Surface braids and mapping class group IV (Pure) Braid groups on surfaces of positive genus: other definitions and topological tools

Paolo Bellingeri

Laboratoire de Mathématiques Nicolas Oresme, Université de Caen

Atlantic Algebra Centre

Surface braids as mapping classes

Let Σ be an oriented surface, $\mathcal{P} = \{p_1, \ldots, p_n\} \subset \Sigma$

Mapping class group of Σ :
$\mathcal{M}(\Sigma) = \{h : \Sigma^+ \to \Sigma^+ ; h_{|\partial\Sigma} = Id\}_{/\sim}$
***n*-punctured Mapping class group of Σ** :

$$\mathcal{M}_n(\Sigma) = \left\{ \begin{array}{l} h : \Sigma^+ \to \Sigma^+ \\ h(p_i) \in \mathcal{P} \ i = 1, \ldots, n \\ h_{|\partial\Sigma} = Id \end{array} \right\}_{/\sim}$$

Let $\psi_n : \mathcal{M}_n(\Sigma) \to \mathcal{M}(\Sigma)$ be the *forgetting map*.

Theorem (Birman).
$1 \to B_n(\Sigma)/ZB_n(\Sigma) \to \mathcal{M}_n(\Sigma) \to \mathcal{M}(\Sigma) \to 1$ if $\Sigma = \mathbb{S}^2$ or $\mathbb{T}^2$.
$1 \to B_n(\Sigma) \to \mathcal{M}_n(\Sigma) \to \mathcal{M}(\Sigma) \to 1$ otherwise.

Applications : automorphisms of surface braids groups

Extended Mapping class group of Σ :

$\mathcal{M}^*(\Sigma) = \{h : \Sigma \to \Sigma; h_{|\partial\Sigma} = Id\}_{/\sim}$

Theorem (B. 2007). Let Σ_g be a closed oriented surface of genus $g > 1$ and $n > 2$.

- The group $Aut(B_n(\Sigma))$ is isomorphic to $\mathcal{M}^*_n(\Sigma_g)$.
- The group $Out(B_n(\Sigma))$ is isomorphic to $\mathcal{M}^*(\Sigma_g)$.

This result has been extended to other orientable surfaces by several authors (An, 2017).

Proposition. Let $g \geq 0$. $B_n(\Sigma_g)$ is hopfian.

Theorem (Kida-Yamagata. 2011). Let $g > 1$: any isomorphism between finite index subgroups of $B_n(\Sigma_g)$ is induced by an extended mapping class.

Theorem (Kida-Yamagata. 2011). Let $g > 1$. $B_n(\Sigma_g)$ is cohopfian.

A group presentation for $P_n(\Sigma_g)$.

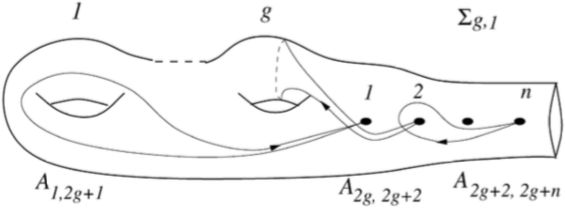
Recall that Fadell-Neuwirth fibration
$p : \mathbb{F}_{n+1}\Sigma \to \mathbb{F}_n\Sigma, \qquad p((x_1, \dots x_n, x_{n+1})) = (x_1, \dots x_n)$ implies that :

$$(PBS) \quad 1 \to \pi_1(\Sigma \setminus \{n \text{ points}\}) \to P_{n+1}(\Sigma) \stackrel{\pi_{n,1}}{\to} P_n(\Sigma) \to 1;$$

Using (PBS) and Lindon-Schupp method we can obtain group presentations for $P_m(\Sigma)$.

The group $P_n(\Sigma_{g,1})$ admits the following presentation :

- **Generators :** $\{A_{i,j} \mid 1 \leq i \leq 2g+n-1, 2g+1 \leq j \leq 2g+n, i < j\}$.



Only non trivial strands of $A_{i,j}$ are presented

The group $P_n(\Sigma_{g,1})$ admits the following presentation :

- **Generators :** $\{A_{i,j} \mid 1 \leq i \leq 2g+n-1, 2g+1 \leq j \leq 2g+n, i<j\}$.
- **Relations :**

(PR1) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s}$ if $(i<j<r<s)$ or $(r+1<i<j<s)$, or $(i=r+1<j<s$ for even $r<2g$ or $r>2g)$;

(PR2) $A_{i,j}^{-1} A_{j,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1}$ if $(i<j<s)$;

(PR3) $A_{i,j}^{-1} A_{i,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$ if $(i<j<s)$;

(PR4) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1} A_{j,s}^{-1} A_{r,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$ if $(i+1<r<j<s)$ or $(i+1=r<j<s$ for odd $r<2g$ or $r>2g)$;

(ER1) $A_{r+1,j}^{-1} A_{r,s} A_{r+1,j} = A_{r,s} A_{r+1,s} A_{j,s}^{-1} A_{r+1,s}^{-1}$ if r odd and $r<2g$;

(ER2) $A_{r-1,j}^{-1} A_{r,s} A_{r-1,j} = A_{r-1,s} A_{j,s} A_{r-1,s}^{-1} A_{r,s} A_{j,s} A_{r-1,s} A_{j,s}^{-1} A_{r-1,s}^{-1}$ if r even and $r<2g$.

Back to Fadell-Neuwirth

- The sequence

$$1 \to P_m(\mathbb{D}^2 \setminus \{n \text{ points}\}) \to P_{n+m} \stackrel{\pi_{n,m}}{\to} P_n \to 1$$

splits (i.e. there exists a section) and $P_{n+m}(\Sigma)$ in an *almost* direct product of $P_m(\mathbb{D}^2 \setminus \{1 \text{ points}\})$ and P_n.

- The sequence

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \stackrel{\pi_{n,m}}{\to} P_n(\Sigma) \to 1$$

splits for $n = 1$ and $P_{1+m}(\Sigma)$ in an *almost* direct product of $P_m(\Sigma \setminus \{1 \text{ points}\})$ and $P_1(\Sigma)$.

Back to Fadell-Neuwirth

- The sequence

$$1 \to P_m(\mathbb{D}^2 \setminus \{n \text{ points}\}) \to P_{n+m} \overset{\pi_{n,m}}{\to} P_n \to 1$$

splits (i.e. there exists a section) and $P_{n+m}(\Sigma)$ in an *almost* direct product of $P_m(\mathbb{D}^2 \setminus \{1 \text{ points}\})$ and P_n.

- The sequence

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \overset{\pi_{n,m}}{\to} P_n(\Sigma) \to 1$$

splits for $n = 1$ and $P_{1+m}(\Sigma)$ in an *almost* direct product of $P_m(\Sigma \setminus \{1 \text{ points}\})$ and $P_1(\Sigma)$.

BUT :

Theorem (Gonçalves-Guaschi 2004). Let Σ_g be a closed surface of genus $g \geq 2$. The sequence (PBS) does not split when $n > 1$.

Back to Fadell-Neuwirth

• The sequence

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \stackrel{\pi_{n,m}}{\to} P_n(\Sigma) \to 1$$

splits for $n = 1$ and $P_{1+m}(\Sigma)$ in an *almost* direct product of $P_m(\Sigma \setminus \{1 \text{ points}\})$ and $P_1(\Sigma)$.

BUT :

Theorem (Gonçalves-Guaschi 2004). Let Σ_g be a closed surface of genus $g \geq 2$. The sequence (PBS) does not split when $n > 1$.

Similarly, given the sequence :

$$(MBS) \quad 1 \to B_m(\Sigma \setminus \{n \text{ points}\}) \to B_{n,m}(\Sigma) \stackrel{\pi_{n,m}}{\to} B_n(\Sigma) \to 1$$

we have that

Proposition (B-Godelle-Guaschi 2017) Let Σ_g of genus $g > 1$; then the sequence (MBS) does not split when $n > 1$.

Let Σ be a surface with boundary.

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \stackrel{\pi_{n,m}}{\to} P_n(\Sigma) \to 1$$

Let Σ be a surface with boundary.

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \overset{\pi_{n,m}}{\to} P_n(\Sigma) \to 1$$

Good point : If Σ has boundary components the "natural" section is well defined. (we can see elements of $P_n(\Sigma)$ as elements of $P_{n+m}(\Sigma)$ adding m vertical strands "at the infinity").

Let Σ be a surface with boundary.

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \stackrel{\pi_{n,m}}{\to} P_n(\Sigma) \to 1$$

Good point : If Σ has boundary components the "natural" section is well defined. (we can see elements of $P_n(\Sigma)$ as elements of $P_{n+m}(\Sigma)$ adding m vertical strands "at the infinity").

Bad point : If Σ has boundary components (and genus $g > 0$) there is no section defining an action of $P_n(\Sigma)$ on $P_m(\Sigma \setminus \{n \text{ points}\})$ which is trivial on the abelianization.

Let Σ be a surface with boundary.

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \stackrel{\pi_{n,m}}{\to} P_n(\Sigma) \to 1$$

Good point : If Σ has boundary components the "natural" section is well defined. (we can see elements of $P_n(\Sigma)$ as elements of $P_{n+m}(\Sigma)$ adding m vertical strands "at the infinity").

Bad point : If Σ has boundary components (and genus $g > 0$) there is no section defining an action of $P_n(\Sigma)$ on $P_m(\Sigma \setminus \{n \text{ points}\})$ which is trivial on the abelianization.

Theorem (B.-Gervais-Guaschi 2008 ; B.-Bardakov 2009)

Let $\Sigma \neq \mathbb{S}^2$ be an oriented surface (possibly with boundary). $P_n(\Sigma)$ is residually torsion-free nilpotent.

$P_n(\Sigma)$ is residually torsion-free nilpotent

"Proof" : case with one boundary component :

$$1 \to \mathbb{Z}^n \to \mathcal{M}(\Sigma_{g,n+p}) \to \mathcal{PM}_n(\Sigma_{g,p}) \to 1$$

There is no section (homology !) but the restriction to

$$P_n(\Sigma_{g,1}) \subset \mathcal{PM}_n(\Sigma_{g,1})$$

has a section. Finally ; gluing holed torus on boundary disks we realize $P_n(\Sigma_{g,1})$ as a subgroup of $\mathcal{M}(\Sigma_{g+n,1})$ and moreover $P_n(\Sigma_{g,1}) \subset \mathcal{T}(\Sigma_{g+n,1})$, the **Torelli group of $\Sigma_{g+n,1}$**.

$P_n(\Sigma)$ is residually torsion-free nilpotent

"Proof" : closed case :

Recall that

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{1 \text{ points}\}) \to P_{m+1}(\Sigma) \stackrel{\pi_{m,1}}{\to} P_1(\Sigma) \to 1$$

The sequence splits and defining a possible section (Guaschi-Gonçalves 2004, B-Bardakov 2009, Gonzáles Meneses-Silvero 2018...) one can verify that $P_{m+1}(\Sigma)$ is an almost direct product of RTFN groups ; then it is also RTFN.

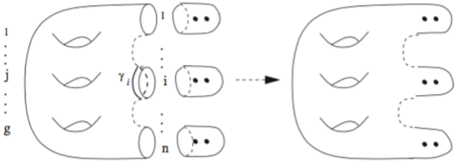
An extension for closed surfaces

nth framed pure braid group of Σ_g (B.-Gervais)
$FP_n(\Sigma_g) = ker\Phi_n,\ \Phi_n : \mathcal{M}(\Sigma_{g,n}) \to \mathcal{M}(\Sigma_g)$

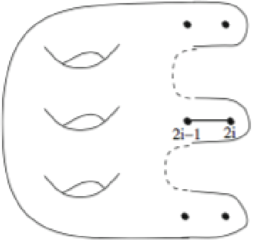
- Let $U\Sigma_g$ the unitary tangent space of Σ_g and $\pi : U\Sigma_g \to \Sigma_g$ the natural projection. Let $\mathrm{T}_n\Sigma_g = (\pi^n)^{-1}(\mathbb{F}_n\Sigma_g)$. $\mathrm{T}_n\Sigma_g$ is a classifying space and $\pi_1(\mathrm{T}_n\Sigma_g) \simeq FP_n(\Sigma_g)$. Therefore we can define :
- **nth framed braid group of Σ_g** : $FB_n(\Sigma_g) = \pi_1(\mathrm{T}_n\Sigma_g/S_n)$

- $FP_1(\Sigma_g) \simeq \pi_1(U\Sigma_g)$;
- $FB_n(\mathbb{D}^2)$ is isomorphic to the framed braid group $\mathcal{F}_n := \mathbb{Z}^n \rtimes B_n$ (Ko-Smolinski 1992) ;
- $FB_n(\Sigma_g)$ can be characterised in terms of *framed* mapping class groups.

- $FP_n(\Sigma_g)$ can be seen as a subgroup of $P_{2n}(\Sigma_g)$



- $FB_n(\Sigma_g)$ can be seen as a particular subgroup of $B_{2n}(\Sigma_g)$.



Theorem (B.-Gervais 2012).
1) $FP_n(\Sigma)$ is a central extension of $P_n(\Sigma)$ by $\mathbb{Z}^n$:

$$(*) \quad 1 \to \mathbb{Z}^n \to FP_n(\Sigma) \to P_n(\Sigma) \to 1$$

($\longrightarrow$ group presentation for $FP_n(\Sigma)$)
2) $(*)$ splits if and only if Σ has boundary or $\Sigma = \mathbb{T}^2$: in particular, if Σ has boundary, $FP_n(\Sigma) \simeq \mathbb{Z}^n \times P_n(\Sigma)$.
3) The inclusion of $\Sigma_{g,n}$ into $\Sigma_{g,n+m}$ induces a section for the exact sequence induced by the "framed" Fadell-Neuwirth fibration $p : \mathbb{T}_{n+m}\Sigma_g \to \mathbb{T}_n\Sigma_g$:

$$1 \longrightarrow FP_m(\Sigma_{g,n}) \longrightarrow FP_{n+m}(\Sigma_g) \longrightarrow FP_n(\Sigma_g) \longrightarrow 1$$

The sequence

$$1 \longrightarrow FP_m(\Sigma_{g,n}) \longrightarrow FP_{n+m}(\Sigma_g) \longrightarrow FP_n(\Sigma_g) \longrightarrow 1$$

has the same behavior than the (PBS) sequence in the case of surfaces with boundary.
Nevertheless :

Proposition (B.-Gervais)

Let $\Sigma \neq \mathbb{S}^2$ be an oriented surface (possibly with boundary). $FP_n(\Sigma)$ is residually torsion-free nilpotent.