Surface braids and mapping class group I
A survey on (classical) braid groups.

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Plan

Braids as paths : group presentations and first results

Braids and configurations spaces : exact sequences and applications

Braids as automorphisms of free groups : Burau representation and outer automorphisms

Braids as mapping classes

Incomplete-web-scholar bibliography

About braids :

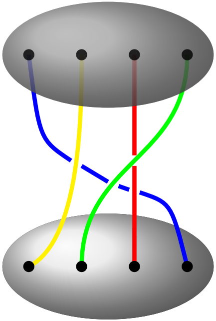
- Ester Dalvit, *Braids, a movie*
- Joan Birman and Tara Brendle, *Braids : A Survey*, math.GT/0409205
- Juan Gonzalez-Meneses, *Basic results on braid groups*, arXiv :1010.0321
- Luis Paris, *Braid groups and Artin Tits Groups*, arXiv :0711.2372
- Luis Paris, *From braid groups to mapping class groups*, math.GR/0412024
- Dale Rolfsen, *Tutorial on the braid groups*, arXiv :1010.4051

Braids as collections of paths

$\mathcal{P} = \{x_1, \ldots, x_n\} \subset \mathbb{D}^2$

Geometric braid (on n strands on $\mathbb{D}^2$) : $\beta = (\psi_1, \ldots, \psi_n)$, $\psi_i : [0,1] \to \mathbb{D}^2 \times [0,1]$

- $\psi_i(0) = (x_i, 0)$ and $\psi_i(1) \in \mathcal{P} \times \{1\}$ $\forall i = 1 \ldots, n$;
- $\psi_i(t) \neq \psi_j(t)$ for $i \neq j$ and $\psi_i(t) \in \mathbb{D}^2 \times \{t\}$.



Braids as collections of paths

Braids are considered up to isotopy :

Isotopy : $\beta_0 \sim \beta_1$ if it exists a **continuous** family of **geometric braids** $\beta_t, \quad t \in [0, 1]$.

The usual composition of paths induces a structure of group on equivalence classes of braids on n strands :

$\{$ Geometric braids (on n strands)$\}_{/\sim} \simeq B_n$,

Braid group on n strands.

The Artin group presentation of B_n

Theorem (Artin, 1925 and 1947)

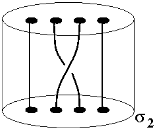
B_n admits a presentation with generators :

$$\sigma_1, \ldots, \sigma_{n-1}$$

and relations :

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \;\; 1 \leq i \leq n-2,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \;\; |i-j| > 1, \;\; 1 \leq i,j \leq n-1.$$

- Generator σ_i is represented by a half twist between ith and $(i+1)$th strands.



Back to 1947

E. Artin, *Theory of braids.* Ann. of Math. 48, (1947) 101–126.

E. Artin, *Braids and permutations.* Ann. of Math. 48, (1947) 643–649.

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The waiting for an answer for the second question has been much longer (Dyer and Grossman, 1981).

Artin Theorem

Definition. A homomorphism $\rho\colon\ B_n \to S_n$ is said to be *transitive* if the action of $\mathrm{Im}\,(\rho)$ on the set $\{1,\ldots,n\}$ is transitive.

Artin Theorem. Let $n \neq 4, 6$. Up to conjugacy, any transitive homomorphism $\phi : B_n \to S_n$ is such that $\mathrm{Im}\,(\phi)$ is a cyclic group of order n or is the canonical projection associating to a braid the corresponding permutation. Four "exotic" homomorphisms exist for $n = 4$ and $n = 6$.

Generalizing Artin Theorem : $n > m$

Artin Theorem brings to natural questions :

- Determine all homomorphisms between B_n and S_m ($n \neq m$).
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The answer to the first question (for $n > m$) was essentially already in Artin's ideas. For the second question :

Gorin-Lin Theorem. Let $n > m$ where $(n, m) \neq (4, 3)$. Any homomorphism $\phi\colon\ B_n \to B_m$ is cyclic (*i.e.* $\phi(B_n)$ is a cyclic group).

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For $(n, m) = (4, 3)$ we have also the surjective homomorphism $\psi : B_4 \to B_3$ such that $\psi(\sigma_1) = \psi(\sigma_3) = \sigma_1$ and $\psi(\sigma_2) = \sigma_2$.

Generalizing Artin Theorem : $n \leq m$

The study of possible homomorphisms between B_n and B_m when $n < m$ is much more complicated : few results for $n \leq m \leq 2n$ and for specific families of homomorphisms have been found by Lin.

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Using the interpretation of braid groups as **mapping classes**, Bell-Margalit (2006) showed that B_n is **quasi-cohopfian** and Castel (2011) determined all homomorphisms when $m = \{n, n+1\}$.

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Using Castel's approach and Nielsen-Thurston classification Chen-Kordek-Margalit (three weeks ago...) determined all homomorphisms between $B_n \to B_{2n}$ (for $n \geq 5$).

Other presentations for B_n

- Presentations with two generators (Artin, 1925 ; Klein, 1926 ; Lin, 2004) ;
- **Birman-Ko-Lee presentation (1998)** :
  Generators : $\sigma_{p,q}$ for $1 \leq p < q \leq n$.
  Relations :

$$\sigma_{p,q}\sigma_{q,r} = \sigma_{q,r}\sigma_{p,r} = \sigma_{p,r}\sigma_{p,q}, \quad 1 \leq p < q < r \leq n$$
$$\sigma_{p,q}\sigma_{r,s} = \sigma_{r,s}\sigma_{p,q}, \quad \text{for } [p,q] \text{ and } [r,s] \text{ disjoint or nested.}$$

- Presentations by graphs, possibly with infinitely many edges (Sergiescu, 1993).

Pure braids

We can associate to a braid the corresponding permutation in the symmetric group S_n. We obtain then :

$$1 \to P_n \to B_n \to S_n \to 1.$$

where P_n is the pure braid group on n strands.

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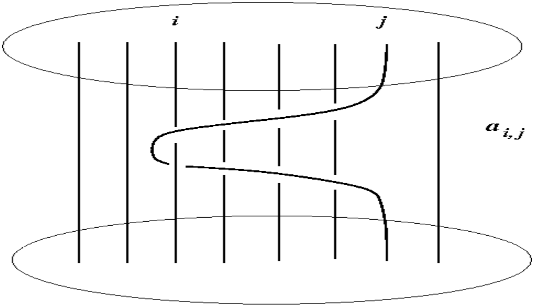
P_n is generated by $a_{i,j} = \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$ for $i < j$ $(a_{i,i+1} = \sigma_i^2)$.



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and defining relations :

$$a_{rs}^{-1}a_{ij}a_{rs} = \begin{cases} a_{ij} & \text{if } i < r < s < j \quad \text{or } r < s < i < j \\ a_{rj}a_{ij}a_{rj}^{-1} & \text{if } r < i = s < j \\ a_{rj}a_{sj}a_{ij}a_{sj}^{-1}a_{rj}^{-1} & \text{if } i = r < s < j \\ a_{rj}a_{sj}a_{rj}^{-1}a_{sj}^{-1}a_{ij}a_{sj}a_{rj}a_{sj}^{-1}a_{rj}^{-1} & \text{if } r < i < s < j \end{cases}$$

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Margalit-McCammond (2006) : "symmetric" group presentations for P_n

Configurations spaces and braid groups

$\mathbb{F}_n\mathbb{C} = \{(x_1, \dots x_n) \in \mathbb{C}^n \,|\, x_i = x_j \iff i = j\}$.

Definition : $\mathbf{P}_n := \pi_1(\mathbb{F}_n\mathbb{C})$

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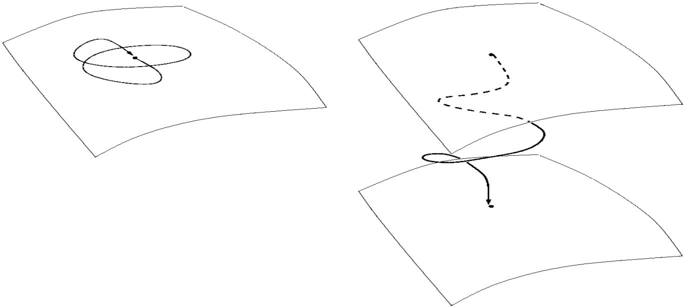
Fact : $\mathbf{P}_n$ is isomorphic to P_n.

Idea : an element of $\mathbf{P}_n$, $\lambda : [0,1] \to \mathbb{F}_n\mathbb{C}$ with base point $(x_{0,1}, \ldots x_{0,n})$ can be seen as a collection of n distinct paths on $\mathbb{C} \times [0,1]$...



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Definition : $\mathbf{B}_n := \pi_1(\mathbb{F}_n\mathbb{C}/S_n)$, where the symmetric group S_n acts by permutation of coordinates.

Proposition : $\mathbf{B}_n$ is isomorphic to B_n.

Configurations spaces and braid groups : a first application

Pure braid group on n strands : $P_n := \pi_1(\mathbb{F}_n\mathbb{C})$

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Fadell-Neuwirth fibration :
$p : \mathbb{F}_{n+1}\mathbb{C} \to \mathbb{F}_n\mathbb{C}\,, \qquad p((x_1, \dots, x_n, x_{n+1})) = (x_1, \dots, x_n)$
Fiber : $\mathbb{C} \setminus n$ points

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$$\xrightarrow{\text{Homotopy exact sequence}}$$

$(BS) \quad 1 \to \pi_1(\mathbb{C} \setminus n \text{ points }) \to P_{n+1} \to P_n \to 1$

$(\pi_1(\mathbb{C} \setminus n \text{ points }) \simeq F_n$ free group on n generators)

Forget configuration spaces...

$$(BS) \quad 1 \to \pi_1(\mathbb{C} \setminus n \text{ points }) \to P_{n+1} \to P_n \to 1$$

- The projection $p_n : P_{n+1} \to P_n$ consists of forgetting last strand, namely $p_n(a_{i,n+1}) = 1$ and $p_n(a_{i,j}) = a_{i,j}$ for $j \neq n+1$.
- The group generated by $\{a_{1,n+1}, \dots a_{n,n+1}\}$ is a free group of rank n and it coincides with $Kerp_n$.
- Iterating (BS) sequence we get that $P_n = F_n \rtimes F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes \mathbb{Z}$
- Using Lindon-Schupp method we get a group presentation for P_n.

Another application : residual properties

Lower central series (LCS) of G :
$\Gamma_1(G) = G$, $\Gamma_i(G) = [G, \Gamma_{i-1}(G)]$ for $i > 1$.

Rational lower central series of G : $D_1(G) = G$,
$D_i(G) = \{\, x \in G \,|\, x^n \in \Gamma_i(G)$ for some $n \in \mathbb{N}^* \,\}$ for $i > 1$.

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Let $\mathcal{P}$ be a group-theoretic property. G is **residually** $\mathcal{P}$ if for any (non-trivial) element $x \in G$, there exists a group H with the property $\mathcal{P}$ and a (surjective) homomorphism $\phi : G \to H$ such that $\phi(x) \neq 1$.

- G is residually nilpotent (RN) if and only if $\bigcap_{i \geq 1} \Gamma_i(G) = \{1\}$;
- G is residually torsion-free nilpotent (RTFN) if and only if $\bigcap_{i \geq 1} D_i(G) = \{1\}$.

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- F_n, free group on n generators, is RTFN.
- Using exact sequence on pure braid groups one can show that P_n is RTFN (Falk-Randell, 1985).

Lower central series : the case of B_n

Let G be a group.

Lower central series (LCS) of G :
$\Gamma_1(G) = G$, $\Gamma_i(G) = [G, \Gamma_{i-1}(G)]$ for $i > 1$.

G is **perfect** if $G = \Gamma_2(G)$.

Notation : $G/\Gamma_2(G) = G^{Ab}$.

Exemples : *mapping class groups* of closed surfaces of genus $g \geq 3$.

Proposition $B_n^{Ab} \cong \mathbb{Z}$ and $\Gamma_2(B_n) = \Gamma_3(B_n)$.

The lower central series of B_n

Proposition $B_n^{Ab} \cong \mathbb{Z}$ and $\Gamma_2(B_n) = \Gamma_3(B_n)$.

Sketch of the proof. We focus on the second statement. Consider the following short exact sequence :

$$1 \to \frac{\Gamma_2(B_n)}{\Gamma_3(B_n)} \to \frac{B_n}{\Gamma_3(B_n)} \xrightarrow{p} \frac{B_n}{\Gamma_2(B_n)} \to 1,$$

By abuse of notation let σ_i be the image of σ_i by $q_3 : B_n \to B_n/\Gamma_3(B_n)$. Since all of the $\sigma_i \in B_n/\Gamma_3(B_n)$ project to the same element of $B_n/\Gamma_2(B_n) \cong \mathbb{Z}$, for each $1 \leq i \leq n-1$, there exists $t_i \in \Gamma_2(B_n)/\Gamma_3(B_n)$ (where $t_1 = 1$) such that $\sigma_i = t_i\sigma_1$. Projecting the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ into $B_n/\Gamma_3(B_n)$, we see that $t_i\sigma_1 t_{i+1}\sigma_1 t_i\sigma_1 = t_{i+1}\sigma_1 t_i\sigma_1 t_{i+1}\sigma_1$. But the t_i are central in $B_n/\Gamma_3(B_n)$, so $t_i = t_{i+1}$, and since $t_1 = 1$, we obtain $\sigma_1 = \ldots = \sigma_{n-1}$. So the surjective homomorphism p is in fact an isomorphism.

Exact sequences of braid groups : how to get representations

Braid group on n strands :$B_n := \pi_1(\mathbb{F}_n\mathbb{C}/S_n)$

Mixed Braid group on $(n,1)$ strands : $B_{n,1} := \pi_1(\mathbb{F}_{n+1}\mathbb{C}/S_n) \subset B_{n+1}$

S_n acting on first n coordinates (in terms of collections of paths we are considering braids on $n+1$ strands where the last one is *pure*).

Fadell-Neuwirth fibration :

$p : \mathbb{F}_{n+1}\mathbb{C} \to \mathbb{F}_n\mathbb{C}\,, \qquad p((x_1,\ldots,x_n,x_{n+1})) = (x_1,\ldots,x_n)$

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Braids as automorphisms of free groups

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Artin representation for B_n

- The action of B_n on $F_n = \langle \xi_1, \ldots, \xi_n \rangle$ is faithful ($\iota : B_n \hookrightarrow Aut(F_n)$).
- Let $\beta \in Aut(F_n)$.
  $\beta \in \iota(B_n) \iff$
  $i)\ \beta(\xi_j) = g_j\, \xi_{\pi(j)}\, g_j^{-1} \qquad g_j \in F_n,\ j = 1, \ldots, n \ \ \pi \in S_n;$
  $ii)\ \beta(\xi_1 \xi_2 \cdots \xi_n) = \xi_1 \xi_2 \cdots \xi_n$

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Remark

Artin representation for B_n $\xrightarrow{\text{Magnus repr.}}$ **Burau representation** for B_n :

$$\rho_B : B_n \to GL_n(\mathbb{Z}[q^{\pm 1}])$$

Braids as automorphisms

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  ii) $\beta(\xi_1 \xi_2 \cdots \xi_n) = \xi_1 \xi_2 \cdots \xi_n$

Easy consequences :

- B_n is hopfian ;
- B_n has solvable word problem.

This interpretation of B_n is the main approach for :

Theorem (Dyer-Grossman, 1981)

The outer automorphism group of B_n, $Out(B_n)$, is isomorphic to $\mathbb{Z}_2$.

Braids as mapping classes

Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of n distinct points on $\mathbb{D}^2$.

***n*-punctured Mapping class group of $\mathbb{D}^2$** :

$$\mathcal{M}(\mathbb{D}^2, \mathcal{P}) = \left\{ \begin{array}{l} h : \mathbb{D}^2 \to \mathbb{D}^2 \text{ orientation preserving} \\ h(p_i) \in \mathcal{P} \; i = 1, \ldots, n \\ h_{|\partial \mathbb{D}^2} = Id \end{array} \right\}_{/\sim}$$

$$\mathcal{PM}(\mathbb{D}^2, \mathcal{P}) = \left\{ \begin{array}{l} h : \mathbb{D}^2 \to \mathbb{D}^2 \text{ orientation preserving} \\ h(p_i) = p_i \; i = 1, \ldots, n \\ h_{|\partial \mathbb{D}^2} = Id \end{array} \right\}_{/\sim}$$

Previous definitions do not depend on the choice of $\mathcal{P}$ but only on its cardinality ; therefore we can define :
$\mathcal{M}_n(\mathbb{D}^2) := \mathcal{M}(\mathbb{D}^2, \mathcal{P})$; $\mathcal{PM}_n(\mathbb{D}^2) := \mathcal{PM}(\mathbb{D}^2, \mathcal{P})$.

Theorem

$\mathcal{M}_n(\mathbb{D}^2) \simeq B_n$ (and $\mathcal{PM}_n(\mathbb{D}^2) = P_n$).

"Proof"

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$\mathcal{M}_n(\mathbb{D}^2) \simeq B_n$.

Sketch of the proof (pure case) :

Set $Homeo^+(\mathbb{D}^2) = \left\{ \begin{array}{l} h : \mathbb{D}^2 \to \mathbb{D}^2 \text{ orientation preserving} \\ h_{|\partial \mathbb{D}^2} = Id \end{array} \right\}$

and

$$Homeo^+(\mathbb{D}^2, \mathcal{P}) = \left\{ \begin{array}{l} h : \mathbb{D}^2 \to \mathbb{D}^2 \text{ orientation preserving} \\ h(p_i) = p_i \; i = 1, \ldots, n \\ h_{|\partial \mathbb{D}^2} = Id \end{array} \right\}$$

They are topological spaces provided with compact-open topology.
$\pi_0(Homeo^+(\mathbb{D}^2, \mathcal{P})) \simeq \mathcal{P}\mathcal{M}_n(\mathbb{D}^2)$

On the other hand, $Homeo^+(\mathbb{D}^2)$ is contractible (Alexander's trick) and $\pi_0(Homeo^+(\mathbb{D}^2)) \simeq \mathcal{M}(\mathbb{D}^2) = 1$.

"Proof"

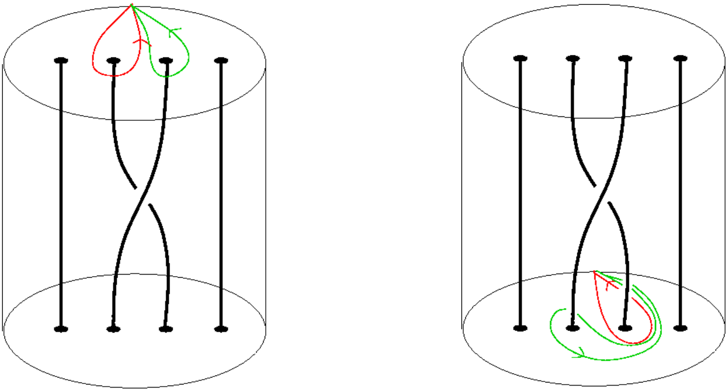
Sketch of the proof (pure case) :
We consider the evaluation map $Ev : Homeo^+(\mathbb{D}^2) \to \mathbb{F}_n(\mathbb{C})$ defined by $Ev(h) = \big(h(p_1), \ldots, h(p_n)\big)$. It is a locally trivial fibration with fiber $Homeo^+(\mathbb{D}^2, \mathcal{P})$. The long exact sequence of homotopy groups of this fibration gives the short exact sequence :

$$1 \to \pi_1(\mathbb{F}_n(\mathbb{C})) \to \pi_0(Homeo^+(\mathbb{D}^2, \mathcal{P})) \to \pi_0(Homeo^+(\mathbb{D}^2)) \to 1$$

Braids act on the punctured disk

From the isomorphism between $\mathcal{M}_n(\mathbb{D}^2)$ and B_n. we deduce that B_n act faithfully on the n punctured disk, call it $\mathbb{D}_n$, and therefore they act on $\pi_1(\mathbb{D}_n) \simeq F_n$. This action coincides with the action by conjugacy defined by the natural section of

$(BS) \quad 1 \to \pi_1(\mathbb{C} \setminus n \text{ points }) \to B_{n,1} \to B_n \to 1$



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Remark : The action of B_n on $\mathbb{D}_n$ (or $\mathbb{C} \setminus n$ points) induces also an action of B_n on $\mathbb{F}_m\mathbb{D}_n = \{(x_1, \dots x_m) \in (\mathbb{D}_n)^{\times n} | x_i = x_j \iff i = j\}$ and $\mathbb{F}_m\mathbb{D}_n/S_m$ for any integer $m \geq 1$.
Therefore B_n acts on the fundamental group of such a configuration space, which is called braid group of the m-punctured disk.
This action is the starting point for Bigelow-Krammer-Lawrence representation(s).