Surface braids and mapping class group III
Braid groups on surfaces of positive genus: first definitions, group presentations and combinatorial tools

Paolo Bellingeri

Laboratoire de Mathématiques Nicolas Oresme, Université de Caen

Atlantic Algebra Centre

Incomplete-web-scholar bibliography

About surface braid groups

- Gwenael Massuyeau, *Lectures on mapping class groups, braid groups and formality*, personal webpage.
- John Guaschi, Daniel Juan-Pineda *A survey of surface braid groups and the lower algebraic K-theory of their group rings* arXiv :1302.6536v1
- Luis Paris, Dale Rolfsen *Geometric subgroups of surface braid groups* https ://arxiv.org/abs/math/9906122

Group presentations for braid groups on surfaces of positive genus

First appearances : Zariski (1936) and... Fadell-Van Buskirk (1961)

Group presentations :

Torus : Birman (1968)

Closed surfaces : Scott (1970)... corrected by Kulikov (1996) ; González Meneses (2001) ; B. (2004)

Surfaces with boundary : B. (2004)

Positive Presentations : B. - Godelle (2007)

New presentations (Torus vs Klein Bottle) : Guaschi-Pereiro (2018)

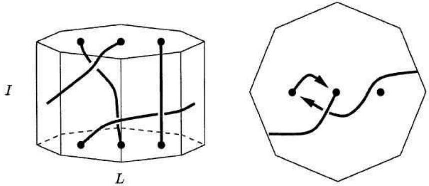
Surface braids as collections of paths

Σ oriented connected surface. $\mathcal{P} = \{x_1, \ldots, x_n\} \subset \Sigma$

Geometric braid (on n strands on Σ) :

$\beta = (\psi_1, \ldots, \psi_n)$,
$\psi_i : [0,1] \to \Sigma \times [0,1]$

- $\psi_i(0) = (x_i, 0)$ and $\psi_i(1) \in \mathcal{P} \times \{1\}$ $\forall i = 1 \ldots, n$;
- $\psi_i(t) \neq \psi_j(t)$ for $i \neq j$ and $\psi_i(t) \in \Sigma \times \{t\}$.

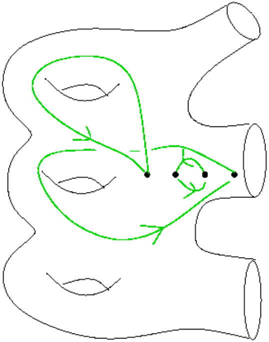


on the left on the fundamental domain times the interval, on the right the projection on the fundamental domain

Surface braids as collections of paths

Σ oriented connected surface. $\mathcal{P} = \{x_1, \ldots, x_n\} \subset \Sigma$

Geometric braid (on n strands on Σ) :



$\beta = (\psi_1, \ldots, \psi_n)$,
$\psi_i : [0,1] \to \Sigma$

- $\psi_i(0) = (x_i)$ and $\psi_i(1) \in \mathcal{P}\ \forall i = 1 \ldots, n$;
- $\psi_i(t) \neq \psi_j(t)$ for $i \neq j$.

Braid group of the surface

Surface braids are considered up to isotopy :

Isotopy : $\beta_0 \sim \beta_1$ if it exists a **continuous** family of **geometric braids** $\beta_t, \quad t \in [0, 1]$.

The usual composition of paths induces a structure of group on equivalence classes of braids on n strands :

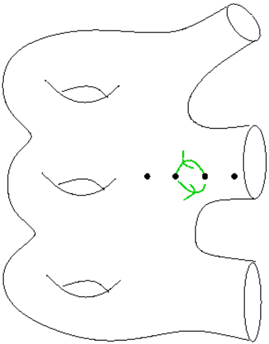
$$\{ \text{ surface geometric braids (on } n \text{ strands) } \}_{/\sim} \simeq B_n(\Sigma),$$

- The subgroup of braids inducing trivial permutation is called pure braid group on Σ, $P_n(\Sigma)$.

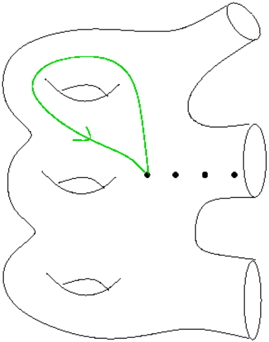
Remark. The embedding of the disk $\mathbb{D}^2$ (with n marked points) in Σ (with n marked points) induces an embedding $B_n \to B_n(\Sigma)$ (and $P_n \to P_n(\Sigma)$) ; more generally Paris and Rolfsen (1999) classified embedding of punctured surfaces inducing embeddings on corresponding braid groups.

Surface braids : generators

Generators "from" B_n :

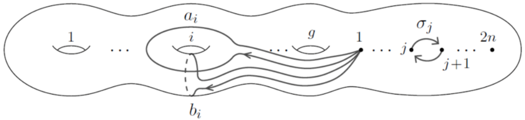


Generators "from" $\pi_1(\Sigma)$:



Closed oriented surface of genus g

Generators for $B_n(\Sigma_g)$: $\sigma_1, \ldots, \sigma_{n-1}, a_1, b_1 \ldots, a_g, b_g$



Closed oriented surface of genus g

Generators for $B_n(\Sigma_g)$: $\sigma_1, \ldots, \sigma_{n-1}, a_1, b_1 \ldots, a_g, b_g$

– Braid relations :

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \;\; 1 \leq i \leq n-2,$
$\sigma_i \sigma_j = \sigma_j \sigma_i, \;\; |i-j| > 1, \;\; 1 \leq i, j \leq n-1,$

– Mixed relations (notation : $c_k \in \{a_k, b_k\}$) :

(R1) $c_r \sigma_i = \sigma_i c_r, \;\; i \neq 1, \;\; 1 \leq r \leq g,$

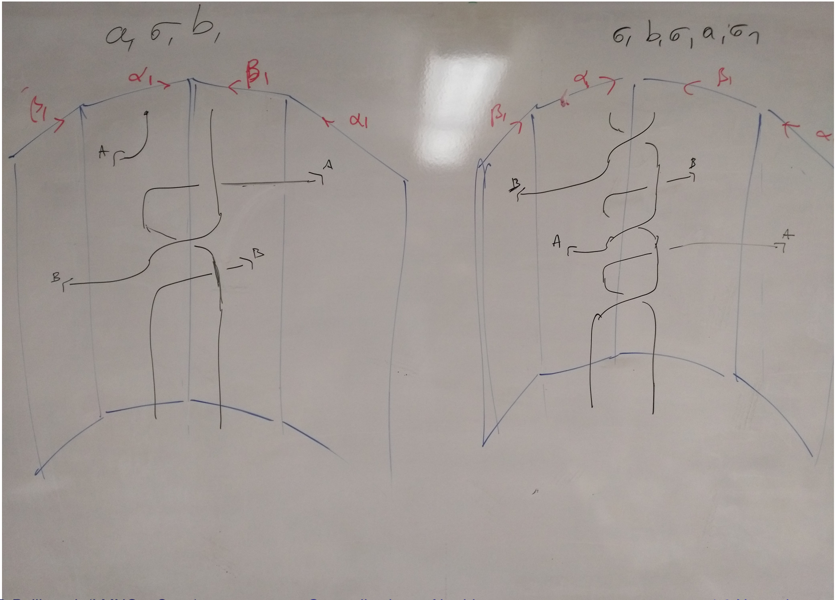
(R2) $\sigma_1 c_r \sigma_1 c_r = c_r \sigma_1 c_r \sigma_1, \;\; 1 \leq r \leq g,$

(R3) $\sigma_1 c_s \sigma_1^{-1} c_r = c_r \sigma_1 c_s \sigma_1^{-1}, 1 \leq s < r \leq g,$

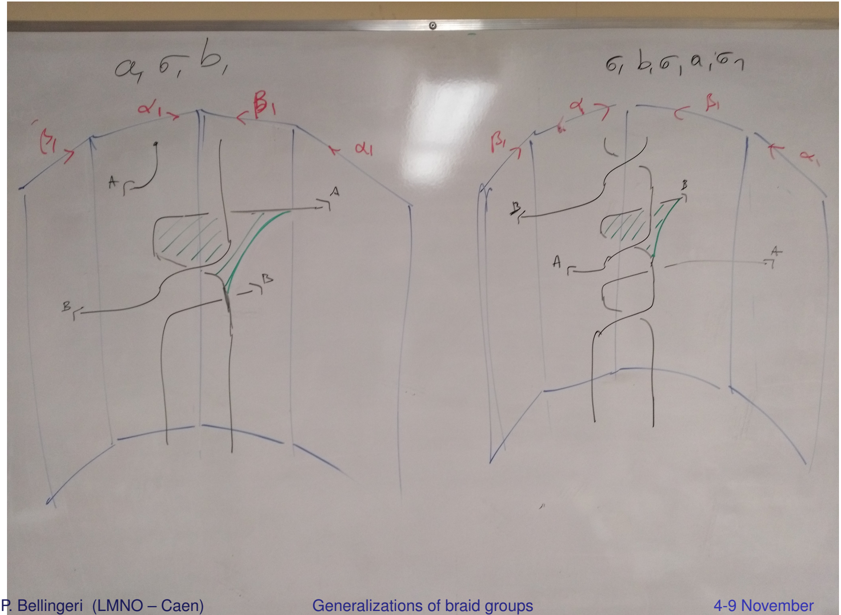
(R4) $\sigma_1 b_r \sigma_1 a_r \sigma_1 = a_r \sigma_1 b_r, \;\; 1 \leq r \leq g,$

(Rπ) $\prod_{i=1}^{g} [a_i^{-1}, b_i] = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1$

A relation

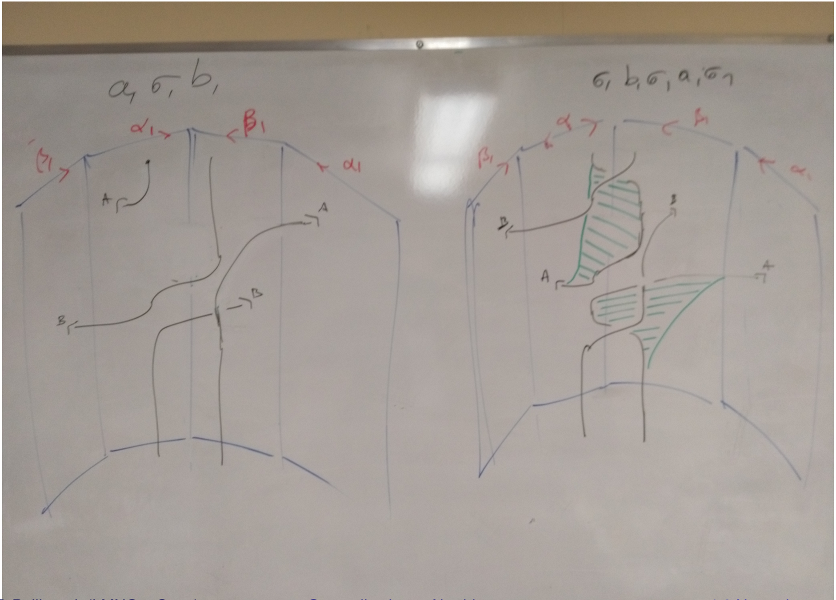


A relation

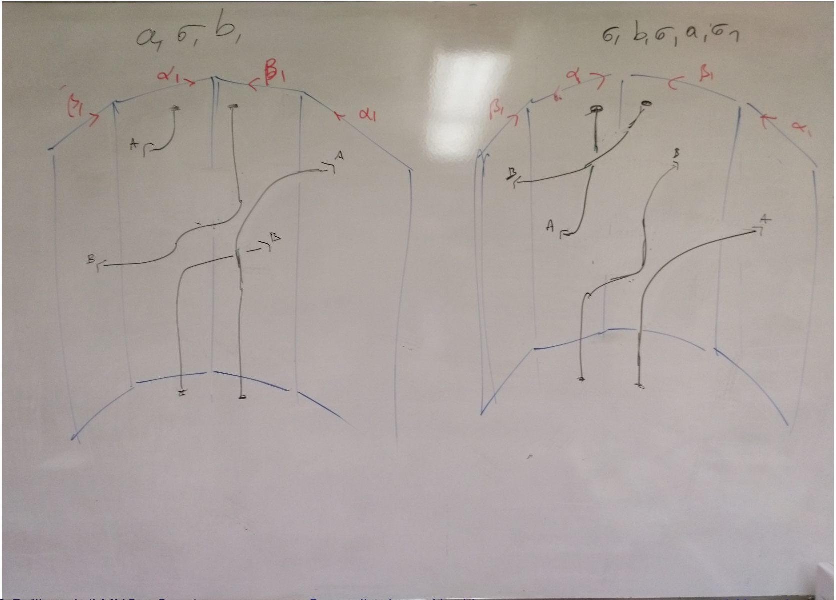


$a_1 \sigma_1 b_1$
α_1
β_1
β_1
α_1
A
A
B
B
$\sigma_1 b_1 \sigma_1 a_1 \sigma_1$
α
β_1
β_1
α_1
B
B
A
A

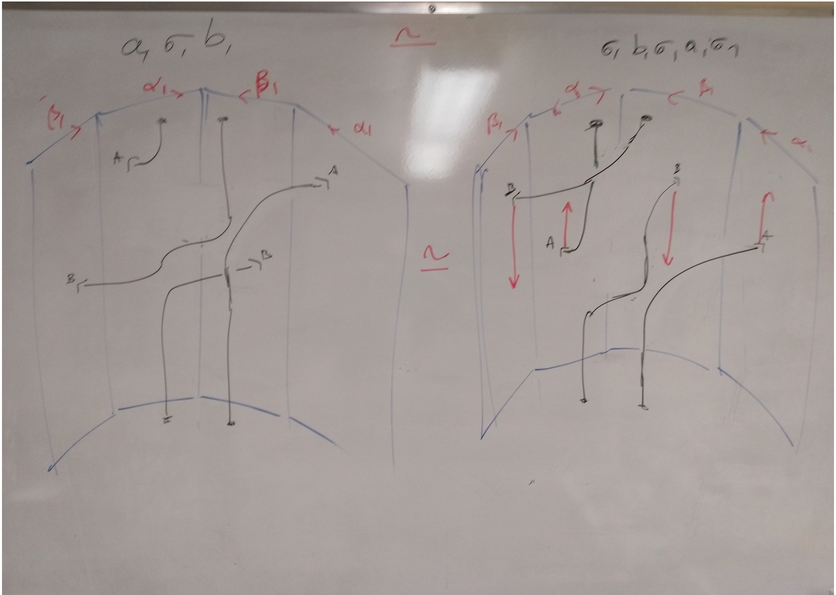
A relation



A relation



A relation



$a_1 \sigma_1 b_1$
$\simeq$
$\sigma_1 b_1 \sigma_1 a_1 \sigma_1$
α_1
β_1
β_1
α_1
A
B
A
B
$\simeq$
β_1
α_1
β_1
α_1
B
A
B
A

Oriented surface of genus g and 1 boundary component, $\Sigma_{g,1}$

Generators for $B_n(\Sigma_{g,1})$: $\sigma_1, \dots, \sigma_{n-1}, a_1, b_1 \dots, a_g, b_g$
– Braid relations :

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \;\; 1 \leq i \leq n-2,$
$\sigma_i \sigma_j = \sigma_j \sigma_i, \;\; |i-j| > 1, \;\; 1 \leq i, j \leq n-1,$

– Mixed relations (notation : $c_k \in \{a_k, b_k\}$) :

(R1) $c_r \sigma_i = \sigma_i c_r, \;\; i \neq 1, \;\; 1 \leq r \leq g,$

(R2) $\sigma_1 c_r \sigma_1 c_r = c_r \sigma_1 c_r \sigma_1, \;\; 1 \leq r \leq g,$

(R3) $\sigma_1 c_s \sigma_1^{-1} c_r = c_r \sigma_1 c_s \sigma_1^{-1}, 1 \leq s < r \leq g,$

(R4) $\sigma_1 b_r \sigma_1 a_r \sigma_1 = a_r \sigma_1 b_r, \;\; 1 \leq r \leq g,$

Oriented surface of with $b > 1$ boundary components

Generators for $B_n(\Sigma_{g,b})$: $\sigma_1, \ldots, \sigma_{n-1}, a_1, b_1 \ldots, a_g, b_g, z_1, \ldots z_{b-1}$

– Braid relations :

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \ 1 \leq i \leq n-2,$
$\sigma_i \sigma_j = \sigma_j \sigma_i, \ \ |i-j| > 1, \ \ 1 \leq i,j \leq n-1,$

– Mixed relations (notation : $c_k \in \{a_k, b_k\}$) :

(R1) $c_r \sigma_i = \sigma_i c_r, \ \ i \neq 1, \ \ 1 \leq r \leq g,$
(R1') $z_r \sigma_i = \sigma_i z_r, \ \ i \neq 1, \ \ 1 \leq r \leq b-1,$

(R2) $\sigma_1 c_r \sigma_1 c_r = c_r \sigma_1 c_r \sigma_1, \ \ 1 \leq r \leq g,$
(R2') $\sigma_1 z_r \sigma_1 z_r = z_r \sigma_1 z_r \sigma_1, \ \ 1 \leq r \leq b-1,$

(R3) $\sigma_1 c_s \sigma_1^{-1} c_r = c_r \sigma_1 c_s \sigma_1^{-1}, 1 \leq s < r \leq g,$
(R3') $\sigma_1 z_s \sigma_1^{-1} z_r = z_r \sigma_1 z_s \sigma_1^{-1}, 1 \leq s < r \leq b-1,$

(R4) $\sigma_1 b_r \sigma_1 a_r \sigma_1 = a_r \sigma_1 b_r, \ \ 1 \leq r \leq g,$

Surface braids in terms of configuration spaces

Σ oriented surface ; $\mathbb{F}_n\Sigma = \{(x_1, \ldots x_n) \in \Sigma^n \,|x_i \neq x_j$ for $i \neq j\}$.

$P_n(\Sigma) \simeq \pi_1(\mathbb{F}_n\Sigma)$;
$B_n(\Sigma) \simeq \pi_1(\mathbb{F}_n\Sigma/S_n)$;

Mixed braid group on n strands of Σ :
$B_{m,n}(\Sigma) \simeq \pi_1(\mathbb{F}_{m+n}\Sigma/(S_m \times S_n))$

Main tools on surface braids : exact sequences

Generalised Fadell-Neuwirth fibrations :

$p : \mathbb{F}_{n+m}\Sigma \to \mathbb{F}_n\Sigma, \qquad p((x_1, \dots x_n, x_{n+1}, \dots, x_{n+m})) = (x_1, \dots x_n)$

$$\xrightarrow{\text{Homotopy exact sequence}}$$

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \stackrel{\pi_{n,m}}{\to} P_n(\Sigma) \to 1$$

and

$$(MBS) \quad 1 \to B_m(\Sigma \setminus \{n \text{ points}\}) \to B_{n,m}(\Sigma) \stackrel{\pi_{n,m}}{\to} B_n(\Sigma) \to 1$$

For $m = 1$ we get :

$$(PBS) \quad 1 \to \pi_1(\Sigma \setminus \{n \text{ points}\}) \to P_{n+1}(\Sigma) \stackrel{\pi_{n,1}}{\to} P_n(\Sigma) \to 1;$$

$$(MBS) \quad 1 \to \pi_1(\Sigma \setminus \{n \text{ points}\}) \to B_{n,1}(\Sigma) \stackrel{\pi_{n,1}}{\to} B_n(\Sigma) \to 1$$

Consequences of Fadell-Neuwirth exact sequences : some exemples

• we can obtain group presentations for $P_n(\Sigma)$ and $B_n(\Sigma)$.

• **Proposition (Paris-Rolfsen 1999).** Let Σ_g be a closed surface of genus $g \geq 1$. $P_n(\Sigma_g)$ and $B_n(\Sigma_g)$ have no torsion (for $B_n(\Sigma_g)$ we need further informations, in particular that $\mathbb{F}_n\Sigma_g$ is a Eilenberg-MacLane space).

• **Theorem (Paris-Rolfsen 1999).** Let Σ_g be a closed surface of genus $g \geq 2$. $P_n(\Sigma_g)$ and $B_n(\Sigma_g)$ have trivial center.

Group presentations for $B_n(\Sigma)$: outlines of proofs

The proof of the presentation for $B_n(\Sigma_g)$ in (González-Meneses 2001) consists to apply iteratively Lindon-Schupp's method to exact sequences of pure braid groups on surfaces and therefore to

$$1 \to P_n(\Sigma_g) \to B_n(\Sigma_g) \to S_n \to 1$$

The proof of the presentation for $B_n(\Sigma_{g,b})$ in (B. 2004) is inspired by Morita's proof of Artin presentation of B_n (Morita, 1992) and is based on the exact sequence :

$$(MBS) \quad 1 \to F_{n+2g+b-1} \to B_{n,1}(\Sigma_{g,b}) \xrightarrow{\pi_{n,1}} B_n(\Sigma_{g,b}) \to 1$$

A consequence of group presentations : the determination of lower central series

Lower central series (LCS) of G :
$\Gamma_1(G) = G$, $\Gamma_i(G) = [G, \Gamma_{i-1}(G)]$ for $i > 1$.

G is **residually nilpotent** if $\bigcap_{i \geq 1} \Gamma_i(G) = \{1\}$;

G is **perfect** if $\Gamma_1(G) = \Gamma_2(G)$.

Reminders. $B_2 = \mathbb{Z}$; if $n \geq 3$, $\Gamma_1(B_n)/\Gamma_2(B_n) = \mathbb{Z}$ and $\Gamma_2(B_n) = \Gamma_3(B_n)$.
$\Gamma_2(B_n)$ is perfect if and only if $n \geq 5$.

LCS for surface braid groups

LCS Theorem (B-Gervais-Guaschi 2008, B-Bardakov 2009, B-Gonçalves-Guaschi 2018). Let $g \geq 1$, and let $n \geq 2$. Then :

1. $B_n(\Sigma_g)/\Gamma_2(B_n(\Sigma_g)) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_2$.
2. $\Gamma_2(B_n(\Sigma_g))/\Gamma_3(B_n(\Sigma_g)) \cong \mathbb{Z}_{n-1+g}$ if $n \geq 3$.
3. Group presentation for $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ if $n \geq 3$.
4. $\Gamma_3(B_n(\Sigma_g)) = \Gamma_4(B_n(\Sigma_g))$ if and only if $n \geq 3$. Moreover $\Gamma_3(B_n(\Sigma_g))$ is perfect if and only if $n \geq 5$.
5. The group $B_2(\Sigma_g)$ is residually nilpotent (but not nilpotent) ; in particular $\Gamma_2(B_2(\mathbb{T}))/\Gamma_3(B_2(\mathbb{T})) \cong \mathbb{Z}_2^3$, and if $g > 1$, $\Gamma_2(B_2(\Sigma_g))/\Gamma_3(B_2(\Sigma_g))$ is a non-trivial quotient of $\mathbb{Z}_2^{2g} \oplus \mathbb{Z}_{g+1}$.

Remarks. The case $n = 1$ is known since long time ($B_1(\Sigma) = \pi_1(\Sigma)$).

LCS, the orientable case with boundary

Let $\Sigma_{g,b}$ be a compact, connected orientable surface of genus $g \geq 1$ and with $b \geq 1$ boundary components.

LCS Theorem (B.-Gervais-Guaschi 2008, B-Gonçalves-Guaschi 2018). Let $n \geq 2$. Then :

1. $B_n(\Sigma_{g,b})/\Gamma_2(B_n(\Sigma_{g,b})) \cong \mathbb{Z}^{2g+b-1} \oplus \mathbb{Z}_2$.
2. $\Gamma_2(B_n(\Sigma_g))/\Gamma_3(B_n(\Sigma_g)) \cong \mathbb{Z}$ if $n \geq 3$.
3. Group presentation for $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ if $n \geq 3$.
4. $\Gamma_3(B_n(\Sigma_{g,b})) = \Gamma_4(B_n(\Sigma_{g,b}))$ if and only if $n \geq 3$. Moreover $\Gamma_3(B_n(\Sigma_{g,b}))$ is perfect if and only if $n \geq 5$.
5. The group $B_2(\Sigma_{g,b})$ is residually nilpotent (but not nilpotent) ; in particular $\Gamma_2(B_2(\Sigma_{g,b}))/\Gamma_3(B_2(\Sigma_{g,b}))$ is a non-trivial quotient of $\mathbb{Z}_2^{2g+b-1} \oplus \mathbb{Z}$.

Remarks and applications

Similar results have been obtained for mixed braid groups $B_{m,n}(\Sigma)$ for m or n greater or equal than 3 (B.-Godelle-Guaschi 2017)

Applications :

- Representations of Torelli groups (Blanchet, in progress)
- Linear representations for surface braid groups (B.-Godelle-Guaschi 2017)
- Surjective morphisms between surface braid groups (B.-Gonçalves-Guaschi 2017).
- Fibrations on curves of genus g (Causin-Polizzi 2019)

Surjective morphisms between surface (pure) braid groups.

Let Σ_g be a compact, connected orientable surface without boundary, of genus $g \geq 0$.

Theorem (Chen, 2017). Let $g > 1$. Any surjection between pure braid groups on Σ_g factors through some forgetful homomorphism.

Conjecture (Chen, 2017). Let $g > 1$ and $n \neq m$. There is not a surjective homomorphism $\phi_{n,m} : B_n(\Sigma_g) \to B_m(\Sigma_g)$.

Surjective morphisms between surface (pure) braid groups.

Theorem (B-Gonçalves-Guaschi, 2018). Let $m, n \in \mathbb{N}$ be such that $m \neq n$.

1. Let $g = 1$. There is a surjective homom. $\rho\colon B_n(\mathbb{T}) \to B_m(\mathbb{T})$ iff $m = 1$.
2. Let $g > 1$. There is not a surjective homom. $\rho\colon B_n(\Sigma_g) \to B_m(\Sigma_g)$.

Corollary. Let $g \geq 1$, and let $n, m \in \mathbb{N}$. There is a surjective homomorphism of $B_n(\Sigma_g)$ onto $P_m(\Sigma_g)$ iff $n = m = 1$ for $g > 1$ and $m = 1$ for $g = 1$.

Remark. Similar results hold for braid groups on surfaces with boundary or non orientable.

Outline of the proof

The proof of our main Theorem is strictly related to the knowledge of the LCS of surface braid groups. For instance :

- Let $n < m$ and $g \geq 1$.

  If $n = 1$, $(B_1(\Sigma_g))_{\mathrm{Ab}} \cong \mathbb{Z}^{2g}$, and $(B_m(\Sigma_g))_{\mathrm{Ab}} \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_2$ by LCS Theorem, which implies that there is no surjective homomorphism from $(B_1(\Sigma_g))_{\mathrm{Ab}}$ onto $(B_m(\Sigma_g))_{\mathrm{Ab}}$, from which it follows there is no surjective homomorphism from $B_1(\Sigma_g)$ to $B_m(\Sigma_g)$.

  If $n = 2$, the result follows from an argument on minimal number of generators $G(B_m(\Sigma_g))$, which is $2g + 1$ for $m = 2$ and $2g + 2$ for $m > 2$.

  If $n \geq 3$, LCS Theorem implies that there is no surjective homomorphism from $\Gamma_2(B_n(\Sigma_g))/\Gamma_3(B_n(\Sigma_g))$ onto $\Gamma_2(B_m(\Sigma_g))/\Gamma_3(B_m(\Sigma_g))$, and hence there is no surjective homomorphism from $B_n(\Sigma_g)$ to $B_m(\Sigma_g)$.

Representations in symmetric groups

Definition. A homomorphism $\rho\colon B_n(\Sigma_g) \to S_n$ is said to be *transitive* if the action of $\mathrm{Im}\,(\rho)$ on the set $\{1, \ldots, n\}$ is transitive.

Definition. A homomorphism $\rho\colon B_n(\Sigma_g) \to S_n$ is said to be *primitive* if the only partitions of the set $\{1, \ldots, n\}$ that are left invariant by the action of $\mathrm{Im}\,(\rho)$ are the set itself, or the partition consisting of singletons.

Ivanov found a family of transitive but imprimitive representations ; all provided examples were *abelian* representations and the last sentence in Ivanov's paper was :

I do not know to what extent these examples exhaust the imprimitive representations.

It turns out that representations of surface braid groups into symmetric groups is much richer than expected...

Representations in symmetric groups : first results

Lemma (B.-Gonçalves-Guaschi 2018). Let $n \geq m \geq 2$, and let $\rho\colon\ B_n(\Sigma_g) \to S_m$ be a homomorphism.

1. If $(n, m) \neq (4, 3)$, $\rho(B_n)$ is a cyclic group.
2. The subgroup $\rho(B_n(\Sigma_g))$ is contained in the centraliser of $\rho(B_n)$.
3. The homomorphism ρ sends $\Gamma_3(B_n(\Sigma_g))$ to the trivial element, so it factors through the quotient $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$.
4. The subgroup $\rho(B_n(\Sigma_g))$ is nilpotent of nilpotency degree at most 2.

Starting from f.p. presentations for $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ (B.-Gervais-Guaschi 2008), it is possible to construct several families of imprimitive non abelian representations $\rho\colon\ B_n(\Sigma_g) \to S_m$.

Non abelian representations : the simplest example

Following (B.-Gervais-Guaschi, 2008) we have a presentation of $B_n(\mathbb{T})/\Gamma_3(B_n(\mathbb{T}))$ given by :

$$\langle a_1, b_1, \sigma | [a_1, \sigma] = [b_1, \sigma] = 1 \,, [a_1, b_1] = \sigma^2 \,, \sigma^{2n} = 1 \rangle.$$

Now, for n even, define $\theta : B_n(\mathbb{T})/\Gamma_3(B_n(\mathbb{T})) \to S_8$ as follows :

$$\theta(a_1) = (1\ 3)(2\ 4) \,, \theta(b_1) = (1\ 5)(2\ 6)(3\ 7)(4\ 8) \,, \theta(\sigma) = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8).$$

By construction θ is transitive, imprimitive and with non-Abelian image.