcal{A}_n) = \mathcal{A}_n \}$

- Hilden (1976) : Finite set of generators for $Hil_n(S^2)$.
- Tawn (2008) Brendle-Hatcher (2011) : Finite type group presentation for $Hil_n(S^2)$.

**Proposition (B.-Cattabriga)** Let $g > 0$. The framed braid group $FB_n(\Sigma_g)$ embeds into $Hil_n(\Sigma_g)$.

**Theorem (B.-Cattabriga)** $Hil_n(\Sigma_g)$ is generated by :

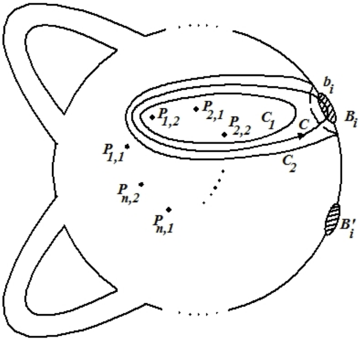
- a set of generators of $FB_n(\Sigma_g)$.

**Theorem (B.-Cattabriga)** $Hil_n(\Sigma_g)$ is generated by :

- a set of generators of $FB_n(\Sigma_g)$.
- Admissible meridian slides ;

**Theorem (B.-Cattabriga)** $Hil_n(\Sigma_g)$ is generated by :

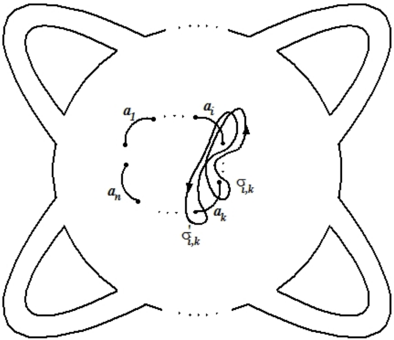
- a set of generators of $FB_n(\Sigma_g)$.
- Admissible meridian slides ;




The slide $M_{i,C} = T_{C_1}^{-1} T_{C_2} T_{b_i}^{-1}$ of the meridian disk $B_i$ along the curve $C$.

**Theorem (B.-Cattabriga)** $Hil_n(\Sigma_g)$ is generated by :

- a set of generators of $FB_n(\Sigma_g)$.
- Admissible meridian slides ;
- slides of arc $a_1$ around $P_{i,1}$ for $i = 2, \ldots, n$.



# Generalized Hilden groups and $(g, b)$-links

**$(g, n)$-decomposition of a link :**
Let $L$ be a link in a 3-manifold $M$. $L$ is a $(g, n)$-link if there exists a genus $g$
Heegaard surface $S$ for $M$ (i. e. $M = \mathrm{H}_g \cup_\phi \bar{\mathrm{H}}_g$) s. t. :

(i) $L$ intersects $S$ transversally and

(ii) $L \cap \mathrm{H}_g$ and $L \cap \bar{\mathrm{H}}_g$ are trivial systems of $n$ arcs.

Let $(\mathrm{H}_g, \mathcal{A}_n)$ and $(\bar{\mathrm{H}}_g, \bar{\mathcal{A}}_n)$ be handlebodies of genus $g$. Let $\tau : \mathrm{H}_g \to \bar{\mathrm{H}}_g$ s t. $\tau(A_i) = \bar{A}_i$, for $i = 1, \ldots, n$.

**Definition** $\mathcal{L}_{g,n}$= set of equivalence classes of $(g, n)$-links.

**Proposition.** Let

$$\Theta_{g,n} : \mathcal{M}_{2n}(\Sigma_g) \longrightarrow \mathcal{L}_{g,n}\,, \quad \Theta_{g,n}(\psi) = L_\psi$$

where $L_\psi$ is the $(g, n)$-link in the 3-manifold $M_\psi$ defined by

$$(M_\psi, L_\psi) = (\mathrm{H}_g, \mathcal{A}_n) \cup_{\psi\tau} (\bar{\mathrm{H}}_g, \bar{\mathcal{A}}_n).$$

The map $\Theta_{g,n}$ is well defined and surjective.

Back to the forgetting map :

$$\psi_{g,n} : \mathcal{M}_{2n}(\Sigma_g) \to \mathcal{M}(\Sigma_g)$$

**Remark.** If $\beta_1, \beta_2 \in \mathcal{M}_{2n}(\Sigma_g)$ are s .t. $\psi_{g,n}(\beta_1) = \psi_{g,n}(\beta_2)$ then $L_{\beta_1}$ and $L_{\beta_2}$ belong to the same ambient manifold.

Let $M$ be a closed manifold. Let $\phi \in \mathcal{M}(\Sigma_{g,1}) \subset \mathcal{M}_{2n}(\Sigma_g)$ s.t. $M = M_\phi$. We get a map

$$\Theta^{\phi}_{g,n} : \ker(\psi_{g,n}) \longrightarrow \{(g,n) - \text{links in } M\}$$

Let $M$ be a closed manifold. Let $\phi \in \mathcal{M}(\Sigma_{g,1}) \subset \mathcal{M}_{2n}(\Sigma_g)$ s.t. $M = M_\phi$. We get a map

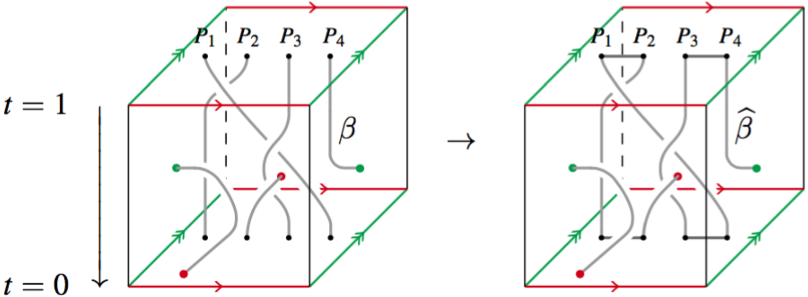
$$\Theta^{\phi}_{g,n} : \ker(\psi_{g,n}) \longrightarrow \{(g,n) - \text{links in } M\}$$

**Question.** Characterize surface braids having isotopic "Heegard" closure (analogous of Birman-Hilden Theorem).

**Proposition (B.-Cattabriga)** Let $\phi \in \mathcal{M}(\Sigma_{g,1}) \subset \mathcal{M}_{2n}(\Sigma_g)$ and $\phi Hil_n(\Sigma_g)\phi^{-1} := Hil_n(\Sigma_g)(\phi)$.

- If $\beta$ and $\delta$ belong to the same left coset of $Hil_n(\Sigma_g)$ in $\ker(\psi_{g,n})$ then $L_\beta$ and $L_\delta$ are isotopic links in $M_\phi$.
- If $\beta$ and $\delta$ belong to the same right coset of $Hil_n(\Sigma_g)(\phi)$ in $\ker(\psi_{g,n})$ then $L_\beta$ and $L_\delta$ are isotopic links in $M_\phi$.

# A new approach



# A new approach

## Theorem (Cattabriga-Gabrovsek 2018, revisited)

Two elements in $\bigcup_{n\in\mathbb{N}} B_{2n}(\Sigma_g)$ determine equivalent links in $\Sigma_g \times I$ if and only if they are connected by a finite sequence of the following moves :

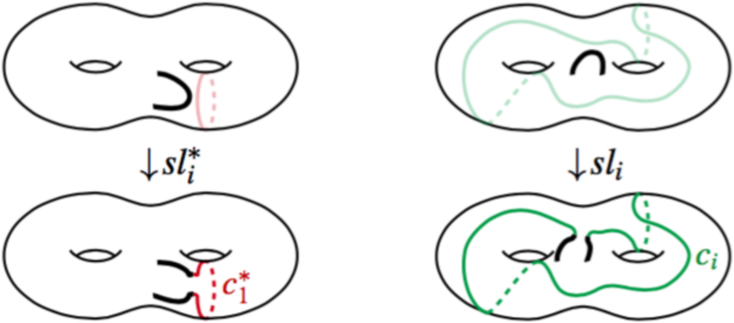
- Double coset moves via framed braids : $\alpha \leftrightarrow h\alpha h'$ for some $h, h' \in FB_n(\Sigma_g)$ ;
- "Sliding" moves : $\sigma_2\sigma_1^2\sigma_2\alpha \leftrightarrow \alpha \leftrightarrow \alpha\sigma_2\sigma_1^2\sigma_2$ ;
- "Stabilization" moves : $\alpha \leftrightarrow T_k(\alpha)\sigma_{2k}$ where $T_k(b_j) = a_j$ $T_k(b_j) = a_j$ for $1 \leq j \leq g$
  and $T_k(\sigma_i) = \begin{cases} \sigma_i & \text{if} \quad i < 2k \\ \sigma_{2k}\sigma_{2k+1}\sigma_{2k+2}\sigma_{2k+1}^{-1}\sigma_{2k}^{-1} & \text{if} \quad i = 2k \\ \sigma_{i+2} & \text{if} \quad i > 2k \end{cases}$

# A new approach

## "Hilden" Theorem (Cattabriga-Gabrovsek 2018)

The plat-closures of two surface braids $\beta_1$ and $\beta_2$ represent isotopic links in $M$ iff $\beta_1$ can be transformed to $\beta_2$ by a finite sequence of :

- isotopies of closed surface braids in $\Sigma_g \times I$
- plat and dual plat slide moves.



# A new approach

"Hilden" Theorem (Cattabriga-Gabrovsek 2018)

The plat-closures of two surface braids $\beta_1$ and $\beta_2$ represent isotopic links in $M$ iff $\beta_1$ can be transformed to $\beta_2$ by a finite sequence of :

- isotopies of closed surface braids in $\Sigma_g \times I$
- plat and dual plat slide moves.

**Goal (B.-Cattabriga-Gabrovsek)** : an "algebraic" version of previous Theorem.