Surface braids and mapping class group II
Braid groups on surfaces of genus 0: group presentations, definitions and torsion elements

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Incomplete-web-scholar bibliography

About braids :

- Ester Dalvit, *Braids, a movie*
- Joan Birman and Tara Brendle, *Braids : A Survey*, math.GT/0409205
- Juan Gonzalez-Meneses, *Basic results on braid groups*, arXiv :1010.0321
- Luis Paris, *Braid groups and Artin Tits Groups*, arXiv :0711.2372
- Luis Paris, *From braid groups to mapping class groups*, math.GR/0412024
- Dale Rolfsen, *Tutorial on the braid groups*, arXiv :1010.4051

About braids on the annulus and the sphere :

- Sofia Lambropoulou, *Diagrammatic representations of knots and links as closed braids*, arXiv :1811.11701
- Daciberg Lima Gonçalves, John Guaschi, *The classification of the virtually cyclic subgroups of the sphere braid groups*, arXiv :1110.6628

Braids on surfaces of genus 0

- B_n : done !
- Braid groups of a surface of genus 0 and $p > 2$ boundary components : group presentation(s).
- Braid groups of the sphere : torsions elements and mapping classes.

Annular Braids

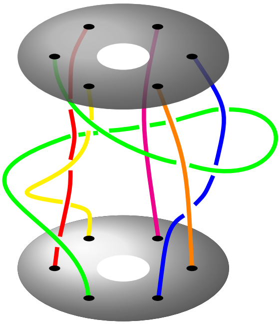
We will restrict our attention on the braid group on n strands of a surface of genus 0 and $p = 2$ boundary components, $\Sigma_{0,2}$. This braid group has several names : braid group of the annulus, circular braid group, Artin group of type $\mathcal{B}$... and it can be defined in terms of collections of paths, configuration spaces, automorphisms of free groups...

Annular Braids

$\mathcal{P} = \{x_1, \ldots, x_n\} \subset \Sigma_{0,2}$

Geometric annular braid : Let $\beta = (\psi_1, \ldots, \psi_n)$,
$\psi_i : [0,1] \to \Sigma_{0,2} \times [0,1]$

- $\psi_i(0) = (x_i, 0)$ and $\psi_i(1) \in \mathcal{P} \times \{1\}$ $\forall i = 1 \ldots, n$;
- $\psi_i(t) \neq \psi_j(t)$ for $i \neq j$ and $\psi_i(t) \in \Sigma_{0,2} \times \{t\}$.



Braid group of the annulus on n strands

Annular braids are considered up to isotopy :

Isotopy : $\beta_0 \sim \beta_1$ if it exists a **continuous** family of **geometric braids** $\beta_t, \quad t \in [0,1]$.

The usual composition of paths induces a structure of group on equivalence classes of braids on n strands :

$$\{ \text{ Annular geometric braids (on } n \text{ strands) } \}_{/\sim} \simeq B_n(\Sigma_{0,2}),$$

Remark. $B_n(\Sigma_{0,2})$ is isomorphic to the subgroup $B_{n,1}$ of B_{n+1} previously defined. More precisely $B_n(\Sigma_{0,2})$ is identified with the subgroup $B_{1,n}$ of B_{n+1} of braid fixing first strand and, evidently, $B_{n,1} = B_{1,n}$ are isomorphic. More generally, for $p \geq 2$, $B_n(\Sigma_{0,p})$ is isomorphic to the subgroup of braids in B_{n+p-1} fixing last (or first...) $p-1$ strands.

Braid group of the annulus : group presentations

Theorem (Chow, 1948)

$B_n(\Sigma_{0,2})$ admits a presentation with generators :

$$\tau, \sigma_1, \ldots, \sigma_{n-1}$$

$$\begin{gathered}
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i \leq n-2, \\
\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| > 1, \quad 1 \leq i,j \leq n-1, \\
\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau, \\
\tau\sigma_j = \sigma_j\tau, \quad 2 \leq j \leq n-1,
\end{gathered}$$

With this presentation the isomorphism with $B_{1,n}$ is realized by $\iota : B_n(\Sigma_{0,2}) \to B_{n+1}$ where $\iota(\tau) = \sigma_1^2$ and $\iota(\sigma_i) = \sigma_{i+1}$ for $1 \leq i \leq n-2$.

Braid group of the annulus : group presentations

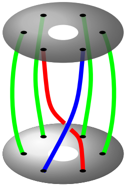
Theorem (tom Dieck 1998 ; Kent 2002)

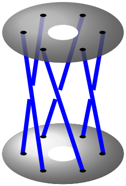
Let

$$CB_n = \left\langle \sigma_1, \ldots, \sigma_n, \zeta \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, 2 \ldots, n, \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \neq 1, \\ \zeta^{-1}\sigma_i\zeta = \sigma_{i+1} & \text{for } i = 1, 2 \ldots, n \end{array} \right\rangle,$$

where indices are defined mod n. The group CB_n (circular braid group) is isomorphic to $B_n(\Sigma_{0,2})$.

Geometric interpretation σ_i (left) and ζ (right) :



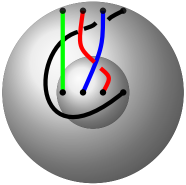


Braid group of the annulus : odds and ends

1. B. and Vershinin (2005) : (bi-colored) graph presentations for $B_n(\Sigma_{0,2})$.
2. REMARK : the center of $B_n(\Sigma_{0,2})$, $ZB_n(\Sigma_{0,2})$, is isomorphic to $\mathbb{Z}$ and generated by ζ^n.
3. Charney and Crisp (2004) : $B_n(\Sigma_{0,2})/ZB_n(\Sigma_{0,2})$ can be characterized as a subgroup of $\mathcal{M}_n(\mathbb{S}^2)$ (see next lecture).
4. EXERCISE : $B_n(\Sigma_{0,2})^{Ab} = \mathbb{Z}^2$ and $\Gamma_2(B_n(\Sigma_{0,2})) = \Gamma_3(B_n(\Sigma_{0,2}))$
5. "Consequence" of 2) –4) : the outer automorphism group of $B_n(\Sigma_{0,2})$, $Out(B_n(\Sigma_{0,2}))$, is isomorphic to $(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$ (Charney and Crisp, 2004).
6. Braids on the annulus are related to links in the solid torus and HOMFLYPT polynomial construction can be generalized via Ariki-Koike Algebras (Lambropoulou (1995–)).

Braids on the sphere

The braid group on n strands of the sphere, $B_n(\mathbb{S}^2)$ is the group of collections of n paths in $\mathbb{S}^2 \times [0, 1]$, monotone, disjoint and with endpoints in $\mathcal{P} \times 0$ and $\mathcal{P} \times 1$



Braids on the sphere

Theorem (Zariski, 1936 ; Fadell-Van Buskirk, 1961)

$B_n(\mathbb{S}^2)$ admits a presentation with generators :

$$\sigma_1, \ldots, \sigma_{n-1}$$

and relations :

$$\begin{gathered}\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \;\; 1 \leq i \leq n-2; \\ \sigma_i\sigma_j = \sigma_j\sigma_i, \;\; |i-j| > 1, \;\; 1 \leq i,j \leq n-1; \\ \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1 = 1.\end{gathered}$$

Braids on the sphere

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- The subgroup of braids inducing trivial permutation is called pure braid group on $\mathbb{S}^2$, $P_n(\mathbb{S}^2)$.
- Set $\theta = \sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_2\sigma_1$; then $B_n/\langle\theta\rangle = B_n(S^2)$.

Braids on the sphere : configuration spaces

$\mathbb{F}_n\mathbb{S}^2 = \{(x_1, \dots x_n) \in \mathbb{S}^2 \,|x_i = x_j \iff i = j\}$.

Definition : $\mathbf{P}_n(\mathbb{S}^2) := \pi_1(\mathbb{F}_n\mathbb{S}^2)$

Fact : $\mathbf{P}_n(\mathbb{S}^2)$ is isomorphic to $P_n(\mathbb{S}^2)$.

Definition : $\mathbf{B}_n(\mathbb{S}^2) := \pi_1(\mathbb{F}_n\mathbb{S}^2/S_n)$.

Proposition : $\mathbf{B}_n(\mathbb{S}^2)$ is isomorphic to $B_n(\mathbb{S}^2)$.

Generalised Fadell-Neuwirth fibrations :

Let $n \geq 3$.

$p : \mathbb{F}_{n+m}\mathbb{S}^2 \to \mathbb{F}_n\mathbb{S}^2 \qquad p((x_1, \ldots x_n, x_{n+1}, \ldots, x_{n+m})) = (x_1, \ldots x_n)$

$$\xrightarrow{\text{Homotopy exact sequence}}$$

$$(PBS) \quad 1 \to P_m(\mathbb{S}^2 \setminus \{n \text{ points}\}) \to P_{m+n}(\mathbb{S}^2) \to P_n(\mathbb{S}^2) \to 1$$

- There is a section at the geometrical level (Fadell-Van Burskirk 1962).

Generalised Fadell-Neuwirth fibration : a first application

Let $m = 1$ and $n \geq 3$.

$$(PBS) \quad 1 \to P_1(\mathbb{S}^2 \setminus \{n \text{ points}\}) \to P_{n+1}(\mathbb{S}^2) \to P_n(\mathbb{S}^2) \to 1$$

Since $P_1(\mathbb{S}^2 \setminus \{n \text{ points}\}) = F_{n-1}$ (free group of rank $n-1$) and (as we will see) $P_3(\mathbb{S}^2) = \mathbb{Z}_2$ we have that $P_{n+1}(\mathbb{S}^2) = F_{n-1} \rtimes \cdots \rtimes F_3 \rtimes \mathbb{Z}_2$.

Theorem (Bardakov, 2005))

$B_n(\mathbb{S}^2)$ is linear.

Braids and mapping classes ont the sphere

Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of n distinct points on $\mathbb{S}^2$.

***n*-punctured Mapping class group of $\mathbb{D}^2$** :

$$\mathcal{M}_n(\mathbb{S}^2) = \left\{ \begin{array}{l} h : \mathbb{S}^2 \to \mathbb{S}^2 \text{ orientation preserving} \\ h(p_i) \in \mathcal{P} \; i = 1, \ldots, n \end{array} \right\}_{/\sim}$$

$$\mathcal{PM}_n(\mathbb{S}^2) = \left\{ \begin{array}{l} h : \mathbb{S}^2 \to \mathbb{S}^2 \text{ orientation preserving} \\ h(p_i) = p_i \; i = 1, \ldots, n \end{array} \right\}_{/\sim}$$

Using (for $n \geq 3$) the evaluation map ($\pi_1(Homeo^+(\mathbb{S}^2)) = \mathbb{Z}_2$) :

Theorem (Magnus (1934) "+" Fadell-Van Buskirk (1961))

$$(BMS) \qquad 1 \to \langle \delta \rangle \to B_n(\mathbb{S}^2) \to \mathcal{M}_n(\mathbb{S}^2) \to 1$$

where $\delta = (\sigma_1 \sigma_2 \ldots \sigma_{n-1})^n$

About BMS sequence

$$(BMS) \qquad 1 \to \mathbb{Z}_2 \to B_n(\mathbb{S}^2) \to \mathcal{M}_n(\mathbb{S}^2) \to 1$$

- (BMS) does not split (comparison possible order of torsion elements)
- Let $n > 3$ then $Out(B_n(\mathbb{S}^2))$ is the Klein group, otherwise $Out(B_n(\mathbb{S}^2)) = \mathbb{Z}_2$ (B. 2008).
- $B_n(\mathbb{S}^2)$ is hopfian and cohopfian (B. 2008).
- More generally, it plays an important part, notably in the study of the centralizers and conjugacy classes of the finite order elements, and of the finite subgroups.

Braids on the sphere : finiteness and torsion

Theorem (Zariski, 1936 ; Fadell-Van Buskirk, 1961)

$B_n(\mathbb{S}^2)$ admits a presentation with generators :

$$\sigma_1, \ldots, \sigma_{n-1}$$

and relations :

$$\begin{gathered}
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \;\; 1 \leq i \leq n-2; \\
\sigma_i\sigma_j = \sigma_j\sigma_i, \;\; |i-j| > 1, \;\; 1 \leq i,j \leq n-1; \\
\sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_2\sigma_1 = 1.
\end{gathered}$$

- Exercise : $B_2(\mathbb{S}^2) = \mathbb{Z}_2$ and (more difficult !) $B_3(\mathbb{S}^2) = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$

Braids on the sphere : finiteness and torsion

- Exercise : $B_2(\mathbb{S}^2) = \mathbb{Z}_2$ and $B_3(\mathbb{S}^2) = \mathbb{Z}_4 \rtimes \mathbb{Z}_3$
- $B_n(\mathbb{S}^2)$ is not finite for $n > 3$ but has torsion elements.
  In particular $ZB_n(\mathbb{S}^2) = \langle \delta \rangle = \mathbb{Z}_2$ where $\delta = (\sigma_1 \sigma_2 \ldots \sigma_{n-1})^n$
  (Gilette-Van Buskirk, 1968).
  (Recall that $ZB_n = \langle \delta \rangle = \mathbb{Z}$).

Theorem (Gonçalves-Guaschi, 2004)

δ is the only torsion element of $P_n(\mathbb{S}^2)$ and the only element of order 2 of $B_n(\mathbb{S}^2)$.

Braids on the sphere : finiteness and torsion

Theorem (Murasugi, 1982)

Let $n \geq 3$; then the torsion elements of $B_n(\mathbb{S}^2)$ are conjugate of the following three elements :

a) $\alpha_0 = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}$ (which is of order $2n$) ;
b) $\alpha_1 = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2$ (which is of order $2(n-1)$) ;
c) $\alpha_2 = \sigma_1 \cdots \sigma_{n-3}\sigma_{n-2}^2$ (which is of order $2(n-2)$).

Theorem (Gonçalves-Guaschi, 2015)

Complete classification of virtually cyclic subgroups of $B_n(\mathbb{S}^2)$.

Braids on the sphere : finiteness and torsion

Previous results are the main ingredient for the following :

Theorem (B.-Gonçalves-Guaschi, 2018)

Let $n \neq m > 1$; $B_n(\mathbb{S}^2)$ surjects on $B_m(\mathbb{S}^2)$ if and only if $m = 2$.

Question

Let $n > 2$ and $n > m$; when $B_n(\mathbb{S}^2)$ can embed in $B_m(\mathbb{S}^2)$? Embeddings are all "geometric" ? Partial results : Gonçalves-Guaschi (2005) and Chen-Salter (2018).