LECTURE 2

1. Root datum and Coxeter groups

We start from the following classical motivating example.

Consider the Euclidean space $\mathbb{R}^N$, $N \geq 2$. Let $e_1, \ldots, e_N$ denote the standard basis. Let $H \subset \mathbb{R}^N$ be the hyperplane consisting of vectors whose sum of coordinates is 0, i.e.

$$H := \{\sum_{i=1}^{N} a_i e_i \mid \sum_{i=1}^{N} a_i = 0\}.$$

Let Σ denote the subset of H consisting of all differences of standard vectors

$$\Sigma := \{e_i - e_j \mid i \neq j\}.$$

By definition, we have

$$\Sigma \cap m\Sigma \neq \emptyset,\ m \in \mathbb{Z} \quad \Longrightarrow \quad m = \pm 1.$$

and Σ splits into two disjoint subsets Σ^+ and Σ^-, where

$$\Sigma^+ := \{e_i - e_j \mid i < j\} \text{ and } \Sigma^- := \{e_i - e_j \mid i > j\}.$$

Each vector α from Σ can be uniquely expressed as a $\mathbb{Z}$-linear combination of vectors from the subset

$$\Pi := \{e_i - e_{i+1} \mid 1 \leq i \leq N-1\}.$$

Moreover, if $\alpha = \sum_i c_i(e_i - e_{i+1})$, then either all $c_i \geq 0$, or all $c_i \leq 0$. We also have

$$|\Sigma^+| = |\Sigma^-| = \binom{N}{2}.$$

Let Λ denote the $\mathbb{Z}$-linear span of Π. Observe that Λ is a free $\mathbb{Z}$-module of rank $N-1$ and $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq H$. The subset Σ of the finitely generated free $\mathbb{Z}$-module Λ provides an example of a *root system* of type A and of rank $N-1$:

- elements of Σ are called roots,
- elements of Π are called simple roots,
- elements of Σ^+ and Σ^- are called positive and negative roots, respectively.
- The $\mathbb{Z}$-module Λ is called root lattice.

Given $\alpha \in \Sigma$ consider an orthogonal reflection s_α which fixes the hyperplane orthogonal to $\alpha = e_i - e_j$. It is given by the following general formula

$$s_\alpha(\beta) := \beta - \tfrac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha, \qquad \text{where } (-,-) \text{ is the usual dot-product in } \mathbb{R}^N.$$

Observe that s_α simply switches e_i and e_j (the ith and the jth coordinates) and, hence, it leaves Σ invariant:

$$s_\alpha(\beta) \in \Sigma, \qquad \text{for all } \alpha, \beta \in \Sigma.$$

Let W be the group (called the Weyl group) generated by reflections s_α, $\alpha \in \Sigma$ where the multiplication is given by the composite of reflections. It acts by permutations of the set $\{e_1, \ldots, e_N\}$, i.e. it can be identified with the symmetric group S_N on the set of indices $\{1, \ldots N\}$. Clearly, it leaves Σ invariant.

1.1. *Root datum.* We now provide the following generalization of the previous example (see e.g. [**?**, Exp. XXI, §1.1]).

We define a *root datum* to be a nonempty finite subset Σ of a free finitely generated $\mathbb{Z}$-module Λ together with a set inclusion

$$\Sigma \hookrightarrow \Lambda^\vee, \quad \alpha \mapsto \alpha^\vee$$

into the dual $\Lambda^\vee = Hom_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ such that

(1) $\Sigma \cap m\Sigma \neq \emptyset$, $m \in \mathbb{Z} \Longrightarrow m = \pm 1$,
(2) $\alpha^\vee(\alpha) = 2$ for all $\alpha \in \Sigma$, and
(3) $\beta - \alpha^\vee(\beta)\alpha \in \Sigma$ and $\beta^\vee - \beta^\vee(\alpha)\alpha^\vee \in \Sigma^\vee$ for all $\alpha, \beta \in \Sigma$,

where $\Sigma^\vee$ denotes the image of Σ in $\Lambda^\vee$. The elements of Σ (resp. $\Sigma^\vee$) are called roots (resp. coroots).

The $\mathbb{Z}$-submodule of Λ generated by Σ is called the *root lattice* and is denoted by Λ_r. A root datum is called semisimple if

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda_r \otimes_{\mathbb{Z}} \mathbb{Q}.$$

From now on by a root datum we will always mean a semisimple one.

The $\mathbb{Z}$-submodule of $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by all $\lambda \in \Lambda_{\mathbb{Q}}$ such that $\alpha^\vee(\lambda) \in \mathbb{Z}$ for all $\alpha \in \Sigma$ is called the *weight lattice* and is denoted by Λ_w.

By definition we have

$$\Lambda_r \subseteq \Lambda \subseteq \Lambda_w \quad \text{and} \quad \Lambda_r \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda_{\mathbb{Q}} = \Lambda_w \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The $\mathbb{Q}$-rank of $\Lambda_{\mathbb{Q}}$ is called the *rank* of the root datum.

1.2. *Simple roots, fundamental weights and the Cartan matrix.* It can be shown that the root lattice Λ_r admits a $\mathbb{Z}$-basis $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ such that each $\alpha \in \Sigma$ is a linear combination of α_i's with either all positive or all negative coefficients and n is the rank of the root datum. So the set Σ splits into two disjoint subsets $\Sigma = \Sigma_+ \amalg \Sigma_-$, where Σ_+ (resp. Σ_-) is called the set of positive (resp. negative) roots. The roots α_i are called *simple roots*.

Given the set Π we define the set of *fundamental weights* $\{\omega_1, \ldots, \omega_n\} \subset \Lambda_w$ as

$$\alpha_i^\vee(\omega_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. Fundamental weights form a basis of the weight lattice Λ_w.

We denote $c_{ij} = \alpha_j^\vee(\alpha_i)$, $i, j = 1 \ldots n$. The matrix $C = (c_{ij})$ is called the *Cartan matrix* of the root datum. By definition we have

$$\alpha_i = \sum_{j=1}^{n} c_{ij}\omega_j,$$

i.e. the Cartan matrix expresses simple roots in terms of fundamental weights.

Observe that that the determinant of the Cartan matrix coincides with the number of elements in the quotient group Λ_w/Λ_r.

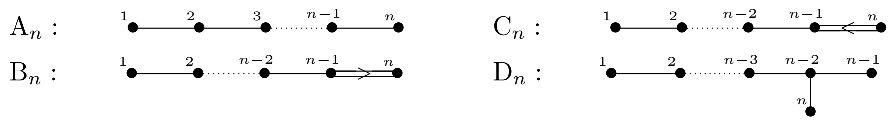
1.3. *The Dynkin diagram.* A root datum is called *irreducible* if it can not be represented as a direct sum of root data, i.e. Λ can not be written as $\Lambda = \Lambda_1 \oplus \Lambda_2$, where $\Sigma_1 \subset \Lambda_1$, $\Sigma_2 \subset \Lambda_2$ are the root data.

To any irreducible root datum we associate a graph called the *Dynkin diagram* $\mathcal{D}$. Its vertices are in 1-1 correspondence with the set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ and the number of edges connecting two different vertices α_i, α_j is given by $c_{ij} \cdot c_{ji}$.

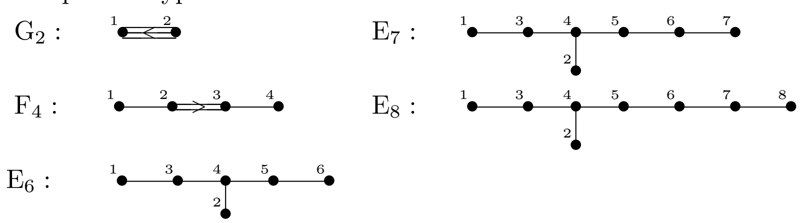
Moreover, if there are more than one edge connecting α_i and α_j we put $' <'$ between α_i and α_j if $c_{ij} < c_{ji}$. Note that the latter inequality leads to $(\alpha_i, \alpha_i) < (\alpha_j, \alpha_j)$.

All Dynkin diagrams are classified and consist of the following types (our enumeration of vertices follows Bourbaki; the lower index is the rank n):

Classical types:



Exceptional types:



It can be shown that an irreducible root datum is determined uniquely by

- its Dynkin diagram and
- the intermediate lattice $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$.

If $\Lambda = \Lambda_w$ (resp. $\Lambda = \Lambda_r$), then the root datum is called simply connected (resp. adjoint) and will be denoted by $\mathcal{D}_n^{sc}$ (resp. $\mathcal{D}_n^{ad}$), where $\mathcal{D} = \mathrm{A, B, C, D, E, F, G}$ is one of the Dynkin diagrams and n is its rank.

1.4. *The Weyl group.* A $\mathbb{Z}$-linear map $s_\alpha \colon \Lambda_w \to \Lambda_w$, $\alpha \in \Sigma$, defined by

$$s_\alpha(\lambda) := \lambda - \alpha^\vee(\lambda)\alpha, \quad \lambda \in \Lambda_w,$$

is called the *reflection* corresponding to the root α. Observe that by definition we have

$$s_\alpha \circ s_\alpha = \mathrm{id}.$$

The group W generated by all reflections s_α is called the *Weyl group* of the root datum. It can be shown that

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $s_i = s_{\alpha_i}$ is the reflection corresponding to the simple root α_i, the *Coxeter exponents* $m_{ii} = 1$ and m_{ij} for $i \neq j$ depend on the values of c_{ij} as follows:

$$m_{ij} = 2 \text{ if } c_{ij}c_{ji} = 0 \text{ (no edge)},$$
$$m_{ij} = 3 \text{ if } c_{ij}c_{ji} = 1 \text{ (single edge)},$$
$$m_{ij} = 4 \text{ if } c_{ij}c_{ji} = 2 \text{ (double edge)},$$
$$m_{ij} = 6 \text{ if } c_{ij}c_{ji} = 3 \text{ (triple edge)}.$$

The group W is a finite group which acts by permutations on Σ. It provides an example of the so called *Coxeter group*.

Each element of W can be written as a product of simple reflections

$$w = s_{i_1} s_{i_2} \ldots s_{i_r}.$$

The smallest such r is called the *length of* w and is denoted by $\ell(w)$. Presentation of w as a product of precisely $\ell(w)$ simple reflections is called *reduced* presentation (or *reduced word*).

1.5. *Geometric example/realization.* We can realize roots Σ (and the root lattice Λ_r) geometrically as vectors (and their $\mathbb{Z}$-linear combinations) in a finite dimensional Euclidean space $\mathbb{R}^N$:

Namely, we may view Σ as a subset of (non-zero) vectors in $\mathbb{R}^N$ and for each $\alpha \in \mathbb{R}^N$ we define $\alpha^\vee \in (\mathbb{R}^N)^*$ by

$$\alpha^\vee(x) := \tfrac{2(\alpha,x)}{(\alpha,\alpha)}, \quad x \in \mathbb{R}^N.$$

If $\alpha^\vee(\beta) \in \mathbb{Z}$ for all $\alpha, \beta \in \Sigma$, then $\alpha^\vee$ defines an element in the dual $\mathbb{Z}$-module $\Lambda_r^\vee$. Hence, it gives the inclusion $\Sigma \hookrightarrow \Lambda_r^\vee$, $\alpha \mapsto \alpha^\vee$ and, therefore, it provides an example of a root datum. In this way we have also realized the Weyl group W as the group generated by the usual orthogonal reflections:

> the operator s_α is the reflection along the vector α as it fixes the hyperplane orthogonal to α and sends α to $-\alpha$.

We call such example/realization of a root datum the *finite crystallographic root system* see e.g. [**?**, §2.9].

Observe that if we identify the dual space $(\mathbb{R}^N)^*$ with $\mathbb{R}^N$ by $e_i^* \mapsto e_i$, where $\{e_1, \ldots, e_N\}$ is the standard basis, then the coroots correspond to the vectors

$$\alpha^\vee \mapsto \tfrac{2}{(\alpha,\alpha)}\alpha \in \Lambda_r \otimes_{\mathbb{Z}} \mathbb{R}.$$

1.6. *Finite real root systems.* We can replace the coefficient rings $\mathbb{Z}$ and $\mathbb{Q}$ by $\mathbb{R}$ in the definitions of the root datum and of the associated lattices (i.e. by considering the $\mathbb{R}$-vector space Λ instead of a $\mathbb{Z}$-module Λ and its $\mathbb{R}$-linear dual $\Lambda^\vee$). In this case, we obtain a root datum over $\mathbb{R}$, where $(\Lambda)_r = \Lambda = (\Lambda)_w$ and $\alpha^\vee(\beta)$ is not necessarily in $\mathbb{Z}$.

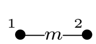
Following example 1.5, we can realize Σ as a subset in $\mathbb{R}^N$ with $(\alpha,\alpha) = 1$ and $\alpha^\vee(x) := 2(\alpha, x)$. We call such an example/realization the *finite (normalized) real root system*, see e.g. [**?**, §1]. The Weyl group then turns into a finite real reflection group which is a Coxeter group

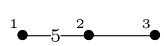
$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

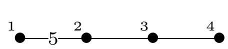
for some exponents m_{ij}.

All finite real root systems are classified by the so called *Coxeter diagrams*. Similar to the Dynkin digram its vertices are in 1-1 correspondence with the set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$. Different vertices α_i and α_j are connected by an edge only if $m_{ij} \geq 3$. In addition, if $m_{ij} \geq 4$, then the respective edge is labelled by the Coxeter exponent m_{ij}. Observe that $\alpha_j^\vee(\alpha_i) = -2\cos(\pi/m_{ij})$ and the respective symmetric matrix $C = (\alpha_j^\vee(\alpha_i))$ is called the Schläfli matrix.

If W has exponents $m_{ij} = 2, 3, 4, 6$ only, then (up to rescaling root vectors) we obtain the usual crystallographic root system. In all other cases (called non-crystallographic), the finite real root systems are classified by the Coxeter diagrams of the following types (here $m = 5$ or $m \geq 7$):

$\mathrm{I}_2(m)$:


H_3 :


H_4 :


Remark 1.1. Observe that we may think of $\mathrm{I}_2(2)$, $\mathrm{I}_2(3)$, $\mathrm{I}_2(4)$ and $\mathrm{I}_2(6)$ as the normalized crystallographic root systems of type $\mathrm{A}_1 \times \mathrm{A}_1$, A_2, B_2 $(= \mathrm{C}_2)$ and G_2 respectively. Indeed, these are obtained by scaling one of the simple roots by $-2\cos(\pi/m_{ij}) = 0, -1, -\sqrt{2}$ and $-\sqrt{3}$, respectively

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