The Equivariant Derived Category and an Introduction to Schemes

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April 26, 2022

Last compiled: April 26, 2022 at 9:12 Atlantic Daylight Time.

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# Introduction and Structure of the Notes

These lecture notes serve four main purposes:

1. First, to help me focus and guide *exactly* what I'd like to say during the mini-course and hence provide me a place to which I can point when there are elaborate, difficult, or just plain tedious proofs that would take up too much time to do live.

2. Second, as a place to provide complete details and constructions regarding some of the things we'll look at. While I'll certainly try to give context around what we discuss, it is much easier to cite historical literature and contextual literature using BibTEX and also to provide places the curious or interested can turn (probably to find an introduction cleaner than my own).

3. Third, this gives me a place to point to the explicit background we need. While I'm trying very hard to make this accessible, by nature it's a difficult topic and there is much background that goes into the various/myriad constructions we consider. I've tried to streamline things as much as possible, but these notes let me point out various background references that may be helpful for the reader/attendee (especially if the reader/attendee is a graduate student).

4. Fourth, and perhaps most importantly, these give a guide to the work we'll be doing and covering!

5. While I'll try to point out any actual homological algebra we need as we use it, I've included a thourough introduction with exercises in the appendices. This introduction is helpful for graduate students or people who need/want a self-contained refresher.

6. I've tried to give a more or less complete introduction to the basics of scheme theory in Chapter 3. This chapter will hold many technical results and proofs, but I've tried to put them here for the sake of completeness and to give a full picture of what (foundational) scheme theory looks like. Special attention has been paid to (co)reflective subcategories and to the categorical techniques that make scheme theory work; however, my recommendation for at least the first five sections of Chapter 3, is to look through the results and understand them but skip over the proofs unless you want to see instructions on how to do Zariski descent. Instead, I recommend looking through the sections on separated, reduced, and finite type schemes and see how to use the gadgets in proofs before focusing on the in-depth constructions of spectra.

7. These notes also give a nice place to discuss the difference between quasi-coherent sheaves, étale sheaves, and some of the ring/module-theoretic subtleties that arise in algebraic geometry. Chapter **??** in particular, while not used explicitly in the lectures themselves, can be seen as a place for the basics and developments of quasi-coherent sheaves and étale sheaves to be done for those who want to see them in one self-contained place.

8. Sadly, at the moment these notes are incomplete. Currently there are some floating references, some orange boxes reminding me of things, and other such eyesores. Additionally, Chapter **??** is missing some of the basic results on étale sheaves I was hoping to have included. I plan on having a working version of Chapter **??** done by the end of the week and continually update these with references and the like until the book is done.

9. If you have any questions or comments, please email me here `gmvooys@dal.ca`. I welcome any pointed out typos or communcations!

## Some Notes on Background and Terminology

In these notes I assume the following background and will make the following conventions:

- I assume relative familiarity with category theory. In particular, I assume that the reader is familiar with categories, adjunctions, and (co)limits together with how they interact (although, to be fair, I have tried to reference somewhat technical facts as necessary).
- I assume relative familiarity with algebra at the graduate level, but not much further. In particular, I assume some background with rings, modules, and tensor products.
- I have made some comments in the footnotes of the article. Some of these are notes for me, silly jokes[1], discussions of ideas and techniques, ideas towards further reading or techniques, and occasionally side arguments for statements made in proofs.
- I have attempted to provided an index (both of notation and of terminology) in this book. Some terms are missing, but propositions, lemmas, and corollaries are at least all hyper-linked for easier reading and reference in PDF.
- If $\mathscr{C}$ is a category we write $\mathscr{C}_0 = \mathbb{O}(\mathscr{C})$ for the class of objects in $\mathscr{C}$ and $\mathscr{C}_1$ for the class of morphisms in $\mathscr{C}$.
- If $G$ is a group and $N \trianglelefteq G$ is a normal subgroup then we write $G/N$ is the quotient group of $G$ by $N$. For example, $\mathbb{Z}/p\mathbb{Z}$ is the cyclic group with $p$ elements and is isomorphic to $\mathbb{F}_p$.
- If we are in a place where explicit representations are not important, we write $\mathbb{F}_{p^n}$ for the field with $p^n$ elements.
- In general script fonts $(\mathscr{C}, \mathscr{D}, \cdots)$ denote 1-categories and German fonts $(\mathfrak{C}, \mathfrak{D}, \cdots)$ denote 2-categories. Named categories (such as **Set**, **Cring**, **Ring**, **Ab**, **Cat**) denote 1-categories and the fraktur versions of the same category denote the 2-categorical version. In particular, **Ab** denotes the category of Abelian groups, **Cring** denotes the category of commutative unital rings, **Ring** denotes the category of unital rings, and **Set** denotes the category of sets.
- We write **Cat** for the category of small categories and $\mathfrak{Katze}$[2] for the 2-category of small 1-categories; **CAT** is the (meta)category of all categories and $\mathfrak{KATZE}$ is the (meta)2-category of all 1-categories.
- Generally the notation $[-,-]$ denotes an internal hom functor of some sort. I've tried to make each contextually clear, but examples of different uses of this notation are the functor category $[\mathscr{C}, \mathscr{D}]$ of functors from $\mathscr{C}$ to $\mathscr{D}$ in **Cat**, the internal hom in a Cartesian closed category, or even $[M, N]$ for the internal hom in the symmetric monoidal closed category $A$-**Mod** of modules over a commutative unital ring.
- Note that we'll abuse notation and occasionally write $\mathbf{Cring}_{/A}$ for the coslice/over category $A \downarrow \mathbf{Cring}$. We don't have a good excuse for this other than when we get to schemes $A \downarrow \mathbf{Cring}$ is opposite equivalent to the category of affine schemes over $A$, and it's convenient to use term/notation as much as possible.

[1] Algebraic geometry is dry enough that at times reading a geometry book is like eating six saltine crackers successively in under a minute. I hope these jokes, when they are jokes, are nice ways to breathe a little life into the subject and help give humanizing perspectives towards things. They also give me places to rant and discuss mathematical philosophy in places where it is relevant without having to break the flow of the main text.

[2] If by writing things more German we make them 2-categories, then turning English words auf Deutsch makes 1-categories even more 2-categorical.

Discuss some of the highlights, why we care about the EDC, where it comes from, and motivation.

Give some words of what I expect as background knowledge. Point out references and such.

Motivate the importance of the EDC in rep theory, geometry, Langlands Programme, topology, descent theory, etc. Give many ref's!

Give a quick nod to what a sheaf is until we have time to go into sheaves in detail.

# Chapter 1

# What Should an Equivariant Derived Category be?

As I mentioned[1] in the introduction above, the equivariant derived category $D^b_G(X)$ is a very important object of study in (equivariant) algebraic geometry, representation theory, the Langlands Programme, descent theory, and various areas around these subjects. While there have been many distinct definitions of these categories that were at best simply treated to be equivalent to each other, it was only finally explicitly and carefully proved in [77] that the four most frequently encountered incarnations are actually equivalent to each other in the algebro-geometric case. In particular, it is shown there that it is ultimately and essentially the theory of equivariant descent that determines and builds the equivariant derived category. However, because of these myriad different perspectives on the equivariant derived category we can[2] ask ourselves "What are the properties of the EDC that are *really* important? What is it that makes the equivariant derived category an equivariant derived category?" We'll begin our journey by asking these questions while largely treating many of the important properties that arise as black boxes (for now, at least).

Perhaps the most important place to being our study of the equivariant derived category is by seeing what we know about the *usual* derived category (we'll discuss the derived category of a variety shortly after). Let's first define the derived category of an Abelian category before describing it's history and giving an interpretation of what it's doing. To give a complete introduction to the derived category correctly requires a good knowledge of homological algebra and would take too much time and space to develop here. We'll discuss the basics of what is needed to define cohomology and quasi-isomorphisms, but not much else.[3] For the reader interested in seeing all the ingredients that go into the construction of cohomology in detail, please see [34], [39], [79], or Appendix C.

**Definition 1.0.1.** A category $\mathscr{A}$ is said to be an Abelian category if the following hold:

- $\mathscr{A}$ is enriched in $\mathbf{Ab}$, i.e., for every $A, B \in \mathscr{A}_0$ the hom-class $\mathscr{A}(A,B)$ is actually an Abelian group (and hence a set) and composition is bilinear in the sense that given a diagram of the form

$$A \xrightarrow{f} B \xrightarrow{g+h} C \xrightarrow{k} D$$

we have

$$k \circ (g+h) \circ f = k \circ g \circ f + k \circ h \circ f.$$

[1] Or will mention. Part of the fun of scientific writing inovolves actually writing the introduction to something last but always referring to the introduction as if it has already been written and hoping that the seeming time paradox in the writing only hurts the *author's* head and not the reader. I apologize if my tense (past, future, present, future perfect, etc.) is ever off and/or confusing in this regard.

[2] And perhaps, at least to the category theorists reading this, should.

[3] With great apologies. Understanding and seeing what goes into allowing us to define and construct cohomology in an Abelian category (or more accurately the chain complexes of an Abelian category) is very worthwhile and important for learning the standard techniques used in algebraic geometry and algebraic topology. Of course, reading through the appendices is a good start in seeing this done completely universally and without choosing elements (especially Appendix C and D).

- $\mathscr{A}$ has a zero object 0 and finite products.
- $\mathscr{A}$ is (finitely) complete and cocomplete.
- Every monic is an equivalizer and every epimorphism is a coequalizer.

In Abelian categories we have notions of both kernels and cokernels of morphisms. The kernel of a map $f : A \to B$ is the equalizer of the pair

$$\begin{tikzcd} \operatorname{Ker}(f) \arrow[r, "\ker f"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

where the zero map is the unique map factoring as:

$$\begin{tikzcd} A \arrow[r, dashed, "\exists!"] \arrow[dr, "0"'] & 0 \arrow[d, dashed, "\exists!"] \\ & B \end{tikzcd}$$

We'll write $\operatorname{Ker}(f)$ for the object of the kernel and $\ker(f) : \operatorname{Ker}(f) \to A$ for the morphism. Dually, the cokernel of a morphism $f : A \to B$ is the coequalizer:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \arrow[r, "\operatorname{coker}(f)"] & \operatorname{Coker}(f) \end{tikzcd}$$

We'll write $\operatorname{Coker}(f)$ for the object of the cokernel and $\operatorname{coker}(f) : B \to \operatorname{Coker}(f)$ for the canonical morphism. These two pieces of technology together allow us to define the image of a morphism $f : A \to B$; this is the subobject of $B$ defined by

$$\operatorname{Im}(f) := \operatorname{Ker}(\operatorname{coker}(f)).$$

**Definition 1.0.2.** Let $\mathscr{A}$ be an Abelian category. The category of chain complexes in $\mathscr{A}$ is the category $\mathbf{Ch}(\mathscr{A})$ where:

- Objects: Pairs $(A^\bullet, \partial_n)$ where $A^\bullet = \{A^n \mid n \in \mathbb{Z}, A^n \in \mathscr{A}_0\}$ is an integer sequence of objects in $\mathscr{A}$ and for all $n \in \mathbb{Z}$ $\partial_n \in \mathscr{A}(A^n, A^{n+1})$ is a morphism (called the $n$-th differential) with the property that $\partial_{n+1} \circ \partial_n = 0$, where 0 here denotes the zero map $A^n \to 0 \to A^{n+2}$. We visualize these as:

$$\begin{tikzcd} \cdots \arrow[r] & A^{n-1} \arrow[r, "\partial_{n-1}"] & A^n \arrow[r, "\partial_n"] & A^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

- Morphisms: A chain map $f : A^\bullet \to B^\bullet$ is a sequence of maps $f_n : A^n \to B^n$ in $\mathscr{A}$ such that for all $n \in \mathbb{Z}$ the diagram

$$\begin{tikzcd} A^n \arrow[r, "\partial_n^A"] \arrow[d, "f_n"'] & A^{n+1} \arrow[d, "f_{n+1}"] \\ B^n \arrow[r, "\partial_n^B"'] & B^{n+1} \end{tikzcd}$$

commutes in $\mathscr{A}$.

- Composition: Degree-wise, i.e., given $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to C^\bullet$, $(g \circ f)_n := g_n \circ f_n$.

Let $A^\bullet \in \mathbf{Ch}(\mathscr{A})_0$ and let $n \in \mathbb{N}$. It can be shown that the diagram

$$\begin{tikzcd} A^n \arrow[r, "\partial_n"] \arrow[dr, two heads] & A^{n+1} \arrow[r, "\partial_{n+1}"] & A^{n+1} \\ & \operatorname{Im}\partial_n \arrow[u, tail] \arrow[ur, "0"'] & \end{tikzcd}$$

commutes. As such there is a unique morphism $\gamma_{n+1} : \operatorname{Im}\partial_n \to \operatorname{Ker}\partial_{n+1}$ making the diagram

$$\begin{tikzcd} \operatorname{Im}\partial_n \arrow[r, "\gamma_{n+1}"] \arrow[dr, tail] & \operatorname{Ker}\partial_{n+1} \arrow[d, tail] \\ & A^{n+1} \end{tikzcd}$$

commute in $\mathscr{A}$. In fact, $\gamma_{n+1}$ is monic and whether or not it is an isomorphism can be checked by determining if the quotient object $\operatorname{Coker}(\gamma_{n+1}) \overset{?}{\cong} 0$; if this isomorphism holds, then $\gamma_{n+1}$ is an isomorphism and $A^\bullet$ is exact at $A^{n+1}$ (and vice-versa). This process defines the $(n+1)$-th cohomology of $A^\bullet$ valued in $\mathscr{A}$.

**Definition 1.0.3.** Let $A^\bullet \in \mathbf{Ch}(\mathscr{A})_0$ for $\mathscr{A}$ an Abelian category. Then the $n$-th cohomology of $\mathscr{A}$ is the object $H^n(A^\bullet \in \mathscr{A}_0$ given by

$$H^n(A^\bullet) := \operatorname{Coker}(\gamma_n)$$

where $\gamma_n$ is the unique comparison map $\gamma_n : \operatorname{Im}\partial_{n-1} \to \operatorname{Ker}\partial_n$.

Following this definition, we define the $n$-th cohomology of a morphism $f : A^\bullet \to B^\bullet$ to be the unique morphism $H^n(f) := \sigma$ making the diagram

$$\begin{tikzcd} \operatorname{Im}\partial^A_{n-1} \arrow[r, "\tilde{f}"] \arrow[d, "\gamma^A_n"'] & \operatorname{Im}\partial^B_{n-1} \arrow[d, "\gamma^B_n"] \\ \operatorname{Ker}\partial^A_n \arrow[r, "\overline{f}"'] \arrow[d] & \operatorname{Ker}\partial^B_n \arrow[d] \\ \operatorname{Coker}\gamma^A_n \arrow[r, dashed, "\exists!\sigma"'] & \operatorname{Coker}\gamma^B_n \end{tikzcd}$$

commute. That this is functorial and defined for any $n \in \mathbb{Z}$ is routine to verify (cf. Lemma D.0.9).

**Definition 1.0.4.** Let $\mathscr{A}$ be an Abelian category. A quasi-isomorphism is a morphism $f : A^\bullet \to B^\bullet$ for which $H^n(f) : H^n(A) \to H^n(B)$ is an isomorphism for all $n \in \mathbb{Z}$. The class of all quasi-isomorphisms in $\mathbf{Ch}(\mathscr{A})$ will be denoted as $Q$.

**Definition 1.0.5.** Let $\mathscr{A}$ be an Abelian category. The derived category of $\mathscr{A}$, $D(\mathscr{A})$, is the category

$$D(\mathscr{A}) := Q^{-1}\,\mathbf{Ch}(\mathscr{A}),$$

i.e., $D(\mathscr{A})$ is the localization of $\mathbf{Ch}(\mathscr{A})$ at the class of quasi-isomorphisms.

**Remark 1.0.6.** For the category theorists: the localization $D(\mathscr{A}) = Q^{-1}\,\mathbf{Ch}(\mathscr{A})$ is not an Ore localization in general. It does satisfy the (two-sided) Ore square condition, but the equalizer/coequalizer condition need not be satisfied.

**Remark 1.0.7** (Achtung!)**.** Even if $\mathbf{Ch}(\mathscr{A})$ is a locally $\mathscr{U}$-small category in some Grothendieck universe $\mathscr{U}$, the category $D(\mathscr{A})$ need not exist in $\mathscr{U}$ (cf. [39, Section 13.1, Page 319], [79, Remark 10.3.3]). However, we will not worry about these set-theoretic difficulties in these notes aside from warning you to be careful in practice and contributing to yet another example of an algebraic geometer not respecting set-theoretic foundations.

The category $D(\mathscr{A})$ is indispensible in modern homological algebra. If we only care about a chain complex up to its cohomology $H^*(A^\bullet)$, then $D(\mathscr{A})$ is the most natural category we can work with. Because of the localization that defines it, $D(\mathscr{A})$ can be thought of as taking the category $\mathbf{Ch}(\mathscr{A})$ and amending it by adding the relations that $A^\bullet \cong B^\bullet$ if and only if $H^n(A^\bullet) \cong H^n(B^\bullet)$ for all $n \in \mathbb{Z}$. However, doing this makes the category $D(\mathscr{A})$ much more difficult to work with: $D(\mathscr{A})$ is rarely an Abelian category, and is not Abelian in most of the cases we could care about (such as the categories $\mathbf{Ab}, \mathbf{Ab}(X), \mathbf{Ab}(Z, \text{ét}), \mathbf{QCoh}(Z)$ for a topological space $X$ and a scheme $Z$ with $X$ and $Z$ non-pathological).[4]

The benefits of working with the derived category are not only that objects become isomorphic whenever they have isomorphic cohomology, but also that we can work with singular geometric and topological spaces in meaningful ways. In fact, in [6] Beilinson, Bernstein, Deligne, and Gabber[5] showed that we can find a full Abelian subcategory $\mathbf{Per}(X)$ of $D(X)$ together with cohomological functors ${}^pH^* : D(\mathscr{A}) \to \mathbf{Per}(X)$ (for $X$ a topological space or variety) which records the intersection cohomology of $X$. This category is the category of perverse sheaves[6], which is a very important category modern representation theory and especially in the representation theory of $p$-adic groups; cf. [14], for instance, for more details.

Consider the Abelian category $\mathbf{Ab}(X)$ of sheaves (of Abelian groups) defined on a scheme/topological space $X$ and assume that $X$ has an action by either an algebraic group or a topological group (depending only on whether $X$ is a scheme or topological space). If we try to say that $G$-equivariant sheaves are those sheaves $\mathscr{F}$ for which $\mathscr{F}(gU) = \mathscr{F}(U)$ for all open subsets $U$ of $X$ (or étale opens if $X$ is a scheme) and for all $g \in G$, we quickly assert that either the group action $G \times X \to X$ stabilizes all open sets or the sheaf is trivial along all open orbits! Both of these conditions are far too restrictive[7] in practice, so we need to have a notion of what it means for a sheaf to be equivariant in more "external" terms.

In [57], Mumford and Fogarty discovered how to characterize equivariant sheaves on a space/scheme. The first trick is to note that if $X$ has an action of a group $G$, then there are two natural maps to consider from $G \times X$ to $X$: the action map $\alpha_X : G \times X \to X$ (this is the map $(g, x) \mapsto gx$) and the projection $\pi_2 : G \times X \to X$ (this is the map $(g, x) \mapsto x$). For a sheaf to be $G$-equivariant, $\mathscr{F}$ needs to be functionally "the same" upon being pulled back to $G \times X$ along either path, i.e., there needs to be an isomorphism $\theta : \alpha_X^* \mathscr{F} \xrightarrow{\cong} \pi_2^* \mathscr{F}$ of sheaves on $G \times X$ which . The reason why we ask for this is that for any point $(g, x) \in G \times X$, the map $\theta_{(g,x)}$ gives us an isomorphism

$$\mathscr{F}_{gx} = \mathscr{F}_{\alpha_X(g,x)} \cong \alpha_X^* \mathscr{F}_{(g,x)} \xrightarrow[\theta_{(g,x)}]{\cong} \pi_2^* \mathscr{F}_{(g,x)} \cong \mathscr{F}_{\pi_2(g,x)} = \mathscr{F}_x .$$

This implies that at least along the $G$-orbits $Gx$ in $X$, there are stalk-wise isomorphisms $\mathscr{F}_{gx} \cong \mathscr{F}_x$ for equivariant sheaves. We additionally ask that $\theta$ satisfy a cocycle condition upon being pulled back to $G \times G \times X$ which amounts to saying that $\theta$ is compatible with the group action of $G$ on $X$ when considering elements of the form $(gh)x = g(hx)$. We'll explore this more in detail in Definition **??**, but for now we'll leave this cocycle condition nebulous. What matters, however, is that this definition of equivariance lifts nicely to the cases when we replace $\mathbf{Ab}(X)$ with the categories $\mathbf{Per}(X)$ of perverse sheaves on $X$ or even $\mathbf{Loc}(X)$ of local systems (locally constant sheaves of Abelian groups) or $\mathscr{O}_X$-$\mathbf{Mod}$ of sheaves of modules.

[4] This is not a bug, however, as cohomology is a homotopy functor and so localizing $\mathbf{Ch}(\mathscr{A})$ at cohomology necessitates also quotienting at homotopy, which in turn requires us to instead consider homotopy limits and colimits. In fact, for those who are familiar with $\infty$-categories, $D(\mathscr{A})$ arises as the homotopy category of an $\infty$-category $\mathsf{D}(\mathscr{A})$ which records all the various homotopies that $D(\mathscr{A})$ quotients and hides. For details see [50].

[5] In the original 1983 version of *Faisceaux Pervers*, O. Gabber was supposed to be an author but declined to be listed as such. The first paragraph of the introduction opens with "Il avait d'abord été prévu que O. Gabber soit coauteur du présent article. Il a préféré s'en absetenir pour ne pas être coresponsable des erreurs ou impreécisions qui s'y trouvent." In essence, Gabber was supposed to be a coauthor but declined to be listed as such so as not to be repsonsible for the errors and imprecisions in [6]. However, in the 2018 reprint of [6], Gabber was finally listed as a coauthor and BBD has now become BBDG!

[6] Which, as the worst named object in mathematics, are neither perverse nor sheaves; they are complexes of sheaves within certain bounds.

[7] For an explicit example of why the first condition is too restrictive, let $X = \mathbb{C}$, let $G = \mathbb{C}^*$, and equip $X$ with the $G$-action given by $(\lambda, z) \mapsto \lambda z$ for $\lambda \in \mathbb{C}^*$ and $z \in \mathbb{C}$. Then if $D$ is the open unit disk centered at 0 and if $\lambda \in \mathbb{C}^*$ has $|\lambda| > 1$ we have $\lambda D \nsubseteq D$. Note that $X = \mathfrak{gl}(1, \mathbb{C})$ and $G = \mathrm{GL}(1, \mathbb{C})$, so this means the natural action of the Lie group $\mathrm{GL}(1)$ on $\mathfrak{gl}(1)$ cannot hope to satisfy the condition $gU \subseteq U$ for opens $U$!

**Definition 1.0.8.** Let $\mathscr{A}(X)$ be an Abelian category of sheaves (or perverse sheaves) on $X$ and assume that $X$ has a left $G$-action $\alpha_X : G \times X \to X$. We say that a $G$-equivariant object of $\mathscr{A}$ is a pair $(\mathscr{F}, \theta)$ where $\mathscr{F} \in \mathscr{A}_0$ and $\theta$ is an isomorphism

$$\alpha_X^* \mathscr{F} \xrightarrow[\theta]{\cong} \pi_2^* \mathscr{F}$$

in $\mathscr{A}(G \times X)$ satisfying the GIT cocycle condition (cf. Definition **??**).

**Definition 1.0.9.** Let $\mathscr{A}(X)$ be a category of sheaves on $X$. We define the category of $G$-equivariant objects of $\mathscr{A}(X)$, $\mathscr{A}_G(X)$, as follows:

- Objects: Pairs $(\mathscr{F}, \theta)$ satisfying Definition 1.0.8.
- Morphisms: A morphism $\rho : (\mathscr{F}, \theta) \to (\mathscr{G}, \sigma)$ is a morphism $\rho \in \mathscr{A}(X)(\mathscr{F}, \mathscr{G})$ for which the diagram

$$\begin{tikzcd} \alpha_X^* \mathscr{F} \arrow[r, "\theta"] \arrow[d, "\alpha_X^* \rho"'] & \pi_2^* \mathscr{F} \arrow[d, "\pi_2^* \rho"] \\ \alpha_X^* \mathscr{G} \arrow[r, "\sigma"'] & \pi_2^* \mathscr{G} \end{tikzcd}$$

  commutes in $G \times X$.
- Composition: As in $\mathscr{A}(X)$.
- Identities: As in $\mathscr{A}(X)$.

Definition 1.0.9 is nice because it is easy to work with, but the problem is that it does not lift to the derived categorical case. Each of the categories $\mathbf{Ab}(X)$ and $\mathbf{Per}(X)$ arise as (full) Abelian subcategories of $D(\mathbf{Ab}(X))$, so it is natural to ask if we can realize $\mathbf{Ab}_G(X)$ and $\mathbf{Per}_G(X)$ arise as a (full) subcategory of complexes $(A^\bullet, \theta)$ in $D(\mathbf{Ab}(X))$ together with cohomological functors $D_G(\mathbf{Ab}(X)) \to \mathbf{Ab}_G(X)$ and $D_G(\mathbf{Ab}(X)) \to \mathbf{Per}_G(X)$ which satisfy an analogue of Definition 1.0.8. To see that this is in fact **not** the case, we follow an example (cf. Example 1.0.10) that relies upon a single observation that we will not prove or go into depth: the derived category $D(\mathscr{A})$ is a triangulated category. This was originally proved by J.-L. Verdier in his thesis and seen again in [74], but ultimately means that $D(\mathscr{A})$ has a notion of cohomology long exact sequences that can be given in spite of the fact that $D(\mathscr{A})$ does not have kernels or cokernels. Our strategy in following [22] is to show that the category of pairs $(A^\bullet, \theta)$ with $A^\bullet$ cannot have such a structure. The facts we will use regarding triangulated categories are the of such a structure we will break is the following:

- Every triangulated category $\mathscr{T}$ is an additive category equipped with a suspension autoequivalence $[1] : \mathscr{T} \to \mathscr{T}$ and a collection of maps $X \to Y \to Z \to X[1]$ called distinguished triangles subject to various axioms (cf. [74], ).
- In a triangulated category $\mathscr{T}$ and if $[1] : \mathscr{T} \to \mathscr{T}$ is its suspension autoequivalence, then for any $f : A \to B$ in $\mathscr{T}_1$, there is a distinguised triangle $A \xrightarrow{f} B \to C \to A[1]$ in $\mathscr{T}$.
- In the derived category $D(\mathscr{A})$, a triangle $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A^\bullet[1]$ is distinguished if and only if there is a commuting diagram of the form

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, "\cong"', "\rho"] & B^\bullet \arrow[r, "g"] \arrow[d, "\cong"', "\varphi"] & C^\bullet \arrow[r, "h"] \arrow[d, "\psi"', "\cong"] & A^\bullet[1] \arrow[d, "\cong"', "\rho[1]"] \\ X^\bullet \arrow[r, "k"'] & Y^\bullet \arrow[r, "\alpha_k"'] & \mathrm{Cone}(k) \arrow[r, "p_k"'] & X^\bullet[1] \end{tikzcd}$$

in $D(\mathscr{A})$ where $X^\bullet[1]^n := X^{n+1}$ and $\partial_n^{X[1]} = -\partial_{n+1}^X$. Note that $\operatorname{Cone}(k)$ is the mapping cone of $k$. It has objects given by

$$\operatorname{Cone}(k)^n := X^{n+1} \oplus Y^n$$

for all $n \in \mathbb{Z}$ and its differential is given by

$$\partial_n^{\operatorname{Cone}(k)} := \langle -\partial_{n+1}^X \circ \pi_1, \partial_n^Y \circ \pi_2 - k_{n+1} \circ \pi_1 \rangle : X^{n+1} \oplus Y^n \to X^{n+2} \oplus Y^{n+1}.$$

In set-theoretic terms, this is

$$\partial_n^{\operatorname{Cone}(k)}(x,y) := \big( -\partial_{n+1}^X(x), \partial_n^Y(y) - k_{n+1}(x)\big).$$

The map $\alpha_k : Y^\bullet \to \operatorname{Cone}(k)$ is the morphism induced by $Y^n \to X^{n+1} \oplus Y^n$ given by $(\alpha_q)_n := [0, \mathrm{id}_{Y^n}] : Y^n$ and the map $p_k$ is given degree-wise $p_k = \pi_1 : X[1]^\bullet \oplus Y^\bullet \to X[1]^\bullet$.

**Example 1.0.10** ([22])**.** Let $X = \{*\}$ and consider the category $D(\mathbf{Ab}(*)) \cong D(\mathbf{Ab})$. Let $G$ be a group, let $A = \mathbb{Z}/p\,\mathbb{Z}$ for $p \in \mathbb{N}$ an integer prime. In this case a $G$-equivariant sheaf corresponds to an actual $G$-action on a complex $A^\bullet$ (by virtue of $X = \{*\}$ being trivial). Fix an additive $G$-action on $\mathbb{Z}/p\,\mathbb{Z}$ which does not lift to $\mathbb{Z}/p^2\,\mathbb{Z}$. The short exact sequence of Abelian groups

$$0 \longrightarrow \mathbb{Z}/p\,\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\,\mathbb{Z} \longrightarrow \mathbb{Z}/p\,\mathbb{Z} \longrightarrow 0$$

gives rise to the Bockstein morphism $\beta : X^\bullet \to X^\bullet[1]$ (the connecting map in the cohomology long exact sequence of the diagram above) and $\beta$ has mapping cone $\operatorname{Cone}(\beta) = \mathbb{Z}/p^2\,\mathbb{Z}$. Then we get the mapping cone sequence:

$$\mathbb{Z}/p\,\mathbb{Z} \xrightarrow{\beta} (\mathbb{Z}/p\,\mathbb{Z})[1] \longrightarrow (\mathbb{Z}/p^2\,\mathbb{Z}) \longrightarrow (\mathbb{Z}/p\,\mathbb{Z})[1]$$

in $D(\mathbf{Ab})$. However, because the action of $G$ on $\mathbb{Z}/p\,\mathbb{Z}$ does not lift to any action on $\mathbb{Z}/p^2\,\mathbb{Z}$, there can be no $G$-action on the mapping cone and hence $D_G(\mathbf{Ab})$ cannot be triangulated.

Because of the example above, we find that the equivariant derived category needs to have a more delicate definition. While there have been various different constructions (cf. [7] for a topological equivariant derived category and [1] for the variety-theoretic version of this approach, [18], [51], and [4] contain four different versions of the equivariant derived category on a variety ([7] and [1] are in essence the same version of the equivariant derived category) and see my thesis [77, Chapters 7 – 9] for a comparison of all these different categories), all of them have a common theme: we need to do some notion of equivariant descent through finer and finer resolutions of the group action $G \times X \to X$ in order to determine what $D_G(\mathscr{A})$ should be. In particular, I'd like to explain how this works for varieties, as if this is going to work for algebraic geometry and representation theory we need to understand how things work in the scheme-theoretic situation. We'll begin this journey in the next lecture with an introduction to locally ringed spaces and schemes.

# Chapter 2

# An Introduction to Locally Ringed Spaces

Today we'll learn about locally ringed spaces. These are a class of mathematical objects that contain (complex) analytic spaces, topological spaces, Riemann surfaces, schemes, formal schemes, and more. While our focus will be on the locally ringed spaces that come from algebraic geometry, it is worth seeing how various examples from analytic geometry appear in the theory. For this, however, we'll need to start getting to know sheaves on a topological space in more detail.[1] Let us proceed by recalling what it means to be a sheaf and a presheaf on a space $X$.

## 2.1 Generalities on Sheaves

**Definition 2.1.1.** Let $X$ be a topological space and let $\mathbf{Open}(X)$ denote the category of open sets of $X$. A presheaf (of sets) on $X$ is a functor $P : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ and a sheaf on $X$ is a presheaf $F : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ which satisfies the sheaf condition: for any open cover

$$U = \bigcup_{i \in I} U_i$$

in $\mathbf{Open}(X)$ the induced map $e = \langle F(U \supseteq U_i)\rangle_{i \in I}$ makes the diagram

$$F(U) \xrightarrow{e} \prod_{i \in I} F(U_i) \underset{\langle F(U_j \supseteq U_i \cap U_j)\rangle_{j \in I}}{\overset{\langle F(U_i \supseteq U_i \cap U_j)\rangle_{i \in I}}{\rightrightarrows}} \prod_{i,j \in I} F(U_i \cap U_j)$$

into an equalizer diagram in $\mathbf{Set}$. If each set $F(U)$ is a commutative ring (group, ring, $R$-module, $K$-algebras, etc.) then we say that $F$ is a sheaf of commutative rings (groups, rings, $R$-modules, $K$-algebras, etc.).

**Definition 2.1.2.** If $X$ is a topological space, we will write $\mathbf{Shv}(X)$ for the category of sheaves on $X$. This is the category defined by:

- Objects: Sheaves $F$ on $X$.
- Morphisms: Natural transformations $\alpha : F \Rightarrow G$.
- Composition: As in $[\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]$.
- Identities: As in $[\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]$.

If we need to be explicit, we'll write $\mathbf{Shv}(X, \mathbf{Open}(X))$ to denote this category.

[1] As opposed to the more informal approach we took in the introduction and the total black box apporach we took on Day One.

**Remark 2.1.3.** It is nice to have some additional explanation of the sheaf condition. The sheaf condition, which asks that for any open cover $U = \bigcup_{i \in I} U_i$ the diagram

$$F(U) \xrightarrow{e} \prod_{i \in I} F(U_i) \underset{\langle F(U_j \supseteq U_i \cap U_j)\rangle_{j \in I}}{\overset{\langle F(U_i \supseteq U_i \cap U_j)\rangle_{i \in I}}{\rightrightarrows}} \prod_{i,j \in I} F(U_i \cap U_j)$$

is an equalizer in **Set** says that the functor[2] $F$ gives local information about the global structure of $X$, i.e., it satisfies a descent condition along the topology $\mathbf{Open}(X)$. This means the following conditions (which, in older or less categorically minded references are often given as a definition of a sheaf; cf. [21] and [33], for instance) get satisfied for all open covers $U = \bigcup_{i \in I} U_i$ in $\mathbf{Open}(X)$:

1. If $s, t \in F(U)$ such that for all $i \in I$ $F(U \supseteq U_i)(s) = F(U \supseteq U_j)(t)$ then $s = t$. This is equivalent to asking that the map $e : F(U) \to \prod_{i \in I} F(U_i)$ is monic.
2. If there is an element $(s_i)_{i \in I} \in \prod_{i \in I} F(U_i)$ such that
$$F(U_i \supseteq U_i \cap U_j)(s_i) = F(U_j \supseteq U_i \cap U_j)(s_j)$$
for all $i, j \in I$ then there exists an $s \in F(U)$ for which $F(U \supseteq U_i) = s_i$ for all $i \in I$. This asks that every map equalizing the parallel arrows in the diagram factors through $F(U)$.

Putting these together gives the equivalence of the sheaf condition in Definition 2.1.1 with the classical construction.

**Remark 2.1.4.**

Describe the espace étalé of $\mathscr{F}$ and describe how every sheaf "is" a sheaf of sections.

**Example 2.1.5.** Let $X = \{0, 1\}$ equipped with the discrete topology $\mathbf{Open}(X) = \{\varnothing, \{0\}, \{1\}, X\}$. Then the functor $P : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ given by

$$P(U) := \mathbb{Z}, U \in \mathbf{Open}(X)_0; \quad P(f) = \mathrm{id}_{\mathbb{Z}}, f \in \mathbf{Open}(X)_1;$$

induces a presheaf which is not a sheaf. The functor $F : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$F(U) := \begin{cases} \mathbb{Z} \times \mathbb{Z} & \text{if}\, U = X; \\ \mathbb{Z} & \text{if}\, U = \{0\}, \{1\}; \\ \{*\} & \text{if}\, U = \varnothing; \end{cases}$$

on objects and given by

$$F(f) := \begin{cases} \pi_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} & \text{if}\, f = X \supseteq \{0\}; \\ \pi_2 : Z \times \mathbb{Z} \to \mathbb{Z} & \text{if}\, f = X \supseteq \{1\}; \\ !_{\mathbb{Z}} : \mathbb{Z} \to \{*\} & \text{if}\, f = \{0\} \supseteq \varnothing, \{1\} \supseteq \varnothing \end{cases}$$

on morphisms induces a sheaf on $X$. More generally, if $X$ is a topological space and $S$ is a non-empty set then the functor $F : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ given by

$$F(U) = \prod_{x \in \pi_0(U)} S$$

on objects and defined by, for $V \subseteq U$ open,

$$\prod_{x \in \pi_0(U)} S \to \prod_{y \in \pi_0(V)} S$$

by projecting away the components of the product which lie in $U$ but not in $V$ is a sheaf on $X$.

[2] We use the letter "$F$" (or more frequently the scripty $\mathscr{F}$) to denote sheaves from the French word "Faisceaux" for sheaf. Because this is the terminology Grothendieck and friends use, we keep it in place.

**Example 2.1.6.** Let $S = \{s, \eta\}$ be the Siepinski space, i.e., equip $S$ with the topology $\mathbf{Open}(S) := \{\varnothing, \{\eta\}, S\}$. Define the functor $\mathcal{O}_{\mathbb{Z}_p}$ by

$$\mathcal{O}_{\mathbb{Z}_p}(U) := \begin{cases} \mathbb{Z}_p & \text{if } U = S; \\ \mathbb{Q}_p & \text{if } U = \{\eta\}; \\ 0 & \text{if } U = \varnothing \end{cases}$$

on opens and

$$\mathcal{O}_{\mathbb{Z}_p}(U \supseteq V) := \begin{cases} \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p & \text{if } U = S, V = \{\eta\}; \\ !_{\mathcal{O}_{\mathbb{Z}_p}(U)} : \mathcal{O}_{\mathbb{Z}_p}(U) \to 0 & \text{if } U \in \mathbf{Open}(S), V = \varnothing \end{cases}$$

on inclusions, where $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are the $p$-adic integers and numbers, respectively. Then $\mathcal{O}_{\mathbb{Z}_p}$ induces a sheaf on $S$. More generally, if $A$ is a discrete valuation ring with fraction field $K$, then

$$\mathcal{O}_A := \begin{cases} A & \text{if } U = S; \\ K & \text{if } U = \{\eta\}; \\ 0 & \text{if } U = \varnothing; \end{cases}$$

on opens and

$$\mathcal{O}_A(U \supseteq V) := \begin{cases} A \hookrightarrow K & \text{if } U = S, V = \{\eta\}; \\ !_{\mathcal{O}_A(U)} : \mathcal{O}_A(U) \to 0 & \text{if } U \in \mathbf{Open}(S), V = \varnothing \end{cases}$$

on inclusions induces a sheaf on $S$. While this is actually true for any integral domain $A$, we chose discrete valuations rings because there is then a homeomorphism of $S$ with the Zariski topology on the spectrum of $A$. We'll see later that this is an example of our first (affine) scheme!

**Example 2.1.7.** Let $X$ be a Riemann surface (a connected complex manifold of complex dimension 1). Define the functor $\mathcal{O} : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ of local homolomorphic functions on $X$ by asserting

$$\mathcal{O}(U) := \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}$$

and defining $\mathcal{O}(U \supseteq V)$ via restriction, i.e., if $V \subseteq U$ we define $\mathcal{O}(U \supseteq V) : \mathcal{O}(U) \to \mathcal{O}(V)$ by $f \mapsto f|_V$. Then $\mathcal{O}$ is the sheaf of holomorphic functions on $X$.

Motivated by the last example, we can discuss the "ultra-local[3]" behaviour of any sheaf as it accumulates to a point $x \in X$. In the case of a complex manifold $X$ and the sheaf $\mathcal{O}$ of holomorphic functions, the idea is that as we get closer and closer to $x \in X$ by evaluating $\mathcal{O}(U)$ at smaller and smaller opens $U$ which contain $x$, we get the ring $\mathcal{O}_x$ of germs of functions at $x \in X$ (which describe the colimit of the rings $\mathcal{O}(U)$ as we accumulate to $x$). The ring $\mathcal{O}_x$ is a local ring with maximal ideal $\mathfrak{m}_x$ given by functions whose germs vanish at $x$. This definition motivates the stalk of a presheaf $\mathscr{F}$ at a point $x$, which is of fundamental importance to us in defining locally ringed spaces and hence schemes.

**Definition 2.1.8.** Let $X$ be a topological space and $P : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf of sets on $X$. The stalk of $P$ at $x \in X$, $P_x$, is defined as the colimit of the sets $P(U)$ as the open sets accumulate to $x$, i.e.,

$$P_x := \operatorname*{colim}_{x \in U} P(U) = \varinjlim_{x \in U} P(U).$$

We now present a theorem which shows the importance of stalks in the sheaf theory of topological spaces. They allow us to show that two sheaves are isomorphic if and only if they have a morphism between each other and each stalk map is an isomorphism.[4]

[3] I just mean the "at the limit of local behaviour" by this term.

[4] I like to think of this as "stalking" through the sheaf to find an obstruction to whether or not the sheaf map is an isomorphism.

**Proposition 2.1.9.** *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space $X$. Then $\varphi$ is an isomorphism (respectively monomorphism, epimorphism) if and only if for all $x \in X$ the induced map*

$$\varphi_x : \mathscr{F}_x \to \mathscr{G}_x \,.$$

*is an isomorphism (repsectively monomorphism, epimorphism) of stalks.*

**Proposition 2.1.10.** *Let $\mathscr{F}$ be a sheaf of sets on a topological space $X$, let $U \subseteq X$ be open, and let $\alpha_{U,x} : \mathscr{F}(U) \to \mathscr{F}_x$ be the colimit map for each $x \in U$. Then the map*

$$\mathscr{F}(U) \xrightarrow{\langle \alpha_{U,x} \rangle_{x \in U}} \prod_{x \in U} \mathscr{F}_x$$

*is injective.*

*Proof.* Fix $s, t \in \mathscr{F}(U)$ for which $\alpha_{U,x}(s) = \alpha_{U,x}(t)$ for all $x \in U$. Then for every $x \in U$ there is an open set $U_x \subseteq U$ with $x \in X$ for which $\mathscr{F}(U \supseteq U_x)(s) = \mathscr{F}(U \supseteq U_x)(t)$. However, because

$$U = \bigcup_{x \in U} U_x$$

it follows from the sheaf condition that $s = t$, as desired. $\square$

**Remark 2.1.11.** It can be shown[5] that every sheaf category $\mathbf{Shv}(X)$ arises as a certain reflective subcategory of the presheaf topos[6] $[\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]$. Explicitly this means that the inclusion functor $i : \mathbf{Shv}(X) \to [\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]$ has a left adjoint $(-)^{++} : [\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]$ as in the diagram:

$$\begin{tikzcd} \mathbf{Shv}(X) \arrow[rr, bend right, "i_X"'] \arrow[rr, phantom, "\perp"] & & {[\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]} \arrow[ll, bend right, "(-)^{++}"'] \end{tikzcd}$$

The functor $(-)^{++}$ is called the associated sheaf functor (or the sheafification functor) and if $P$ is a presheaf then the sheaf $P^{++}$ is called the sheafification of $P$. For any presheaf $P$ with sheafification $P^{++}$, it follows from the fact that left adjoints preserve colimits that

$$P_x \cong P_x^{++}.$$

Note also that the sheafififcation functor has the following universal property: for any presheaf $P$ and any sheaf $\mathscr{G}$ on $X$ with a morphism $P \to \mathscr{G}$ there is a unique morphism of sheaves $P^{++} \to \mathscr{G}$ making the diagram

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[dr] & P^{++} \arrow[d, dashed, "\exists!"] \\ & \mathscr{G} \end{tikzcd}$$

commute, where $\eta$ is the unit of adjunction. For a complete description of sheafification, see Appendix A.2.

[5] Not in these notes; it's beyond our scope. For details see [54, Section 3.5] or [9] for the complete and careful construction in modern language. In Appendix A we simply show that every sheaf topos $\mathbf{Shv}(\mathscr{C}, J)$ is a reflective subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, but not the converse.

[6] If you've never met toposes before, please don't worry (although you've likely met them but just not yet been introduced). For the most part you can get away with the word "category" instead as far as these notes are concerned. If you want to meet toposes, however, see Appendix A and Definition A.2.26 for a fast introduction to sheaf toposes or [54] for a gentle introduction to topos theory. A more robust introduction can be found in [36] and the encyclopedic references are, of course, the Elephant: [37], [38].

**Remark 2.1.12.** While I'll try my best not to refer to it too much without an explicit reference tag, I'll likely occasionally refer to the sheafification of a presheaf (cf. Remark 2.1.11 and Appendix A.2) without additional comment.

Before jumping into ringed spaces, we'll need to know about the inverse images and direct images[7] of sheaves under a continuous map $f : X \to Y$. To see how to define this, let $f : X \to Y$ be a continuous function. We want to define functors $f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$ and $f^{-1} : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$ which should on one hand push a sheaf on $X$ forward to $Y$ and on the other hand pull a sheaf back from $Y$ to $X$ along the image of $f$.

The easiest of these functors to define is the pushforward $f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$. To turn any sheaf $\mathscr{F}$ on $X$ into a sheaf on $Y$ by using the continuous map $f : X \to Y$ we simply recall that because $f$ is continuous, for any open set $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open as well. Thus we define the presheaf $f_*\mathscr{F} : \mathbf{Open}(Y)^{\mathrm{op}} \to \mathbf{Set}$ via the assignment

$$f_*\mathscr{F}(-) := \mathscr{F}\left(f^{-1}(-)\right).$$

Similarly, if $\alpha : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the direct image of the morphism $\alpha$ by defining

$$(f_*\alpha)_V := \alpha_{f^{-1}(V)}$$

for all opens $V \subseteq Y$.

**Proposition 2.1.13.** *If $f : X \to Y$ is a continuous map and $\mathscr{F}$ is a sheaf on $X$, the functor $f_*\mathscr{F}$ is a sheaf on $Y$. In particular, $f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$ is a functor.*

*Proof.* It is routine to check that the definition of $f_*\mathscr{F}$ is indeed a functor by using that $\mathscr{F}$ is a functor, so we omit that calculation. Instead, we must show that that $f_*\mathscr{F}$ is a sheaf on $Y$. For this assume that we have a cover $V = \bigcup_{i\in I} V_i$ of opens in $Y$ and consider the commuting diagram:

$$f_*\mathscr{F}(V) \xrightarrow{e} \prod_{i\in I} f_*\mathscr{F}(V_i) \underset{\langle f_*\mathscr{F}(V_j \supseteq V_i\cap V_j)\rangle_{j\in I}}{\overset{\langle f_*\mathscr{F}(V_i \supseteq V_i\cap V_j)\rangle_{i\in I}}{\rightrightarrows}} \prod_{i,j\in I} f_*\mathscr{F}(V_i \cap V_j)$$

Using the definition of $f_*\mathscr{F}$ and the fact that the pre-image function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves arbitrary unions, finite intersections, and inclusions we see that the above diagram is equal to the diagram

$$\mathscr{F}\left(f^{-1}(V)\right) \xrightarrow{e} \prod_{i\in I} \mathscr{F}\left(f^{-1}(V_i)\right) \underset{\left\langle \mathscr{F}\left(f^{-1}(V_j)\supseteq f^{-1}(V_i)\cap f^{-1}(V_j)\right)\right\rangle_{j\in I}}{\overset{\left\langle \mathscr{F}\left(f^{-1}(V_i)\supseteq f^{-1}(V_i)\cap f^{-1}(V_j)\right)\right\rangle_{i\in I}}{\rightrightarrows}} \prod_{i,j\in I} \mathscr{F}\left(f^{-1}(V_i) \cap f^{-1}(V_j)\right)$$

in $\mathbf{Set}$. However, because each of the sets $f^{-1}(V), f^{-1}(V_i)$, and $f^{-1}(V_j)$ are open in $X$ it follows that we have an open cover of $f^{-1}(V) = \bigcup_{i\in I} f^{-1}(V_i)$. Thus because $\mathscr{F}$ is a sheaf we have that the diagram above is an equalizer which shows that $f_*\mathscr{F}$ is a sheaf on $Y$.

Finally the verification that if $\alpha : \mathscr{F} \to \mathscr{G}$ is a natural transformation of sheaves on $X$ then $f_*\alpha : f_*\mathscr{F} \to f_*\mathscr{G}$ is a natural transformation of sheaves on $Y$ is straightforward given the definition of $f_*\alpha$. That this is functorial is immediate from the fact that for all $V \subseteq Y$ open we have

$$f_*\beta_V \circ f_*\alpha_V = \beta_{f^{-1}(V)} \circ \alpha_{f^{-1}(V)} = (\beta\circ\alpha)_{f^{-1}(V)} = f_*(\beta\circ\alpha)_{f^{-1}(V)}$$

for any natural transformations of sheaves $\alpha : \mathscr{F} \to \mathscr{G}, \beta : \mathscr{G} \to \mathscr{H}$. $\square$

[7] These functors are often called the pullback/pushforward of sheaves along $f$, in analogy to the pullback/pushforward of differential forms along smooth maps between smooth manifolds. I tend to prefer that language, but the inverse image/direct image is used more frequently (at least in my experience). I'll try to consistently use this language, but I apologize if the pushforward pushes its way into my writing.

Sadly the inverse image functor $f^{-1} : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$ is more difficult to define, as we have to use sheafification (cf. Remark 2.1.11). To define this functor on presheaves, we first note that it would be nice if we could try a trick like some sort of co-version of the pushforward $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$, i.e., some sort of $f^{-1}P(U) = P(f(U))$. However, this does not work![8] As such, we need to define the pullback of $P$ along $f$ to be essentially the best approximation of $f(U)$ by the opens it contains. As such, our inverse image on presheaves

$$f^{-1}_{\text{pre}} : [\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}] \to [\mathbf{Open}(X)^{\text{op}}, \mathbf{Set}]$$

by asserting that for each $U \subseteq X$ open and each presheaf $P$ on $Y$,

$$f^{-1}_{\text{pre}}P(U) := \operatorname*{colim}_{\substack{f(U)\subseteq V \\ V \in \mathbf{Open}(Y)_0}} P(V)$$

and similarly for morphisms $\alpha : P \Rightarrow Q$:

$$f^{-1}_{\text{pre}}\alpha_U := \operatorname*{colim}_{\substack{f(U)\subseteq V \\ V \in \mathbf{Open}(Y)_0}} \alpha_V.$$

**Definition 2.1.14.** Let $f : X \to Y$ be a continuous map. The inverse image functor $f^{-1} : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$ is defined as the sheafification (cf. Remark 2.1.11) of $f^{-1}_{\text{pre}}$, i.e., $f^{-1} : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$ is defined via the diagram:

$$\begin{tikzcd}
\mathbf{Shv}(Y) \arrow[r, "f^{-1}"] \arrow[d, "i_Y"'] & \mathbf{Shv}(X) \\
{[\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}]} \arrow[r, "f^{-1}_{\text{pre}}"'] & {[\mathbf{Open}(X)^{\text{op}}, \mathbf{Set}]} \arrow[u, "(-)^{++}_X"']
\end{tikzcd}$$

We now need to know two small lemmas before we show how the inverse image and direct image functors are related: $f^{-1}$ is left adjoint to $f_*$. The lemmas below just states that the direct image $f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$ arises as the restriction of a functor $f_*^{\text{pre}} : [\mathbf{Open}(X)^{\text{op}}, \mathbf{Set}] \to [\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}]$ and that $f^{-1}_{\text{pre}} \dashv f_*^{\text{pre}}$. I've included these so that we have a slick proof of the fact that $f^{-1} \dashv f_*$ that is not present in most introductory accounts of algebraic geometry, but the reader who just wants to get to the sheafy and scheme-y goodness that follows can safely ignore the lemmas and just use the theorem (cf. Theorem 2.1.17) as necessary.

**Lemma 2.1.15.** *Let $f : X \to Y$ be a continuous map. There is a functor $f_*^{\text{pre}} : [\mathbf{Open}(X)^{\text{op}}, \mathbf{Set}] \to [\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}]$ defined by, for any presheaf $P$ on $X$ and open $V \subseteq Y$,*

$$f_*^{\text{pre}}P(V) := P(f^{-1}(V))$$

*and similarly for morphisms. In particular, for any sheaf $\mathscr{F}$ on $X$, $f_*^{\text{pre}}\mathscr{F} = f_*\mathscr{F}$ and the diagram*

$$\begin{tikzcd}
{[\mathbf{Open}(X)^{\text{op}}, \mathbf{Set}]} \arrow[r, "f_*^{\text{pre}}"] & {[\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}]} \\
\mathbf{Shv}(X) \arrow[u, "i_X"] \arrow[r, "f_*"'] & \mathbf{Shv}(Y) \arrow[u, "i_Y"']
\end{tikzcd}$$

*commutes.*

[8] Asking this to work asks for every continuous map to be open, which is far too restrictive and unnatural in practice.

*Proof.* That $f_*^{\mathrm{pre}}$ is a functor follows from Proposition 2.1.13, so we only need to verify the second and third claims. First, that $f_*^{\mathrm{pre}}\mathscr{F} = f_*\mathscr{F}$ is immediate from the definition (the only difference between $f_*$ and $f_*^{\mathrm{pre}}$ is domain and codomain, not how they act). For the second claim we just note that for any sheaf $\mathscr{F}$ on $X$,

$$f_*^{\mathrm{pre}}\left(i_X(\mathscr{F})\right) = f_*^{\mathrm{pre}}\mathscr{F} = f_*\mathscr{F} = i_Y\left(f_*\mathscr{F}\right).$$

Note that the cancellation of $i_X$ follows from the fact that $i_X$ is an inclusion functor and the introduction of $i_Y$ follows from the fact that $i_Y$ is an inclusion functor as well and that $f_*\mathscr{F}$ is a sheaf on $Y$ by Proposition 2.1.13. The verification that the diagram commutes on morphisms follows ismilarly and is omitted. $\square$

**Lemma 2.1.16.** *There is an adjunction, for any continuous $f : X \to Y$,*

$$\begin{tikzcd} {[\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]} \arrow[rr, bend right, "f_*^{\mathrm{pre}}"'] \arrow[rr, phantom, "\perp"] & & {[\mathbf{Open}(Y)^{\mathrm{op}}, \mathbf{Set}]} \arrow[ll, bend right, "f_{\mathrm{pre}}^{-1}"'] \end{tikzcd}$$

*Proof.* Let $P \in [\mathbf{Open}(Y)^{\mathrm{op}}, \mathbf{Set}]_0$ and let $Q \in [\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]_0$. We must show that there is a natural isomorphism

$$[\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]\left(f_{\mathrm{pre}}^{-1}P, Q\right) \cong [\mathbf{Open}(Y)^{\mathrm{op}}, \mathbf{Set}]\left(P, f_*^{\mathrm{pre}}Q\right).$$

To do this fix a morphism $\alpha : f_{\mathrm{pre}}^{-1}P \to Q$ in the presheaf topos $[\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]$. Such a morphism is described by commuting diagrams

$$\begin{tikzcd} f_{\mathrm{pre}}^{-1}P(U) \arrow[r, "\alpha_U"] \arrow[d, "f_{\mathrm{pre}}^{-1}Q(U\supseteq U')"'] & Q(U) \arrow[d, "Q(U\supseteq U')"] \\ f_{\mathrm{pre}}^{-1}P(U') \arrow[r, "\alpha_{U'}"'] & Q(U') \end{tikzcd}$$

for all inclusions of opens $U' \subseteq U$ in $\mathbf{Open}(X)$. However, since

$$f_{\mathrm{pre}}^{-1}P(U) = \operatorname*{colim}_{\substack{V\supseteq f(U) \\ V\subseteq Y \text{ open}}} P(V)$$

and

$$f_{\mathrm{pre}}^{-1}P(U') = \operatorname*{colim}_{\substack{V'\supseteq f(U') \\ V'\subseteq Y \text{ open}}} P(V')$$

this is equivalent to asking that we have morphisms $\alpha_V^\sharp : P(V) \to Q(V \times_Y X)$ and $P(V') \to Q(V' \times X)$ which fit into the commuting diagram

$$\begin{tikzcd} P(V) \arrow[r, "\alpha_V^\sharp"] \arrow[d, "P(V\supseteq V')"'] & Q(V \times_Y X) \arrow[d, "Q(V\times_Y X\supseteq V'\times_Y X)"] \\ P(V') \arrow[r, "\alpha_{V'}^\sharp"'] & Q(V' \times_Y X) \end{tikzcd}$$

for all opens $V' \subseteq V \subseteq Y$; note that $V \times_Y X = f^{-1}(V)$ and $V' \times_Y X = f^{-1}(V')$ so both are open subspaces of $X$ with $V \times_Y X \supseteq V' \times_Y X$, as implied in the diagram above. Because of this observation, we can rewrite the diagram above as a diagram

$$\begin{tikzcd}
P(V) \arrow[r, "\alpha^\sharp_V"] \arrow[d, "P(V\supseteq V')"'] & Q(f^{-1}(V)) \arrow[d, "Q(f^{-1}(V)\supseteq f^{-1}(V'))"] \\
P(V') \arrow[r, "\alpha^\sharp_{V'}"'] & Q(f^{-1}(V'))
\end{tikzcd}$$

which in turn describes the object assignments of a natural transformation $\alpha^\sharp : P \to f_*^{\text{pre}}Q$, and hence a morphism, in the presheaf topos $[\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}]$. This gives our function

$$\theta : [\mathbf{Open}(X)^{\text{op}}, \mathbf{Set}](f_{\text{pre}}^{-1}P, Q) \to [\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}](P, f_*^{\text{pre}}Q)$$

defined by $\alpha \mapsto \alpha^\sharp$. Running this same argument in reverse allows us to determine uniquely from a morphism $\beta : P \to f_*^{\text{pre}}Q$ a morphism $\beta^\flat : f_{\text{pre}}^{-1}P \to Q$ which defines a function

$$\sigma : [\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}](P, f_*^{\text{pre}}Q) \to [\mathbf{Open}(X)^{\text{op}}, \mathbf{Set}](f_{\text{pre}}^{-1}P, Q)$$

given by $\beta \mapsto \beta^\flat$. These maps are by construction natural inverses, i.e., $(\alpha^\sharp)^\flat = \alpha$ and $(\beta^\flat)^\sharp = \beta$. This establishes the adjunction. $\square$

We finally prove the push/pull (inverse image/direct image) adjunction.

**Theorem 2.1.17** (Often left as an exercise; cf. [33, Exercise II.1.18] and [73, Exercise 2.7.B])**.** *There is an adjuction, for any continuous map $f : X \to Y$:*

$$\begin{tikzcd}
\mathbf{Shv}(X) \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathbf{Shv}(Y) \arrow[l, bend right, "f^{-1}"']
\end{tikzcd}$$

*Proof.* We verify this by calculating that for any sheaf $\mathscr{F}$ on $Y$ and for any sheaf $\mathscr{G}$ on $X$,

$$\begin{aligned}
\mathbf{Shv}(X)\left(f^{-1}\mathscr{F}, \mathscr{G}\right) = \mathbf{Shv}(X)\left(\left(f_{\text{pre}}^{-1}(i_Y(\mathscr{F}))\right)_X^{++}, \mathscr{G}\right) &\cong [\mathbf{Open}(X)^{\text{op}}, \mathbf{Set}]\left(f_{\text{pre}}^{-1}(i_Y\mathscr{F}), \mathscr{G}\right) \\
&\cong [\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}]\left(i_Y\mathscr{F}, f_*^{\text{pre}}\mathscr{G}\right) = \mathbf{Shv}(Y)\left(\mathscr{F}, f_*\mathscr{G}\right).
\end{aligned}$$

Note that the first isomorphism holds from the $(-)_X^{++} \dashv i_X$ adjunction of Remark 2.1.11, the second isomorphism follows from Lemma 2.1.16, and the final equality follows from Lemma 2.1.15, Proposition 2.1.13, and the fact that as a reflective subcategory of $[\mathbf{Open}(Y)^{\text{op}}, \mathbf{Set}]$ (cf. Corollary A.2.15), $\mathbf{Shv}(Y)$ is a full subcategory. $\square$

We can finally define ringed spaces and their morphisms. After seeing ringed spaces and the category of ringed spaces, we'll also discuss locally ringed spaces before closing today's lecture with some examples.

**Definition 2.1.18.** A ringed space is a pair $X = (|X|, \mathcal{O}_X)$ where $|X|$ is a topological space and where $\mathcal{O}_X$ is a sheaf of rings on $|X|$. A morphism of ringed spaces $f : (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$ consists of a pair $f = (|f|, f^\sharp)$ where $|f| : |X| \to |Y|$ is a continuous morphism of spaces and $f^\sharp : \mathcal{O}_Y \to |f|_*\mathcal{O}_X$ is a morphism of sheaves of rings (often called the comorphism of $f$).

To define the category $\mathbf{RS}$ of ringed spaces, we need one quick observation about how we can compose direct image functors so that we can show how to build the composite $(g \circ f)^\sharp : \mathcal{O}_Z \to (g \circ f)_*\mathcal{O}_X$. Let $f : X \to Y$ and let $g : Y \to Z$ be continuous maps and let $W \subseteq Z$ be an open set. We then calculate that if $\mathscr{F}$ is any sheaf on $X$,

$$g_*\left(f_*\mathscr{F}\right)(W) = f_*\mathscr{F}\left(g^{-1}(W)\right) = \mathscr{F}\left(f^{-1}\big(g^{-1}(W)\big)\right) = \mathscr{F}\left((g \circ f)^{-1}(W)\right) = \big((g \circ f)_*\mathscr{F}\big)(W)$$

so there is an equality of functors $g_* \circ f_* = (g \circ f)_*$. Now, if we have two comorphisms $f^\sharp : \mathscr{O}_Y \to |f|_* \mathscr{O}_X$ and $g^\sharp : \mathscr{O}_Z \to |g|_* \mathscr{O}_Y$ then we can produce a morphism $\mathscr{O}_Z \to (|g| \circ |f|)_* \mathscr{O}_X$ by applying $|g|_*$ to $f^\sharp$ and pre-composing by $g^\sharp$:

$$\mathscr{O}_Z \xrightarrow{g^\sharp} |g|_* \mathscr{O}_Y \xrightarrow{|g|_*(f^\sharp)} |g|_* (|f|_* \mathscr{O}_X) = (|g| \circ |f|)_* \mathscr{O}_X$$

We use this to define our composition of morphisms in the (yet to be defined) category **RS**.

**Definition 2.1.19.** The category **RS** of ringed spaces is defined as follows:

- Objects: Ringed spaces $X$.
- Morphisms: Maps $f = (|f|, f^\sharp) : X \to Y$ of ringed spaces.
- Composition: The composition of maps $f = (|f|, f^\sharp) : X \to Y$ and $g = (|g|, g^\sharp) : Y \to Z$ then the composite $g \circ f$ is defined by
$$g \circ f := (|g| \circ |f|, |g|_*(f^\sharp) \circ g^\sharp).$$
- Identities: The identity on $X$ is $(\mathrm{id}_{|X|}, \mathrm{id}_{\mathscr{O}_X})$.

We will not prove this is a category, but it is a nice exercise in checking that the direct image functors play well with associativity and using the various properties of the preimage. We instead now define an important category of sheaves for ringed spaces: the category of modules for the sheaf of rings $\mathscr{O}_X$. This category arises as the category of modules for the ring object $\mathscr{O}_X$ in the topos of sheaves $\mathbf{Shv}(|X|)$, and will be very important later in these notes when we discuss quasi-coherent sheaves on a scheme.

**Definition 2.1.20.** Let $(|X|, \mathscr{O}_X)$ be a ringed space. We define the category $\mathscr{O}_X$-**Mod** of sheaves of $\mathscr{O}_X$-modules as follows:

- Objects: Sheaves $\mathscr{M}$ of Abelian groups $\mathscr{F}$ on $|X|$ such that for every $U \in \mathbf{Open}(X)_0$, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module and for any inclusion of opens $V \subseteq U$, the map $\mathscr{F}(U) \to \mathscr{F}(V)$ is a morphism of $\mathscr{O}_X(V)$-modules when we equip $\mathscr{F}(U)$ with an $\mathscr{O}_X(V)$-module structure via extension of scalars along the map $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$.
- Morphisms: Sheaf maps $\varphi : \mathscr{M} \to \mathscr{N}$ for which each $\varphi_U : \mathscr{M}(U) \to \mathscr{N}(U)$ is a morphism of $\mathscr{O}_X(U)$-modules for all $U \subseteq |X|$ open.
- Composition and Identities: As in $\mathbf{Shv}(|X|)$.

The category of $\mathscr{O}_X$-modules also has a tensor product, which allows us to give $\mathscr{O}_X$-**Mod** the structure of a symmetric monoidal category in the same way that the tensor product on $A$-**Mod** is a symmetric monoidal category on. We describe this below and describe some of its properties before proceeding.

**Definition 2.1.21.** Let $(|X|, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves in $\mathscr{O}_X$-**Mod**. Then the tensor product of $\mathscr{F}$ and $\mathscr{G}$, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is defined to be the sheafification of the presheaf induced by the assignment
$$U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U).$$
That is, if $\mathscr{F} \otimes_{\mathscr{O}_X}^{\text{pre}} \mathscr{G}$ is the presheaf $(\mathscr{F} \otimes_{\mathscr{O}_X}^{\text{pre}} \mathscr{G})(U :) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$ then

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} := (\mathscr{F} \otimes_{\mathscr{O}_X}^{\text{pre}} \mathscr{G})^{++}.$$

We give two quick facts about the tensor product of sheaves of modules before moving on.

**Proposition 2.1.22.** *Let $(X, \mathscr{O}_X)$ be a ringed space. Then the tensor product of $\mathscr{O}_X$-modules is associative, i.e., if $\mathscr{F}, \mathscr{G}, \mathscr{H} \in \mathscr{O}_X$-$\mathbf{Mod}_0$ then there is a natural isomorphism*

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{H} \cong \mathscr{F} \otimes_{\otimes_X} (\mathscr{G} \otimes_X \mathscr{H}).$$

*Proof.* This follows from the fact that the tensor presheaves satisfy

$$\left((\mathscr{F} \otimes_{\mathcal{O}_X}^{\text{pre}} \mathscr{G}) \otimes_{\mathcal{O}_X}^{\text{pre}} \mathscr{H}\right)(U) = (\mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)) \otimes_{\mathcal{O}_X(U)} \mathscr{H}(U)$$

for all $U \in \mathbf{Open}(X)_0$, so sheafifying gives the result. $\square$

**Proposition 2.1.23.** *If $X$ is a ringed space and $\mathscr{F} \in \mathcal{O}_X\text{-}\mathbf{Mod}_0$ then there are natural isomorphisms*

$$\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathscr{F} \cong \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathscr{F}.$$

*Proof.* This follows immediately from the natural isomorphisms of presheaves

$$(\mathscr{F} \otimes_{\mathcal{O}_X}^{\text{pre}} \mathcal{O}_X)(U) \cong \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \cong \mathscr{F}(U) \cong \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U) = (\mathcal{O}_X \otimes_{\mathcal{O}_X}^{\text{pre}} \mathscr{F})(U)$$

for all $U \in \mathbf{Open}(X)_0$. $\square$

We now will define locally ringed spaces, which allow us to capture many of the fundamental geometric properties of complex manifolds equipped with their sheaves of holomorphic functions. Before this, however, we'll need one definition and one structural lemma regarding how morphisms of ringed spaces interact with stalks.

**Definition 2.1.24.** A commutative ring with identity $A$ is a local ring if it has a unique maximal ideal $\mathfrak{m}$.

**Remark 2.1.25.** While it is possible to have noncommutative local rings, in this set of notes we will only pay attention to the commutative (unital) case so as to not have to add the word "commutative" to every statement we make. That being said, we'll still likely point out the rings we're considering are commutative unital rings from time to time.

**Definition 2.1.26** ([33, Page 74])**.** A local morphism of local rings $A \to B$ is a ring homomorphism $\varphi : A \to B$ for which $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

**Lemma 2.1.27.** *Let $f : X \to Y$ be a continuous function and let $\mathscr{F}$ be a sheaf on $X$ and $\mathscr{G}$ a sheaf on $Y$. Assume that there is a morphism of $Y$-sheaves $\varphi : \mathscr{G} \to f_*\mathscr{F}$. Then for every $x \in X$, there is a morphism of stalks*

$$\varphi_{f(x)} : \mathscr{G}_{f(x)} \to \mathscr{F}_x.$$

*Proof.* Begin by fixing some $V \subseteq Y$ open and note that the morphism $\varphi : \mathscr{G} \to f_*\mathscr{F}$ gives rise to morphisms

$$\varphi_V : \mathscr{G}(V) \to \mathscr{F}(f^{-1}(V)).$$

Now fix $x \in X$. Note that on one hand we have

$$\mathscr{G}_{f(x)} = \operatorname*{colim}_{\substack{V \in \mathbf{Open}(Y)_0 \\ f(x) \in V}} \mathscr{G}(V)$$

while on the other hand

$$(f_*\mathscr{F})_{f(x)} = \operatorname*{colim}_{\substack{V \in \mathbf{Open}(Y)_0 \\ f(x) \in V}} \mathscr{F}(f^{-1}(V)) \cong \operatorname*{colim}_{\substack{f^{-1}(V) \in \mathbf{Open}(X)_0 \\ x \in f^{-1}(V)}} \mathscr{F}(f^{-1}(V)).$$

Since the category of opens of the form $\{f^{-1}(V) \mid V \in \mathbf{Open}(Y)_0\}$ is a full subcategory of $\mathbf{Open}(X)$, we get a canonical map

$$\operatorname*{colim}_{\substack{f^{-1}(V) \in \mathbf{Open}(X)_0 \\ x \in f^{-1}(V)}} \mathscr{F}(f^{-1}(V)) \xrightarrow{\rho} \operatorname*{colim}_{U \in \mathbf{Open}(X)_0 x \in U} \mathscr{F}(U) = \mathscr{F}_x.$$

Composing the colimit of the maps $\varphi_V^{\sharp}$ together with the natural map $\rho$ above gives a composite $\varphi_x$

$$\begin{tikzcd}
\mathscr{G}_{f(x)} \arrow[r, equal] \arrow[drr, bend right, "\varphi_x"'] & \operatorname*{colim}\limits_{\substack{V \in \mathbf{Open}(Y)_0 \\ f(x) \in V}} \mathscr{G}(V) \arrow[r, "\operatorname{colim} \varphi_V"] & \operatorname*{colim}\limits_{\substack{V \in \mathbf{Open}(Y)_0 \\ x \in f^{-1}V}} \mathscr{F}(f^{-1}V) \arrow[d, "\rho"] \\
 & \mathscr{F}_x \arrow[r, equal] & \operatorname*{colim}\limits_{\substack{U \in \mathbf{Open}(X)_0 \\ x \in U}} \mathscr{F}(U)
\end{tikzcd}$$

as desired. ∎

**Proposition 2.1.28.** *Let $f : X \to Y$ be a continuous function and let $x \in X$. Then if $\mathscr{G}$ is a sheaf on $Y$,*

$$(f^{-1}\mathscr{G})_x \cong \mathscr{G}_{f(x)}.$$

*Proof.* This is a routine calculation, as

$$(f^{-1}\mathscr{G})_x = \operatorname*{colim}_{x \in U} \left(f^{-1}\mathscr{G}\right)(U) = \operatorname*{colim}_{x \in U} \left(\operatorname*{colim}_{f(U) \subseteq V} \mathscr{G}(V)\right)^{++} \cong \operatorname*{colim}_{f(x) \in V} \mathscr{G}(V) = \mathscr{G}_{f(x)}.$$

∎

**Definition 2.1.29.** A locally ringed space is a ringed space $X = (|X|, \mathcal{O}_X)$ such that for any $x \in |X|$, the stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. A morphism of locally ringed spaces is a morphism of ringed spaces $f : X \to Y$ such that for any $x \in |X|$ the morphism

$$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a local morphism of local rings. We will also write, for any $x \in |X|$,

$$\kappa(x) := \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x}.$$

**Remark 2.1.30.** If $X$ is a locally ringed space then we will write $\mathfrak{m}_x$ for the maximal ideal of $\mathcal{O}_{X,x}$.

**Definition 2.1.31.** Let $X = (|X|, \mathcal{O}_X)$ be a locally ringed space. Then for any open subset $|U| \subseteq |X|$ with inclusion $|i| : |U| \to |X|$, the pair $U = (|U|, \mathcal{O}_X|_U)$ is a locally ringed space where

$$\mathcal{O}_X|_U := |i|^{-1}\mathcal{O}_X.$$

If the context is clear, we'll often refer to the sheaf $\mathcal{O}_X|_U$ by $\mathcal{O}_U$ instead.

**Example 2.1.32.** Consider the space $|X| = \{s, \eta\}$ with topology $\mathbf{Open}(X) = \{\varnothing, \{\eta\}, X\}$. Induce the sheaf $\mathcal{O}_X$ on $|X|$ by

$$\mathcal{O}_X(U) := \begin{cases} \mathbb{Z}_p & \text{if } U = |X|; \\ \mathbb{Q}_p & \text{if } U = \{\eta\}; \\ 0 & \text{if } U = \varnothing. \end{cases}$$

Then $X = (|X|, \mathcal{O}_X)$ is a locally ringed space, as the stalks are

$$\mathcal{O}_{X,\eta} = \mathbb{Q}_p, \quad \mathcal{O}_{X,s} = \mathbb{Z}_p.$$

If we replace $\mathcal{O}_X$ with the sheaf $\mathscr{F}$ induced by

$$\mathscr{F}(U) = \begin{cases} \mathbb{Z} & \text{if } U = X; \\ \mathbb{Q} & \text{if } U = \{\eta\}; \\ 0 & \text{if } U = \varnothing; \end{cases}$$

then $(X, \mathscr{F})$ is a ringed space which is not a locally ringed space, as the stalk $\mathscr{F}_s = \mathbb{Z}$ is not a local ring.

**Example 2.1.33.** Let $A$ be a commutative ring with identity and let $|X| = \{*\}$. Define the sheaf $\mathscr{A}$ on $X$ by

$$\mathscr{A}(U) = \begin{cases} A & \text{if } U = X; \\ 0 & \text{if } U = \varnothing. \end{cases}$$

Then $X = (|X|, \mathscr{A})$ is a ringed space which is a locally ringed space if and only if $A$ is a local ring. In particular, for any field $K$, the ringed space $X = (\{*\}, \mathscr{K})$ is a locally ringed space.

**Example 2.1.34.** If $X$ is a complex manifold and $\mathcal{O}$ is the sheaf of holomorphic functions on $X$, then $(X, \mathcal{O})$ is a locally ringed space.

**Proposition 2.1.35.** *Let $f : X \to Y$, $g : Y \to Z$ be continuous functions of spaces. Then there is a natural isomorphism of functors*

$$f^{-1} \circ g^{-1} = (g \circ f)^{-1} : \mathbf{Shv}(Z) \to \mathbf{Shv}(X)$$

*and a strict equality*

$$(g \circ f)_* = g_* \circ f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Z).$$

*Proof.* We first calculate the equality for pushforwards[9] on sheaves; because the calculation is similar on morphisms, it is omitted. Fix a sheaf $\mathscr{F}$ on $X$ and an open set $V \subseteq Z$. Then

$$\begin{aligned} \big((g \circ f)_* \mathscr{F}\big)(V) &= \mathscr{F}\left((g \circ f)^{-1}(V)\right) = \mathscr{F}\left(f^{-1}\left(g^{-1}(V)\right)\right) = (f_* \mathscr{F})\left(g^{-1}(V)\right) = (g_*(f_* \mathscr{F}))(V) \\ &= \big((g_* \circ f_*)(\mathscr{F})\big)(V) \end{aligned}$$

so we conclude that $(g \circ f)_* = g_* \circ f_*$.

We now show the natural isomorphism $(g \circ f)^{-1} \cong f^{-1} \circ g^{-1}$. Consider the diagram of adjoint functors:

$$\begin{tikzcd}
\mathbf{Shv}(X) \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathbf{Shv}(Y) \arrow[l, bend right, "f^{-1}"'] \arrow[r, bend right, "g_*"'] \arrow[r, phantom, "\perp"] & \mathbf{Shv}(Z) \arrow[l, bend right, "g^{-1}"']
\end{tikzcd}$$

We know that $f^{-1} \circ g^{-1} \dashv g_* \circ f_*$ because the composition of left adjoints is left adjoint to the composition of right adjoints. We also know that $(g \circ f)^{-1} \dashv (g \circ f)_*$ by Theorem 2.1.17. However, because $(g \circ f)_* = g_* \circ f_*$ and because left adjoints are unique up to natural isomorphism, we obtain a natural isomorphism $f^{-1} \circ g^{-1} \cong (g \circ f)^{-1}$, as desired. ☐

**Remark 2.1.36.** In what follows we will likely treat composition of inverse image functors as if it were strict, i.e., as if there is an equality $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$. While this is, strictly speaking, a poor convention, it likely will not cause confusion and will simplify our work later on when we get to schemes.[10]

## 2.2 Skyscraper Sheaves

This section can be skipped on a first read, as it is used only peripherally as we proceed (we need skyscraper sheaves to be able to prove that flatness of sheaves of modules may be checked at stalks, for instance, but nowhere earlier; cf. Theorem **??**). I've included many of these for the sake of completeness and to give examples of the funky things that you can do with sheaves, but not so much for the explicit usefulness within the scope of these notes. That being said, these notions are crucial and of high importance for sheaf

[9] Direct images; I'm trying to be coherent in my terminology, but both are used in the literature and to some degree I feel it's important to see both used in practice.

[10] There will be less pre-and-post-composition by natural isomorphisms to get in the way of actually understanding what is going on. I encourage the reader interested in doing things *completely* properly to go through and replace such equalities with natural isomorphisms and check the resulting theory.

theory and doing algebraic geometry in practice, so not including them would be a disservice. The only really helpful and immediately useful fact in this section is Corollary 2.2.6 which says that the stalk functor $(-)_x : \mathbf{Shv}(X) \to \mathbf{Set}$ which sends a sheaf to its stalk at a point $x \in X$ is an exact functor.

We will begin by discussing skyscraper sheaves, whose name describes a good way to think about these gadgets. These are sheaves which may be defined over an entire space $X$, but which start disappearing as the space restricts along opens which do not contain $x$ but remains at opens which do contain $x$. The nice thing is that these sheaves arise as the essential image of the direct image functor of a morphism of ringed spaces. However, to describe this we'll first need a quick lemma that says the category of sheaves of sets on a single point space is isomorphic to the category of sets.

**Lemma 2.2.1.** *There is an isomorphism of categories*

$$\mathbf{Shv}(\{*\}) \cong \mathbf{Set}\,.$$

*Proof.* Begin by recalling that a sheaf $\mathscr{F}$ on $\{*\}$ is completely described by its global sections as in the diagram below

$$\begin{tikzcd} \mathscr{F}(\{*\}) \arrow[d, dashed, "\exists!\,\mathscr{F}(\{*\}\supseteq\varnothing)"] \\ \{\star\} \end{tikzcd}$$

because since $\mathscr{F}$ is a sheaf, $\mathscr{F}(\varnothing) = \{\star\}$. We thus define our functor

$$F : \mathbf{Set} \to \mathbf{Shv}(\{*\})$$

by setting $F(X)$ to be the sheaf on $\{*\}$ with $F(X)(\{*\}) = X$ for sets $X$ and defined similarly on morphisms. The functor

$$G : \mathbf{Shv}(\{*\}) \to \mathbf{Set}$$

is given by $G(\mathscr{F}) := \mathscr{F}(\{*\})$. It then follows that

$$(G \circ F)(X) = G(F(X)) = F(X)(\{*\}) = X$$

while

$$(F \circ G)(\mathscr{F}) = F(\mathscr{F}(X)) = \mathscr{F},$$

as both $F(\mathscr{F}(X))$ and $\mathscr{F}$ have the same values at $\{*\}$ and $\varnothing$. $\square$

Our next goal is to describe how for any ringed space $(X, \mathcal{O}_X)$ and any $x \in X$ there is a morphism of ringed spaces $(\{x\}, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$. For this we first need to establish that the inclusion $\mathrm{incl} : \{x\} \to X$ is continuous before constructing the sheaf morphism $\mathcal{O}_X \to (\mathrm{incl})_* \mathcal{O}_{X,x}$.

**Lemma 2.2.2.** *Let $X$ be a nonempty topological space and let $x \in X$. Then the inclusion* $\mathrm{incl} : \{x\} \to X$ *is continuous.*

*Proof.* Pick an open $U \subseteq X$. If $x \in U$ then

$$\mathrm{incl}^{-1}(U) = \{x\}$$

while if $x \notin U$ then

$$\mathrm{incl}^{-1}(U) = \varnothing.$$

In either case both sets are open, so incl is continuous. $\square$

Let $(X, \mathcal{O}_X)$ be a ringed space and let $x \in X$. We would now like to examine the nature of the pushforward functor $\mathrm{incl}_* : \mathbf{Shv}(\{x\}) \to \mathbf{Shv}(X)$ and its associated inverse image functor $\mathrm{incl}^{-1} : \mathbf{Shv}(X) \to \mathbf{Shv}(\{x\})$. Using Lemma 2.2.1 we will denote a sheaf on $\{x\}$ by its set of global sections; while this is a bit of an abuse of notation, the isomorphism of Lemma 2.2.1 tells us that this does in fact determine the sheaf. Let $A$ be a set and regard $A$ as a sheaf on $\{x\}$. We then find that for any $U \subseteq X$ open,

$$\mathrm{incl}_*(A)(U) = A\big(\mathrm{incl}^{-1}(U)\big) = \begin{cases} A & \text{if } x \in U; \\ \{*\} & \text{if } x \notin U; \end{cases}$$

while the restriction maps $\mathrm{incl}_* A(U \supseteq V)$ take the form

$$(\mathrm{incl}_* A)(U \supseteq V) = \begin{cases} \mathrm{id}_A & \text{if } x \in U, V; \\ !_A : A \to \{*\} & \text{if } x \in U, x \notin V; \\ \mathrm{id}_{\{*\}} & \text{if } x \notin U, V. \end{cases}$$

To determine a map from a sheaf $\mathscr{F}$ on $X$ to the sheaf $\mathrm{incl}_* A$ is the same as defining functions $\alpha_U : \mathscr{F}(U) \to A$ for all $U \subseteq X$ open with $x \in U$ for which the diagrams

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "\mathscr{F}(U \supseteq V)"'] & A \arrow[d, equal] \\ \mathscr{F}(V) \arrow[r, "\alpha_V"'] & A \end{tikzcd}$$

commute whenever $x \in V \subseteq U$. The reason why this suffices is because whenever $V \subseteq X$ with $x \notin V$ we have a unique morphism

$$!_{\mathscr{F}(V)} : \mathscr{F}(V) \to \{*\}.$$

Putting this altogether we not only get the definition of what it means to be a skyscraper sheaf on a space, but we also construct morphisms of ringed spaces of the form $\mathrm{incl} : (\{x\}, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$ for any $x \in X$. After this we will write down an observational lemma before proceeding to calculate the stalks of skyscraper sheaves.

**Proposition 2.2.3.** *Let $X$ be a ringed space and let $x \in X$. There is then a morphism of ringed spaces $(\mathrm{incl}, \mathrm{incl}^\sharp) : (\{x\}, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$ where $\mathrm{incl}^\sharp : \mathcal{O}_X \to \mathrm{incl}_* \mathcal{O}_{X,x}$ is given on opens $U \subseteq X$ by $\mathcal{O}_X(U) \to 0$ if $x \notin U$ and the colimit map $\alpha_U : \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ if $x \in U$.*

*Proof.* The map incl is already known to be continuous, so we only need to prove that $\mathrm{incl}^\sharp$ is a morphism of sheaves of rings on $X$. Moreover, from the comments preceding the statement of the proposition, it suffices to prove that for any $U \supseteq V$ open with $x \in V, U$ the diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[r, "\alpha_U"] \arrow[d, "\mathcal{O}_X(U \supseteq V)"'] & \mathcal{O}_{X,x} \arrow[d, equal] \\ \mathcal{O}_X(V) \arrow[r, "\alpha_V"'] & \mathcal{O}_{X,x} \end{tikzcd}$$

commutes. However this follows immediately from the definition of the stalk so we are done. $\square$

**Definition 2.2.4.** Let $X$ be a topological space. A skyscraper sheaf at a point $x \in X$ is a sheaf $\mathscr{F}$ which is isomorphic to $\mathrm{incl}_* A$ for a sheaf $A \in \mathbf{Shv}(\{x\})$

We now close this short section by describing the inverse image of the skyscraper sheaf functor, a consequence of this calculation, and some facts about stalks of skyscraper sheaves. These calculations will be helpful later when we discuss flat sheaves of $\mathcal{O}_X$-modules. We will also present some pictures that help motivate the intuition and naming of skyscraper sheaves.

**Lemma 2.2.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Then if* $\mathrm{incl} : \{x\} \to X$ *is the inclusion of a point $x \in X$,* $\mathrm{incl}^{-1}(\mathscr{F}) = \mathscr{F}_x$ *for any sheaf $\mathscr{F}$ on $X$.*

*Proof.* Once again we use Lemma 2.2.1 and characterize a sheaf on $\{x\}$ completely by its global sections. Now by definition we have

$$\mathrm{incl}^{-1}(\mathscr{F})(\{*\}) = \operatorname*{colim}_{\substack{\mathrm{incl}\{x\} \subseteq U \\ U \subseteq X \text{ open}}} \mathscr{F}(U) = \operatorname*{colim}_{\substack{x \in U \\ U \subseteq X \text{ open}}} = \mathscr{F}_x .$$

$\square$

**Corollary 2.2.6.** *For any $x \in X$ the stalk functor $(-)_x : \mathbf{Shv}(X) \to \mathbf{Set}$ is exact and cocontinuous.*

*Proof.* Because the stalk functor factors as

$$\begin{tikzcd}
\mathbf{Shv}(X) \arrow[r, "\mathrm{incl}^{-1}"] \arrow[dr, "(-)_x"'] & \mathbf{Shv}(\{x\}) \arrow[d, "\simeq"] \\
 & \mathbf{Set}
\end{tikzcd}$$

and both the equivalence of categories and $\mathrm{incl}^{-1}$ are exact and cocontinuous, the result follows. $\square$

**Proposition 2.2.7.** *For any topological space $X$ and any point $x \in X$ with inclusion map* $\mathrm{incl} : \{x\} \to X$, *if $A$ is a sheaf on $\{x\}$ then*

$$\mathrm{incl}^{-1}(\mathrm{incl}_*(A)) = A.$$

*Proof.* We once again use Lemma 2.2.1 to describe a sheaf on $\{x\}$ by its global sections. It follows from the discussion prior to Proposition 2.2.3 and from Lemma 2.2.5 that

$$\mathrm{incl}^{-1}(\mathrm{incl}_*(A)) = (\mathrm{incl}_* A)_x =$$

$\square$

**Proposition 2.2.8.** *If $A$ is a sheaf on a point $\{x\}$ of a topological space $X$ then for any $y \in X$,*

$$(\mathrm{incl}_* A)_y = \begin{cases} A & \mathit{if} y \in \overline{\{x\}}; \\ \{*\} & \mathit{if} y \notin \overline{\{x\}}. \end{cases}$$

*Proof.* If $y \in \overline{\{x\}}$ then every open set $U \subseteq X$ with $y \in U$ also has $x \in X$. Thus

$$(\mathrm{incl}_* A)_y = \operatorname*{colim}_{\substack{y \in U \\ U \subseteq X \text{ open}}} \mathrm{incl}_* A(U) = \operatorname*{colim}_{\substack{y \in U \\ U \subseteq X \text{ open}}} A = A.$$

Alternatively, if $y \notin \overline{\{x\}}$ then there is an open set $U \subseteq X$ for which $y \in U$ but $x \notin U$. Thus for any open $V \subseteq U$ the skyscraper sheaf $(\mathrm{incl}_* A)(V) = \{*\}$; moreover, the class $\mathcal{I}$ of such opens (those which contain $y$ but not $x$) is cofinal in the poset of opens which contain $y$. Thus

$$(\mathrm{incl}_*(A))_y = \operatorname*{colim}_{\substack{y \in U \\ U \subseteq X \text{ open}}} \mathrm{incl}_* A(U) = \operatorname*{colim}_{\substack{y \in U \\ U \in \mathcal{I}}} \mathrm{incl}_* A(U) = \operatorname*{colim}_{\substack{y \in U \\ U \in \mathcal{I}}} \{*\} = \{*\}.$$

$\square$

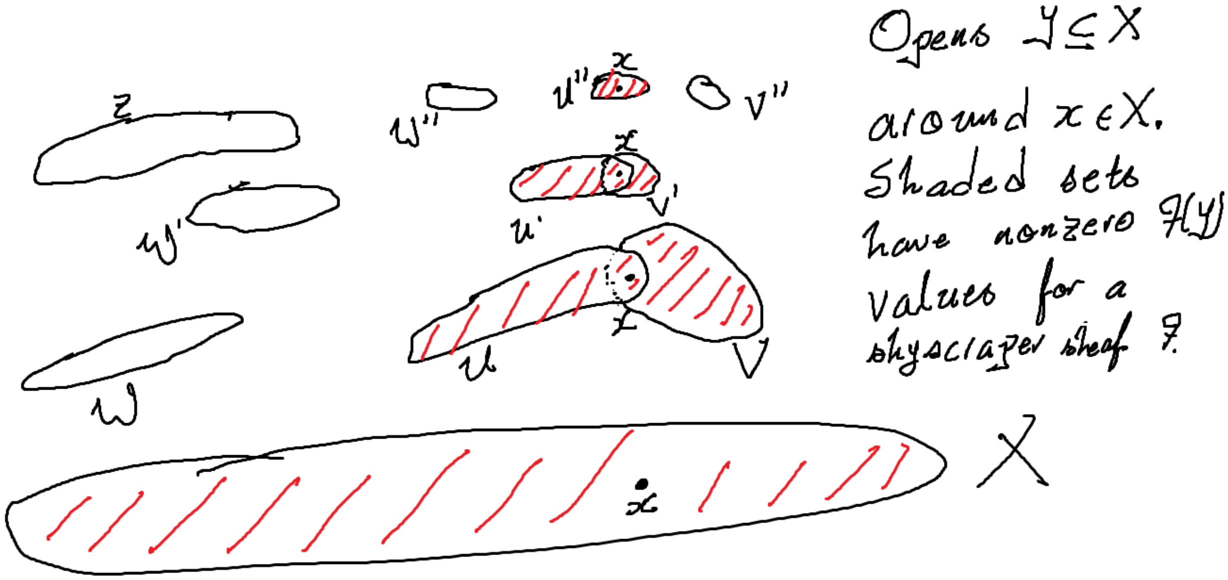


Figure 2.1: An example of open sets in a space $X$ converging to a point $x \in X$ and a coloring of those sets for which a skyscraper sheaf is non-trival.

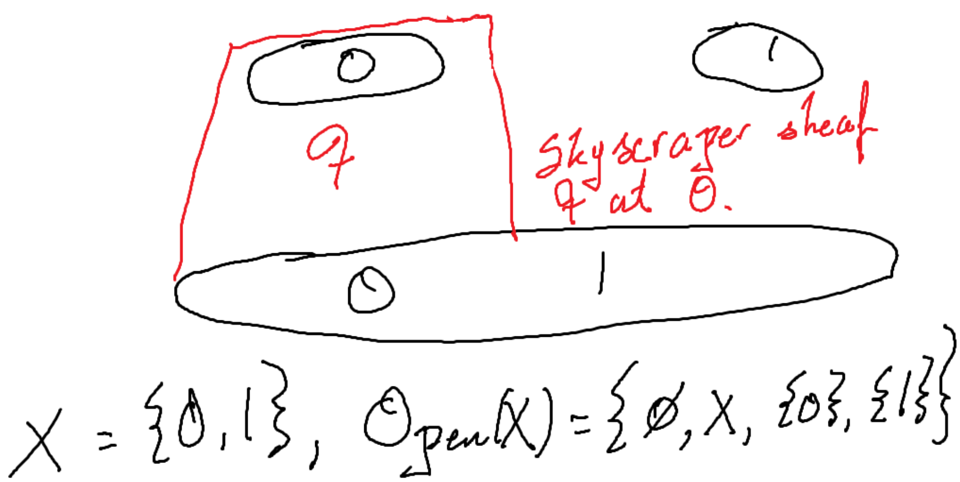


Figure 2.2: An example of those sets of the discrete topological space $X = \{0, 1\}$ which are non-trivial on a skyscraper sheaf at 0.

# Chapter 3

# Schemes and Varieties

We now get to meet the fundamental objects of study in modern algebraic geometry: schemes. Schemes are to commutative rings what open subsets $U \subseteq \mathbb{R}^n$ are to (real) manifolds. The way we'll want to exploit this analogy and observation is to see a manifold as a "space of real spaces.[1]" before building schemes up via taking them to be manifolds made up of affine schemes.

Before diving into things headfirst, I feel it's a good idea to discuss some of the structure of this overly long "day" of lecture notes. In my attempt to make these notes as self-contained as possible (and in particular contain an introduction to schemes from the ground up), the notes got quite long. There are many constructions that go into defining schemes, and including them all has made this day's notes become quite lengthy.[2] In an effort to help you read through these notes I've chopped things up into subsections to help you decide what you want/need to read and help guide you to the wonderful world of schemes.

1. Nearly everyone can skip the proofs of Theorem 3.2.3 and Lemma 3.2.5. These are technical facts that need to be there and their proofs do contain interesting geometric information, but their technicality means you should almost see *how* they're used before going through the proofs in detail.

2. For the person who wants to know EXACTLY what goes into defining schemes in this classical way: first, I applaud your curiousity! Second, I suggest you go through the definitions and poofs starting from right here! This will take some time, however, so if a proof looks too technical I recommend skipping over it and seeing how the result is used.

3. For the person who only wants to see the topological side of things and is willing to black box the sheaf theory a bit, start with Section 3.1, check out Proposition 3.3.2 and Corollary 3.3.4, and then go to 3.4.

4. For the person who wants to focus on the spectrum functor itself and back-reference the needed definitions, go straight to Section 3.3 (and notably look at Corollary 3.3.4) before moving on as normally.

5. For the truly brave reader, the definitions of schemes themselves and a bunch of examples can be found in Section 3.4 (and in particular in Definitions 3.4.1 and 3.4.3).

6. For the braver reader, we give various properties of schemes and the category of schemes in Section 3.5. This is where we describe some categorical properties of the categories of (relative) schemes, such as the fact $\mathbf{Sch}_{/S}$ admits all finite pullbacks but does not admit coequalizers. The fact that coequalizers

[1] This is my attempt to poorly name the fact that a manifold is really a toplogical space equipped with a structure sheaf valued in a category I still need to work out $\mathbf{Open}(\mathbb{R}^n)$.

[2] It's also arguably not the correct approach if you've drank the Grothendieck style functorial Kool-Aid. In an alternate universe it is possible to think of schemes instead as certain functors from $\mathbf{Cring} \to \mathbf{Set}$ and then extract various properties of the Zariski or étale topology from the copresheaf topos $[\mathbf{Cring}, \mathbf{Set}]$. However, this approach is a little abstract and requires significantly more categorical machinery and baggage, so I've elected to take the more traditional approach which is looks closer to manifold theory and complex analytic geometry.

do not exist in $\mathbf{Sch}_{/S}$ is a little bit technical, but the sketch given basically says that some spaces you could try to glue can be too non-separated to possibly be schemes in the end. The last component of this section can be skipped, as the proof that pullbacks exist (cf. Theorem 3.5.12) is technical, but I recommend at least looking through the statements of the lemmas and propositions that go into it (cf. Lemmas 3.5.8, 3.5.7, 3.5.9 and Proposition 3.5.11).

7. For the even braver reader, Sections 3.7, 3.8, and 3.9 go through the main ingredients in defining varieties. We've paid special attention to pullbacks of reduced schemes and drawing some pictures in these sections, as well as focus on giving some examples.

8. For the bravest of all, the definition of varieties can be found in Definition 3.10.1. We've focused in this section on giving many, many examples here and shown how to define certain Lie groups as varieties.

9. Finally, Section 3.11 can be skipped entirely for all but the most adventurous readers. This section discusses quasi-separated morphisms, which are interesting for many algebro-geometric contexts, but are technical and included only for applications to various (particularly general) results on quasi-coherent sheaves.

In any case, I'd recommend at least taking a look at the motivational material describe manifolds as spaced spaces before doing any skipping of material. It's nice to have this intuition in mind, and frankly it's not something I've found written down before.[3] I've also tried to back reference definitions as they get used, as this document is hyperlinked and you can click on the references to go straight to the definition used if you need a refresher.

For what follows we need to define a category.

**Definition 3.0.1.** Let $X$ be a topological space. We define $\mathcal{O}(X)$ to be the category where:

- Objects: Open subsets $U \subseteq X$;
- Morphisms: Continuous maps $U \to V$;
- Composition and Identities: As in **Top**.

Recall that a $\mathcal{C}^p$, for $0 \leq p \leq \infty$, (real) manifold is (cf. [45], [72], and [78], for instance) a topological space $M$ with a countable basis equipped with a collection of open sets $\{U_i \subseteq M \mid i \in I\}$ for which:

- The $U_i$ constitute an open cover of $M$, i.e.,

$$M = \bigcup_{i \in I} U_i.$$

- Fix an $n \in \mathbb{N}$. Then for each $i \in I$ there is a homeomorphism $\varphi_i$ of $U_i$ onto some open subspace $V_i \subseteq \mathbb{R}^n$, i.e., $\varphi_i : U_i \to V_i$ is a homeomorphism.
- For each pair of indices $i, j \in I$, the diagram

$$\begin{tikzcd} U_i \arrow[r, "\varphi_i"] & V_i \\ U_i \cap U_j \arrow[u] \arrow[r, "\varphi_i|_{U_i \cap U_j}"'] & \varphi_i(U_i \cap U_j) \arrow[u] \end{tikzcd}$$

commutes with each vertical arrow an open inclusion, i.e., $\varphi_i(U_i \cap U_j) \subseteq V_i$ is open.

[3] Which likely means that this is folkloric, unhelpful, or most likely that I need to read more.

- For every pair of indices $i, j \in I$ the composite

$$\varphi_i(U_i \cap U_j) \xrightarrow{(\varphi_i|_{U_i \cap U_j})^{-1}} U_i \cap U_j \xrightarrow{\varphi_j|_{U_i \cap U_j}} \varphi_j(U_i \cap U_j)$$

is a $\mathcal{C}^p$ homeomorphism of $\varphi_i(U_i \cap U_j)$ with $\varphi_j(U_i \cap U_j)$.

Putting these observations together we define a functor $\mathfrak{A} : \mathcal{O}(M)^{\mathrm{op}} \to \mathcal{O}(\mathbb{R}^n)$ by gluing the functors $\mathfrak{A}_i : \mathbf{Open}(U_i)^{\mathrm{op}} \to \mathbf{Open}(\mathbb{R}^n$ defined as follows:

- On objects $W \subseteq U_i$ in $\mathcal{O}(U_i)$ we define $\mathfrak{A}_i(W) := \varphi_i(W)$ (note that this is well-defined because, as a $\varphi_i$ is a homeomorphism, it is an open map).

- Given an inclusion of opens $V \subseteq W$ in $\mathcal{O}(U_i)$, we define $\mathfrak{A}_i(W) \to \mathfrak{A}_i(V)$ to be the opposite/restriction of the inclusion $\varphi_i(W) \supseteq \varphi_i(V)$.

From the construction of the axioms of a manifold, each $\mathfrak{A}_i$ is a sheaf on $U_i$ and we in turn can apply the Gluing Lemma (cf. Proposition 3.4.8) to get a sheaf $\mathfrak{A}$ on $M$ which records the local gluing and local homeomorphism structure. In fact, what many people call the atlas of the manifold (as a strictly combinatorial set of data) is actually a sheaf on $M$. This perspective not only helps us see that schemes really are "manifolds of rings" but also shows us how algebraic geometry really is "geometric" in nature.

Let us move on from manifolds to discuss (affine) schemes. The idea here is that we want to think of a commutative ring $A$ as a ring of operators of some kind and then build a topological space $X$ for which the points of $X$ are the spectral values of the operators in $A$.[4] This will in turn require making the prime ideals of $A$ into the sprectral values of $A$ and then equipping this with the Zariski topology, as the prime ideals then look like points along some sort of curve and are topologized based on how these curves behave algebraically, i.e., by asking that the points in the spectra for which certain algebraic subsets of $A$ vanish be closed. Afterwards, we'll need to equip this spectrum with a structure sheaf that reads the elements of $A$ off as the global sections of the spectrum and then records those sections which are invertible around certain points as the local sections (just like how Taylor series allow us to record which functions are locally invertible within some neighborhood of a point in $\mathbb{C}^n$ or $\mathbb{R}^n$).

## 3.1 The Commutative Algebra and Zariski Topological Background

Before beginning this task in earnest, we'll need to recall some definitions and results from commutative algebra before continuing. While I do assume the reader is familiar with basic algebra, think of this next little bit as codifying commutative algebra notation and making sure that we are on the same page algebraically.

**Definition 3.1.1.** Let $A$ be a commutative ring with identity and let $\mathfrak{a} \trianglelefteq A$ be an ideal. Then:

- The ideal $\mathfrak{a}$ is prime if $\mathfrak{a} \neq A$ and the quotient ring $A/\mathfrak{a}$ is an integral domain. Equivalently, this asks that $1 \notin \mathfrak{a}$ and that for any $a, b \in A$ the product $ab \in \mathfrak{a}$ if and only if $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$. We will usually denote prime ideals by the Fraktur/Gothic letters $\mathfrak{p}$ and $\mathfrak{q}$.[5]

- The ideal $\mathfrak{a}$ is maximal if $\mathfrak{m} \neq A$ and the quotient ring $A/\mathfrak{a}$ is a field. Equivalently, this asks that $1 \notin \mathfrak{a}$ and if $\mathfrak{b} \trianglelefteq A$ is any ideal for which $\mathfrak{a} \subseteq \mathfrak{b} \subsetneq A$ then $\mathfrak{b} = \mathfrak{a}$. We will usually denote maximal ideals with the letters $\mathfrak{m}$ and $\mathfrak{n}$.

[4] It is not too surprising, given this intuition and inspiration, that Grothendieck's PhD thesis was actually in functional analysis. In fact, one of Grothendieck's PhD supervisors was Laurent Schwartz, the person for whom Schwartz spaces (an important class of spaces in harmonic analysis) are named.

[5] My family is of German descent, so I try to write in Fraktur fonts as much as I can. It's just also convenient that the use of Fraktur letters to denote ideals is still very common in algebraic geometry.

- The radical of the ideal $\mathfrak{a}$ is the set

$$\sqrt{\mathfrak{a}} := \bigcap_{\substack{\mathfrak{p} \text{ prime ideal} \\ \mathfrak{a} \subseteq \mathfrak{p}}} \mathfrak{p}\,.$$

Equivalently,

$$\sqrt{\mathfrak{a}} = \{a \in A \mid \exists n \in \mathbb{N}\,.\, a^n \in \mathfrak{a}\}.$$

**Remark 3.1.2.** For any ideal $\mathfrak{a}$ of a commutative ring with identity, $\sqrt{\mathfrak{a}}$ is an ideal of $A$. The proof that

$$\{a \in A \mid \exists n \in \mathbb{N}\,.\, a^n \in \mathfrak{a}\} = \sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{p} \text{ prime ideal} \\ \mathfrak{a} \subseteq \mathfrak{p}}} \mathfrak{p}$$

follows from [20, Proposition 15.2.9]

**Definition 3.1.3.** If $S \subseteq A$ is a multiplicatively closed set (so $s, t \in S$ implies $st \in S$; we also assume that $1 \in S$ for convenience, but this is technically far from necessary) then the localization of $A$ at $S$, $S^{-1}A = AS^{-1}$, is the ring generated by elements of the form

$$S^{-1}A := \left\{ \frac{a}{s} \mid a \in A, s \in S \right\}$$

where $a/s = b/t$ for $a, b \in A$ and $s, t \in S$ if and only if there exists an $r \in S$ for which $r(at - bs) = 0$.[6] This is a commutative ring with identity where

$$1_{S^{-1}A} = \frac{1}{1} = \frac{s}{s}$$

for any $s \in S$ and the addition and multiplication are generated by the rules

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \quad \left(\frac{a}{s}\right)\left(\frac{b}{t}\right) = \frac{ab}{st}$$

for $a, b \in A$ and $s, t \in S$. There is also a canonical localization map $\lambda_S : A \to S^{-1}A$ given by[7]

$$a \mapsto \frac{a}{1}.$$

The ring $S^{-1}A$ has the universal property where if $\varphi : A \to B$ is any commutative ring homomorphism for which $\varphi(s) \in B$ is a unit for all $s \in S$ then there is a unique morphism $\overline{\varphi} : S^{-1}A \to B$ making

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\lambda_S"'] & & B \\ & S^{-1}A \arrow[ur, dashed, "\overline{\varphi}"'] & \end{tikzcd}$$

commute in **Cring**. If the multiplicatively closed set $S = \{f^n \mid n \in \mathbb{N}\}$ for some $n \in \mathbb{N}$ we'll abuse notation somewhat and write $S^{-1}A = A[f^{-1}] = A_f$ depending on the situation. Similarly, if $\mathfrak{p}$ is a prime ideal, we write

$$A_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1}A.$$

[6]The reason we ask for such an $r$ is so that if $A$ has zero divisors and $at = 0$, we can potentially zero divide out $bs$ to make the "expected" cross-multiplication identity $at - bs = 0$ work even if $at = 0$. In fact, if $A$ is an integral domain, asking for such an $r$ is superfluous and unnecessary. Moreover if $S = \{f^n \mid n \in \mathbb{N}\}$ then $r = f^k$ for some $k \in \mathbb{N}$.

[7]This is where it is convenient to assume $1 \in S$. If $1 \notin S$ the identity in $S^{-1}A$ is the equivalence class $s/s$ and the canonical map is $a \mapsto as_0/s_0$ where $s_0 \in S$ is some chosen element.

**Remark 3.1.4.** If $0 \in S$ it is a nice check to show that $S^{-1}A \cong 0$; this shows that the only algebraic way we can sensibly[8] divide by zero is if the only number is zero.

**Definition 3.1.5.** We define the prime spectrum of $A$ to be the set

$$\operatorname{Spec} A := \{\mathfrak{p} \trianglelefteq A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

The maximal ideal spectrum is the set

$$\operatorname{Specm} A := \{\mathfrak{m} \triangleleft A \mid \mathfrak{m} \text{ is a maximal ideal}\}.$$

These sets already contains some interesting information about the ring $A$, although to make it of maximal[9] interest we need to focus on the prime spectrum topologize and sheafify it in a sensible way. However, before introducing the Zariski topology, I want to record some useful observations about the points of the spectrum $\operatorname{Spec} A$ and the maximal ideal spectrum $\operatorname{Specm} A$.

**Proposition 3.1.6.** *The zero ideal* $(0)$ *is a point in* $\operatorname{Spec} A$ *if and only if* $A$ *is an integral domain.*

*Proof.* This is equivalent to saying that $(0)$ is a prime ideal if and only if $A$ has no nontrivial zero divisors. However, this equivalence is immediate from asking whether or not $A \cong A/(0)$ is an integral domain. $\square$

**Lemma 3.1.7.** *Let* $A$ *be a (two-sided) Artinian ring. Then* $A$ *has finitely many maximal two-sided ideals.*

*Proof.* Assume for the purpose of deriving a contradiction[10] $\{\mathfrak{m}_n \mid n \in \mathbb{N}\}$ be an infinite set of pairwise distinct maximal two-sided ideals of $A$. We then see that the chain

$$\mathfrak{m}_0 \supsetneq \mathfrak{m}_0 \cap \mathfrak{m}_1 \supsetneq \mathfrak{m}_0 \cap \mathfrak{m}_1 \cap \mathfrak{m}_2 \supsetneq \cdots$$

is an infinite descending chain of two-sided ideals of $A$. This contradicts that $A$ is Artinian, so no such infinite set may exist. $\square$

**Lemma 3.1.8.** *Let* $A$ *be a commutative unital ring. Then* $A$ *is Artinian if and only if* $\operatorname{Spec} A = \operatorname{Specm} A$.

*Proof.* $\implies$ : Let $A$ be an Artinian ring, let $\mathfrak{p} \in \operatorname{Spec} A$, and let $\operatorname{Specm} A = \{\mathfrak{m}_0, \cdots, \mathfrak{m}_n\}$; note such an assumption is valid by Lemma 3.1.7. Because every nilpotent element of $A$ is contained in every prime ideal $\mathfrak{p}$ and in the Jacobson radical $J(A)$, it follows that

$$\mathfrak{p} \supseteq J(A) = \bigcap_{i=0}^{n} \mathfrak{m}_i \,.$$

Furthermore, because $\mathfrak{m}_0 \cap \cdots \cap \mathfrak{m}_n \supseteq \mathfrak{m}_0 \cdots \mathfrak{m}_n$ we also have that $\mathfrak{p} \supseteq \mathfrak{m}_0 \cdots \mathfrak{m}_n$. This implies moreover that there exists an $0 \leq i \leq n$ for which $\mathfrak{m}_i \subseteq \mathfrak{p}$ so we conclude that $\mathfrak{p} = \mathfrak{m}_i$ by the maximality of $\mathfrak{m}_i$. Thus in this case $\operatorname{Spec} A = \operatorname{Specm} A$.

$\impliedby$ : Assume that $\operatorname{Specm} A = \operatorname{Spec} A$ and note that this implies that $A$ has Krull dimension zero. Since zero dimensional commutative unital rings are Artinian, this completes the proof. $\square$

In defining and proving that the Zariski topology exists and is a topology on $\operatorname{Spec} A$, it is most convenient and helpful to use a basis of open sets (as opposed to the full definition, as the basis of opens is particularly well-suited to defining sheaves). I'll take this approach in defining the Zariski topology, but keep in mind that many classical references (cf. [33, Section II.2, Page 70], for instance) define the Zariski topology in terms of its closed sets, as these describe the primes $\mathfrak{p}$ where ideals or elements vanish.

[8] Sensibly here means put a ring (structure) on it, as we like algebra so we follow Beyoncé's advice.

[9] Pun intended. The way this scientific/philosophical writing differs from Derridian texts is that you know I know I'm making puns and that's okay.

[10] I'd like to introduce the word "Aftpodac" as a word which means "Assume for the purpose of deriving a contradiction" to mathematical literature. It is a distinct word that is not an acronym nor initialism of "Assume for the purpose of deriving a contradiction," is pronounced "aft-poe-dak," and can be easily spoken while teaching or just more generally at the blackboard explaining proof strategies. It's also fun to write!

**Definition 3.1.9.** If $S \subseteq A$, define the non-vanishing set of $A$ as

$$D(S) := \{\mathfrak{p} \in \operatorname{Spec} A \mid S \nsubseteq \mathfrak{p}\}.$$

In particular, for any $f \in A$ we abuse notation and write

$$D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$$

for the non-vanishing set of $A$. Dually, the vanishing set for $S \subseteq A$ is

$$V(S) = \{\mathfrak{p} \in \operatorname{Spec} A \mid S \subseteq \mathfrak{p}\}$$

and if $f \in A$ we abuse notation and write

$$V(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \in \mathfrak{p}\}.$$

**Remark 3.1.10.** It's worth remarking/noting that there is an equality of sets

$$D(f) = D\big((f)\big)$$

where $(f)$ is the ideal in $A$ generated by $f$: $(f) = Af$. This follows because for any prime ideal $\mathfrak{p}$ for which $(f) \nsubseteq \mathfrak{p}$, we must have that $f \notin \mathfrak{p}$; similarly, if $f \notin \mathfrak{p}$ then $(f) \nsubseteq \mathfrak{p}$ as well.

**Lemma 3.1.11.** *For any $f \in A$ and any positive $n \in \mathbb{N}$, $D(f^n) = D(f)$.*

*Sketch.* This proof is a routine check using the fact that $f^n \notin \mathfrak{p}$ if and only if $f \notin \mathfrak{p}$ for any prime ideal $\mathfrak{p}$ and for any $n \geq 1$. □

**Lemma 3.1.12.** *Let $A$ be a commutative ring with identity. Then:*

1. *If $\mathfrak{a}, \mathfrak{b} \trianglelefteq A$, $D(\mathfrak{a}\,\mathfrak{b}) = D(\mathfrak{a}) \cap D(\mathfrak{b})$.*
2. *For any* **Set***-indexed collection of ideals $\{\mathfrak{a}_i \mid i \in I\}$,*
$$D\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcup_{i \in I} D(\mathfrak{a}_i).$$
3. *For any ideals $\mathfrak{a}, \mathfrak{b} \trianglelefteq A$, $D(\mathfrak{a}) \subseteq D(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}}$.*
4. *If $f, g \in A$ then $D(f) \subseteq D(g)$ if and only if $g$ is a unit in the localization $A_f = A[f^{-1}]$.*

*Proof.* (1): Begin by letting $\mathfrak{p} \in D(\mathfrak{a}\,\mathfrak{b})$. Then $\mathfrak{a}\,\mathfrak{b} \nsubseteq \mathfrak{p}$ so there is a pair $ab \in \mathfrak{a}\,\mathfrak{b}$ such that $ab \notin \mathfrak{p}$. However, since $ab \notin \mathfrak{p}$ and the ideal $\mathfrak{p}$ is prime, we necessarily have that both $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$ so $\mathfrak{p} \in D(\mathfrak{a}) \cap D(\mathfrak{b})$. Alternatively, if $\mathfrak{p} \in D(\mathfrak{a}) \cap D(\mathfrak{b})$ then there exist $a \in \mathfrak{a}, b \in \mathfrak{b}$ for which $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$. But then, once again from $\mathfrak{p}$ being prime, we have that $ab \notin \mathfrak{p}$ so $\mathfrak{a}\,\mathfrak{b} \nsubseteq \mathfrak{p}$. Thus $D(\mathfrak{a}\,\mathfrak{b}) = D(\mathfrak{a}) \cap D(\mathfrak{b})$.

(2): We calculate that

$$\begin{aligned} D\left(\sum_{i \in I} \mathfrak{a}_i\right) &= \left\{\mathfrak{p} \in \operatorname{Spec} A \;:\; \sum_{i \in I} \mathfrak{a}_i \nsubseteq \mathfrak{p}\right\} = \{\mathfrak{p} \in \operatorname{Spec} A \mid \exists i \in I.\ \mathfrak{a}_i \nsubseteq \mathfrak{p}\} \\ &= \bigcup_{i \in I} \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a}_i \nsubseteq \mathfrak{p}\} = \bigcup_{i \in I} D(\mathfrak{a}_i). \end{aligned}$$

(3): Because the radicals of $\mathfrak{a}$ and $\mathfrak{b}$ are the intersections of all primes which contain $\mathfrak{a}$ and $\mathfrak{b}$ respectively, it follows immediately that $\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}}$ if and only if $D(\mathfrak{a}) \subseteq D(\mathfrak{b})$.

(4): $\implies$: Assume that $f, g \in A$ and $D(f) \subseteq D(g)$. Then $\sqrt{(f)} \subseteq \sqrt{(g)}$ by Part (3) so, since $(f) \subseteq \sqrt{(f)}$, we have that $f \in \sqrt{(g)}$. Thus from the alternative characterization of the radical as the elements of $A$ whose residues in $A/(g)$ are nilpotent we have that there is some $n \in \mathbb{N}$ for which $f^n \in (g)$. Consequently find an $a \in A$ for which $f^n = ag$. Now we calculate that in $A[f^{-1}]$,

$$g\left(\frac{a}{f^n}\right) = \left(\frac{g}{1}\right)\left(\frac{a}{f^n}\right) = \frac{ag}{f^n} = \frac{f^n}{f^n} = 1_{A[f^{-1}]}$$

so $g$ is indeed a unit in $A[f^{-1}]$.

$\impliedby$: Assume that $g$ is a unit in $A[f^{-1}] = A_f$. Then there exists some fraction $\alpha = b/f^m$ for $b \in A$ and $m \in \mathbb{N}$ for which

$$g\alpha = 1_{A[f^{-1}]} = g\left(\frac{b}{f^m}\right).$$

But then we have that

$$g\left(\frac{b}{f^m}\right) = \frac{gb}{f^m} = 1_{A[f^{-1}]}$$

so it follows after left-multiplying the above expression by $f^m$ that $gb = f^m$. But then $f^m \in (g)$ so $f \in \sqrt{(g)}$ and hence $D(f) \subseteq D(g)$. □

**Remark 3.1.13.** A corollary of Part (3) of Lemma 3.1.12 is that if $f, g \in A$ then $D(f) \subseteq D(g)$ if and only if there is an $n \in \mathbb{N}$ for which $f^n \in (g)$. In particular, this means that in the Zariski topology that we can include the non-vanishing set of $f$ into the non-vanishing set of $g$ if and only if $g$ looks like it's invertible within an infinitesimal neighborhood of $f$.

**Proposition 3.1.14.** *The collection of sets $\mathcal{Z} := \{D(\mathfrak{a}) \mid \mathfrak{a} \trianglelefteq A\}$ forms a topology on* $\operatorname{Spec} A$ *and the set* $\mathcal{B} := \{D(f) \mid f \in A\}$ *forms a basis for this topology.*

*Proof.* We first verify that $\operatorname{Spec} A, \varnothing \in \mathcal{Z}$. For this note that since $0 \in \mathfrak{a}$ for any ideal $\mathfrak{a} \trianglelefteq A$, $(0) \subseteq \mathfrak{a}$. Thus $D(0) = \{\mathfrak{p} \in \operatorname{Spec} A \mid (0) \not\subseteq \mathfrak{p}\} = \varnothing$ so $\varnothing \in \mathcal{Z}$. Dually, since every prime ideal $\mathfrak{p}$ of $A$ is a proper ideal, there is no prime ideal of $A$ which contains 1. Thus $D(1) = \{\mathfrak{p} \in \operatorname{Spec} A \mid A \not\subseteq \mathfrak{p}\} = \operatorname{Spec} A$ so $\operatorname{Spec} A \in \mathcal{Z}$.

We now verify that $\mathcal{Z}$ is closed under arbitrary unions. Let $I$ be an index set and consider the collection of sets $\{D(\mathfrak{a}_i) \mid i \in I, \mathfrak{a}_i \trianglelefteq A\}$. From Part (2) of Lemma 3.1.12 we get that

$$\bigcup_{i \in I} D(\mathfrak{a}_i) = D\left(\sum_{i \in I} \mathfrak{a}_i\right)$$

so the union of the $D(\mathfrak{a}_i)$ is indeed in $\mathcal{Z}$. Thus $\mathcal{Z}$ is closed under unions.

Finally we verify that finite (nonempty) intersections of sets in $\mathcal{Z}$ remain in $\mathcal{Z}$. As usual, however, it suffices to prove this for binary intersections by virtue of a routine induction. Thus let $D(\mathfrak{a}), D(\mathfrak{b}) \in \mathcal{Z}$. By Part (1) of Lemma 3.1.12 we get that

$$D(\mathfrak{a}) \cap D(\mathfrak{b}) = D(\mathfrak{a}\,\mathfrak{b})$$

so $D(\mathfrak{a}) \cap D(\mathfrak{b}) \in \mathcal{Z}$. Thus we conclude that $\mathcal{Z}$ is a topology on $\operatorname{Spec} A$.

We now show that the set $\mathcal{B} = \{D(f) \mid f \in A\}$ forms a basis to $\mathcal{Z}$. For this we first show that the $D(f)$ cover $\operatorname{Spec} A$. Let $\mathfrak{p} \in \operatorname{Spec} A$. Because $\mathfrak{p} \subsetneq A$, there is an $f \in A$ for which $f \notin \mathfrak{p}$ and so $\mathfrak{p} \in D(f)$. Thus, since $\mathfrak{p}$ was arbitrary,

$$\operatorname{Spec} A = \bigcup_{f \in A} D(f)$$

so $\mathcal{B}$ covers $\operatorname{Spec} A$.

We now snow that if $\mathfrak{p} \in D(f) \cap D(g)$ then there is a $D(h) \in \mathcal{B}$ for which $D(h) \subseteq D(f) \cap D(g)$ and $\mathfrak{p} \in D(h)$. For this note that because $A$ is a commutative ring with identity, $(f)(g) = (fg)$ as ideals. Thus by Part (1) of Lemma 3.1.12 we have that $D(f) \cap D(g) = D(fg)$ and so setting $h = fg$ gives the desired subset of $D(f) \cap D(g)$. □

**Definition 3.1.15.** The topology $\mathcal{Z}$ on $\operatorname{Spec} A$ constructed in Proposition 3.1.14 is the Zariski topology on $\operatorname{Spec} A$.

We now show that for ring morphism between two commutative rings with identity, $\varphi : A \to B$, induces a continuous function between spectra via taking preimages. This will allow us to deduce that there is a functor $\mathbf{Cring}^{\mathrm{op}} \to \mathbf{Top}$ given by taking spectra.

**Proposition 3.1.16.** *Let $\varphi \in \mathbf{Cring}(A, B)$. Then the map $\varphi^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous in the Zariski topology.*

*Proof.* First let us verify that the preimage $\varphi^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$ is well-typed, i.e., that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of $A$ for any $\mathfrak{p} \in \operatorname{Spec} B$. For this we begin by noting 1the verification that $\varphi^{-1}(\mathfrak{p})$ is an ideal of $A$ is trivial; as such, we only need verify that it is prime. To this end we consdier the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{a, b \in A.\, ab \in \varphi^{-1}(\mathfrak{p})}{\varphi(ab) \in \mathfrak{p}}}{\varphi(a)\varphi(b) \in \mathfrak{p}}\ \varphi \text{ is a ring homomorphism so } \varphi(ab) = \varphi(a)\varphi(b)}{\big(\varphi(a) \in \mathfrak{p}\,\big) \vee \big(\varphi(b) \in \mathfrak{p}\,\big)}\ \mathfrak{p} \text{ is prime}}{\big(a \in \varphi^{-1}(\mathfrak{p})\big) \vee \big(b \in \varphi^{-1}(\mathfrak{p})\big)}$$

Thus $\varphi^{-1}(\mathfrak{p})$ is a prime ideal in $A$ and $\varphi^{-1}$ is well-typed.

We now show that $\varphi^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous. Since the set $\mathcal{B} = \{D(f) \mid f \in A\}$ is a basis for the Zariski topology on $\operatorname{Spec} A$ by Proposition 3.1.14, it suffices to show that the preimage of $D(f)$ is open in $\operatorname{Spec} B$. For this we calculate

$$\begin{aligned}\left(\varphi^{-1}\right)^{-1}(D(f)) &= \left\{\mathfrak{p} \in \operatorname{Spec} B \mid \varphi^{-1}(\mathfrak{p}) \in D(f)\right\} = \left\{\mathfrak{p} \in \operatorname{Spec} B \mid f \notin \varphi^{-1}(\mathfrak{p})\right\} = \{\mathfrak{p} \in \operatorname{Spec} B \mid \varphi(f) \notin \mathfrak{p}\} \\ &= D\big(\varphi(f)\big)\end{aligned}$$

which is open in $\operatorname{Spec} B$ also by Proposition 3.1.14. Thus $\varphi^{-1}$ is continuous. ☐

For later use we now provide an important proposition without proof. The basic idea here is that the spectrum of a localization map $\lambda_f : A \to A[f^{-1}]$ for any $f \in A$ picks out the basic open subset $D(f)$.

**Proposition 3.1.17.** *Let $A$ be a commutative ring with identity with $f \in A$ and let $\lambda_f : A \to A[f^{-1}]$ be the localization map. Then the image of the spectral map $\operatorname{Spec} \lambda_f : \operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$ satisfies*

$$\operatorname{Im}(\operatorname{Spec} \lambda_f) = D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}.$$

*In particular, $\operatorname{Spec} \lambda_f$ is monic and homeomorphic to $D(f)$ equipped with its subspace topology.*

We will now record four important facts (without proof for the first two) about the Zariski topology on $\operatorname{Spec} A$. The first two facts will show us that the Zariski topology is quasi-compact[11] and very rarely Hausdorff (in fact, the condition asking for $\operatorname{Spec} A$ to be Hausdorff is a little bit arcane and insane). The final facts we present deal with the closed points in $\operatorname{Spec} A$ (they correspond to maximal ideals) and when certain points can be dense in $\operatorname{Spec} A$.

**Proposition 3.1.18.** *Let $A$ be a commutative ring with identity. Then $\operatorname{Spec} A$ is qusi-compact, i.e., any open cover of $\operatorname{Spec} A$ admits a finite refinement.*

[11] I really mean that arbitrary open covers admit finite refinements. However, because the language "quasi-compact" is embedded deeply in the algebraic geometry literature I have continued to use it. It is also worth noting that the term quasi-compact is used because schemes are very rarely Hausdorff, and classically "compact" also meant that arbitrary open covers of a space admitted finite refinements as well as that the underlying space was Hausdorff.

*Sketch.* The trick here is to take an arbitrary open cover by open sets $D(\mathfrak{a}_i)$ and write it instead as an open cover in terms of the basic opens $D(f_j)$. Once we've written

$$\operatorname{Spec} A = \bigcup_{j \in J} D(f_j)$$

we can show that the above equality holds if and only if the $f_j$ generate the unit ideal in $A$, i.e., if and only if $(1) = (f_j : j \in J)$. However this implies that there is a (necessarily finite) linear combination

$$1 = \sum_{k=1}^{n} a_{j_k} f_{j_k}$$

and so $(1) = (f_{j_k} : 1 \leq k \leq n)$. But then

$$\operatorname{Spec} A = \bigcup_{k=1}^{n} D(f_{j_k})$$

and we're done. □

**Proposition 3.1.19.** *The topological space* $\operatorname{Spec} A$ *is Hausdorff if and only if the ring* $A/\sqrt{(0)}$ *is von Neumann regular.*

For the next two facts we'll need some short observations. First, note that the sets $V(\mathfrak{a})$ are the closed complements of the non-vanishing sets $D(\mathfrak{a})$. As such, it can be shown that for any prime ideal $\mathfrak{p}$, $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, i.e., the sets $V(\mathfrak{p})$ are the closures of the points $\{\mathfrak{p}\}$. In this way we find that every maximal ideal $\mathfrak{m}$ is closed by virtue of

$$V(\mathfrak{m}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{m} \subseteq \mathfrak{p}\} = \{\mathfrak{m}\}$$

and the fact that $\mathfrak{m}$ is maximal.

**Proposition 3.1.20.** *The closed points in* $\operatorname{Spec} A$ *are exactly the points* $\{\mathfrak{m}\}$ *for* $\mathfrak{m} \trianglelefteq A$ *maximal.*

*Proof.* That $\{\mathfrak{m}\} = V(\mathfrak{m})$ whenever $\mathfrak{m}$ is maximal is described just prior to the proposition, so we only need to show that if $\{\mathfrak{p}\} = V(\mathfrak{p})$ then $\mathfrak{p}$ is maximal. For this assume that $\mathfrak{a} \trianglelefteq A$ is an ideal with $\mathfrak{p} \subseteq \mathfrak{a} \subsetneq A$. Now since any ideal is contained in some maximal ideal[12], there is a maximal ideal $\mathfrak{m}$ for which $\mathfrak{a} \subseteq \mathfrak{m}$. However, since $\mathfrak{m}$ is a prime ideal $\mathfrak{p} \subseteq \mathfrak{a} \subseteq \mathfrak{m}$, we must have that $\mathfrak{m} \in V(\mathfrak{p})$. However, since $V(\mathfrak{p}) = \{\mathfrak{p}\}$, we have that $\mathfrak{m} = \mathfrak{p} = \mathfrak{a}$ and so $\mathfrak{p}$ is maximal. □

**Proposition 3.1.21.** *In any integral domain* $A$ *the ideal* $(0)$ *is dense, i.e., the point* $(0)$ *is a generic point for* $\operatorname{Spec} A$*. In particular, no integral domain* $A$ *is* $T_1$ *unless* $A$ *is a field.*

*Proof.* The density of $(0)$ follows from the fact that if $(0)$ is prime then

$$\overline{\{(0)\}} = V((0)) = \{\mathfrak{p} \in \operatorname{Spec} A \mid 0 \in \mathfrak{p}\} = \operatorname{Spec} A.$$

The fact that $\operatorname{Spec} A$ is not $T_1$ unless $A$ is a field follows from the fact that if $A$ is not a field then $A$ has at least one nonzero prime ideal and hence $(0)$ is not maximal. In this case because $(0)$ is not maximal it is not closed and hence $\operatorname{Spec} A$ cannot be $T_1$. □

The last point brings us to the last step we need to make affine schemes, which are to schemes what open subsets of $\mathbb{R}^n$ are to manifolds. It is unfortunately not enough to classify commutative rings just by their spectra. In fact, in complete generality we have

$$\operatorname{Spec} A = \operatorname{Spec}\left(\frac{A}{\sqrt{(0)}}\right)$$

[12]For the constructive mathematicians out there, this is equivalent to the Ultrafilter Principle.

and hence for all fields $K$ and all of the rings $K[x]/(x^n)$ for $n \geq 1$,

$$\operatorname{Spec} K \cong \{*\} \cong \operatorname{Spec}\left(\frac{K[x]}{(x^n)}\right).$$

As such, in order to be able to geometrically distinguish these spaces, we need to equip them with a structure sheaf that will record, for instance when two fields are distinct or when the ring possesses nontrivial nilpotents. Before describing the structure sheaf we equip $\operatorname{Spec} A$ with, however, we will give one sheaf-theoretic lemma that makes defining the sheaf $\mathcal{O}_A$ much easier by allowing us to define $\mathcal{O}_A$ on a basis of $\operatorname{Spec} A$.

## 3.2 Sheaf-Theoretic Background: Extending From Bases to Spaces and the Spectral Structure Sheaf

Begin our sheafy journey by letting $X$ be a topological space and letting $\mathcal{B}$ be a basis of opens for $X$. Because $\mathcal{B} \subseteq \mathbf{Open}(X)$, $\mathcal{B}$ is a poset ordered by inclusion $\subseteq$. We thus regard $\mathcal{B}$ as a category and note that from the inclusion of posets $\mathcal{B} \subseteq \mathbf{Open}(X)$ being monotonic increasing, we get a fully faithful functor $\mathcal{B} \to \mathbf{Open}(X)$ which also op's to a fully faithful functor $\mathcal{B}^{\mathrm{op}} \to \mathbf{Open}(X)^{\mathrm{op}}$. As such defining a sheaf $\mathscr{F} : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ in terms of a functor $F : \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ means two things:

1. First that when we define a presheaf $P : \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ there is a presheaf $\mathscr{P} : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ making the diagram
$$\begin{tikzcd} \mathcal{B}^{\mathrm{op}} \arrow[rr] \arrow[dr, "P"'] & & \mathbf{Open}(X)^{\mathrm{op}} \arrow[dl, "\mathscr{P}"] \\ & \mathbf{Set} & \end{tikzcd}$$
commute up to natural isomorphism.[13]

2. There is a sheaf condition on $\mathcal{B}^{\mathrm{op}}$ which extends to $\mathbf{Open}(X)^{\mathrm{op}}$ in the senese that if $F : \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ is a $\mathcal{B}$-sheaf then there is a unique sheaf $\mathscr{F}$ on $X$ making
$$\begin{tikzcd} \mathcal{B}^{\mathrm{op}} \arrow[rr] \arrow[dr, "F"'] & & \mathbf{Open}(X)^{\mathrm{op}} \arrow[dl, dashed, "\exists!\,\mathscr{F}"] \\ & \mathbf{Set} & \end{tikzcd}$$
commute up to natural isomorphism.

In other words, this asks for a notion of sheaf on the category $\mathcal{B}$ which allows us to descend the sheaf theory of $X$ to the base $\mathcal{B}$.[14]

**Definition 3.2.1** ([73, Page 91])**.** A presheaf on the basis $\mathcal{B}$ is a functor $P : \mathscr{B}^{\mathrm{op}} \to \mathbf{Set}$. A sheaf on the basis $\mathcal{B}$ is a functor $F : \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ such that if $U \in \mathcal{B}$ and

$$U = \bigcup_{i \in I} U_i$$

where $U_i \in \mathcal{B}$ for all $i \in I$:

[13] In the future, we will say that diagrams which commute up to natural isomorphism form invertible 2-cells (in the 2-category $\mathfrak{Kat}_{\mathfrak{ze}}$ of categories) and draw the 2-categorical information. I did not want to lead with this, as it makes the diagrams more intimidating, but it's a nice visual aide to realize that while the diagram doesn't have to commute on the nose as far as names are concerned, it does commute up to a change in labeling.

[14] For those in the know, this essentially asks for $\mathcal{B}$ to be a pretopology on $X$. There is a strong formal analogy in this basis descent technique and the relation ship between a Grothendieck pretopology and a Grothendieck topology on a category $\mathscr{C}$ (with pullbacks).

- If there are $f, g \in F(U)$ such that $F(U \supseteq U_i)(f) = F(U \supseteq U_i)(g)$ for all $i \in I$ then $f = g$.

- If there is an element $(f_i)_{i\in I}$ in $\prod_{i\in I} F(U_i)$ such that for any pair of indices $i, j \in I$ and any $W \subseteq U_i \cap U_j$ with $W \in \mathcal{B}$ we have $F(U_i \supseteq W)(f_i) = F(U_j \supseteq W)(f_j)$ then there exists an $f \in F(U)$ for which $F(U \supseteq U_i)(f) = f_i$ for all $i \in I$.

Equivalently, the diagram

$$\begin{tikzcd} F(U) \arrow[r, "\langle F(U\supseteq U_i)\rangle_{i\in I}"] & \prod\limits_{i\in I} F(U_i) \arrow[r, shift left, "\langle F(U_i\supseteq U_i\cap U_j\supseteq W)\rangle_{i,j\in I}"] \arrow[r, shift right, swap, "\langle F(U_j\supseteq U_i\cap U_j\supseteq W)\rangle_{i,j\in I}"] & \prod\limits_{\substack{i,j\in I;\, W\in\mathcal{B} \\ W\subseteq U_i\cap U_j}} F(W) \end{tikzcd}$$

is an equalizer in **Set**. Finally, the stalk of a $\mathcal{B}$-sheaf at a point $x \in X$ is defined by

$$F_x := \operatorname*{colim}_{x\in U, U\in\mathcal{B}} F(U).$$

**Remark 3.2.2.** Note that the factorization of set inclusions $U_i \supseteq U_i \cap U_j \supseteq W$ appearing in the diagram

$$\begin{tikzcd} F(U) \arrow[r, "\langle F(U\supseteq U_i)\rangle_{i\in I}"] & \prod\limits_{i\in I} F(U_i) \arrow[r, shift left, "\langle F(U_i\supseteq U_i\cap U_j\supseteq W)\rangle_{i,j\in I}"] \arrow[r, shift right, swap, "\langle F(U_j\supseteq U_i\cap U_j\supseteq W)\rangle_{i,j\in I}"] & \prod\limits_{\substack{i,j\in I;\, W\in\mathcal{B} \\ W\subseteq U_i\cap U_j}} F(W) \end{tikzcd}$$

generically only happens in $\mathbf{Open}(X)$. However, because both $\mathcal{B}$ and $\mathbf{Open}(X)$ are posets and because the inclusion $\mathcal{B} \to \mathbf{Open}(X)$ is fully faithful, the factored inclusion $U_i \supseteq U_i \cap U_j \supseteq W$ is equal to the "diagonal" inclusion $U_i \supseteq W$; the long form of writing the arrow $W \subseteq U_i$ or $W \subseteq U_j$ is just a way for us to record that the sets $W$ chosen are contained in the intersection $U_i \cap U_j$ and that we are restricting from either $U_i$ or $U_j$, respectively.

**Theorem 3.2.3** ([73, Theorem 2.7.1],[21, Proposition I-12])**.** *Suppose $\mathcal{B}$ is a base on $X$ and $F : \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ is a $\mathcal{B}$ sheaf. Then there exists a sheaf $\mathscr{F}$ on $X$ giving rise to the invertible 2-cell:*

$$\begin{tikzcd} \mathcal{B}^{\mathrm{op}} \arrow[rr] \arrow[dr, swap, "F"] \arrow[rr, phantom, shift right=4ex, "\Downarrow\cong"] & & \mathbf{Open}(X)^{\mathrm{op}} \arrow[dl, "\mathscr{F}"] \\ & \mathbf{Set} & \end{tikzcd}$$

*Furthermore, the sheaf $\mathscr{F}$ is unique up to unique isomorphism.*

With the Theorem above about generating a sheaf on a space by defining it on a base to the topology, we now proceed to define the structure sheaf on $\operatorname{Spec} A$.

**Definition 3.2.4.** Let $A$ be a commutative ring with identity and consider the base $\mathcal{D} := \{D(f) \mid f \in A\}$ to $\operatorname{Spec} A$. Define the structure presheaf $\mathcal{O}_A : \mathcal{D}^{\mathrm{op}} \to \mathbf{Cring}$ by

$$\mathcal{O}_A(D(f)) := A[f^{-1}]$$

and the morphism $\mathcal{O}_A(D(g)) \to \mathcal{O}_A(D(f))$ for any $D(f) \subseteq D(g)$ to be given by the map

$$A[g^{-1}] \to A[f^{-1}], \frac{a}{g} \mapsto \frac{a}{g}.$$

**Lemma 3.2.5.** *The structure presheaf $\mathcal{O}_A : \mathcal{D}^{\mathrm{op}} \to \mathbf{Cring}$ of Definition 3.2.4 is a $\mathcal{D}$-sheaf on* $\operatorname{Spec} A$*.*

*Proof.* Because $\mathcal{D}$ is closed under intersections by Proposition 3.1.14, it suffices to verify that if we have any covering

$$D(f) = \bigcup_{i \in I} D(f_i)$$

then the diagram

$$A[f^{-1}] \longrightarrow \prod_{i \in I} A[f_i^{-1}] \rightrightarrows \prod_{i,j \in I} A[(f_i f_j)^{-1}]$$

is an equalizer to verify that $\mathcal{O}_A$ is a $\mathcal{D}$-sheaf. Note that we used Proposition 3.1.14 to conclude $D(f_i) \cap D(f_j) = D(f_i f_j)$.

We first verify that the arrow $e : A[f^{-1}] \to \prod_{i \in I} A[f_i^{-1}]$ is monic. For this we will show that $\mathrm{Ker}(e) = (0)$. Let $a \in \mathrm{Ker}(e)$ so that $e(a) = \langle \lambda_{f_i}(a) \rangle_{i \in I} = (0_{A[f_i^{-1}]})_{i \in I}$. Because $\lambda_{f_i}(a) = 0$ we have that for all $i \in I$ there exists some $m_i \in \mathbb{N}$ for which

$$f_i^{m_i} a = 0$$

in $A[f^{-1}]$.

Now observe that by Lemma 3.1.11

$$D(f) = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} D(f_i^{m_i}).$$

It then follows from Propositions 3.1.17 and 3.1.18 that the $f_i^{m_i}$ generate the unit ideal in $A[f^{-1}]$, i.e.,

$$A[f^{-1}] = \left( \frac{f_i^{m_i}}{1} : i \in I \right).$$

In particular, we can find a finite list of the $f_i^{m_i}$, say $f_{i_1}^{m_{i_1}}, \cdots, f_{i_n}^{m_{i_n}}$, and fractions $\alpha_1, \cdots, \alpha_n \in A[f^{-1}]$ such that

$$1_{A[f^{-1}]} = \sum_{k=1}^{n} \alpha_k f_{i_k}^{m_{i_k}}.$$

We then calculate that

$$a = 1_{A[f^{-1}]} a = \left( \sum_{k=1}^{n} \alpha_k f_{i_k}^{m_{i_k}} \right) a = \sum_{k=1}^{n} \alpha_k \left( f_{i_k}^{m_{i_k}} a \right) = \sum_{k=1}^{n} \alpha_k \cdot 0 = 0.$$

Thus $\mathrm{Ker}(e) = (0)$ and hence $e$ is monic and hence injective. In particular, we deduce that for any cover $D(f) = \bigcup_{j \in J} D(h_j)$ we have that $\langle \lambda_{h_j} \rangle_{j \in J}$ is monic and hence injective.[15]

We will now prove that any element

$$\left( \frac{a_i}{f_i^{m_i}} \right)_{i \in I} \in \prod_{i \in I} A[f_i^{-1}]$$

for which

$$\lambda_{f_i f_j} \left( \frac{a_i}{f_i^{m_i}} \right) = \lambda_{f_i f_j} \left( \frac{a_j}{f_j^{m_j}} \right)$$

[15]Monics in **Cring** are automatically injective because there is an algebraic theory $\mathscr{C}$ of commutative unital rings and **Cring** $= \mathscr{C}(\mathbf{Set})$ is the category of set-theoretic models of $\mathscr{C}$ and so the forgetful functor $\mathscr{C}(\mathbf{Set}) \to \mathbf{Set}$ creates all limits. Because monics in **Set** are injective, the same is true for (commutative unital) rings.

for all $i, j \in I$ must factor through $A[f^{-1}]$, i.e., they both arise as the image of some $a \in A[f^{-1}]$.[16] We begin by observing that the condition

$$\frac{a_i}{f_i^{m_i}} = \lambda_{f_i f_j}\left(\frac{a_i}{f_i^{m_i}}\right) = \lambda_{f_i f_j}\left(\frac{a_j}{f_j^{m_j}}\right) = \frac{a_j}{f_j^{m_j}}$$

in $A[(f_i f_j)^{-1}]$ implies from the definition of the localization that there is a power $m_{ij} \in \mathbb{N}$ for which

$$(f_i f_j)^{m_{ij}}(f_j^{m_j} a_i - f_i^{m_i} a_j) = 0$$

for all $i, j \in I$. We simplify our notation now by setting, for all $i \in I$, $f_i^{m_i} := g_i$. Using Lemma 3.1.11 to deduce that

$$D(f_i) = D(f_i^{m_i}) = D(g_i)$$

and then further deducing from Proposition 3.1.14 that for any $i, j \in I$

$$D(f_i f_j) = D(f_i) \cap D(f_j) = D(g_i) \cap D(g_j) = D(g_i g_j)$$

we can, after potentially redefining $m_{ij}$, rewrite the equation $(f_i f_j)^{m_{ij}}(f_j^{m_j} a_i - f_i^{m_i} a_j) = 0$ as

$$(g_i g_j)^{m_{ij}}(g_j a_i - g_i a_j) = 0.$$

We now use the cover

$$D(f) = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} D(g_i)$$

and Proposition 3.1.18 to find a finite refinement

$$D(f) = \bigcup_{k=1}^{n} D(g_{i_k}).$$

We use the finiteness of the cover above to define $m := \max\{m_{i_k i_\ell} \mid 1 \leq k, \ell \leq n\}$. With this choice of constant $m$, we now note that in $A[f^{-1}]$ we have that for any $1 \leq k, \ell \leq n$ the equations below hold in $A[f^{-1}]$:

$$\frac{\dfrac{(g_{i_k} g_{i_\ell})^m (g_{i_\ell} a_{i_k} - g_{i_k} a_{i_\ell}) = 0}{g_{i_k}^m g_{i_\ell}^m (g_{i_\ell} a_{i_k} - g_{i_k} a_{i_\ell}) = 0}}{g_{i_k}^m g_{i_\ell}^{m+1} a_{i_k} = g_{i_k}^{m+1} g_{i_\ell}^m a_{i_\ell}}$$

Now for all $1 \leq k \leq n$ define $h_{i_k} := g_{i_k}^{m+1}$ and $b_{i_k} := g_{i_k} a_{i_k}$. The above equality may now be written as

$$h_{i_\ell} b_{i_k} = h_{i_k} b_{i_\ell}$$

for $1 \leq k, \ell \leq n$. Note also that $D(g_{i_k}) = D(h_{i_k})$ for all $1 \leq k \leq n$. Using that

$$D(f) = \bigcup_{i=1}^{n} D(h_{i_k})$$

[16] There are going to be some tricks in what follows this footnote in the proof that have a very "' feel to them. We'll present some combinatorial simplifications justified by vague isomorphisms of localizations likely not spelled out explicitly but implied by various equalities of open sets in the Zariski topology. I urge you, however, to go through it carefully. It essentially is all about rewriting things in $f_i f_j$-local coordinates to $f_i^{m_i} f_j^{m_j}$-local coordinates and using these to give vast simplifications of the given material; essentially, sometimes the coordinates in which we work at first are painful (or difficult to work with) algebraically, so if we just change our perspective on the space to a more amenable one we can use the algebra to show the geometry that was hiding beneath the combinatorial nightmare of bad coordinates.

find $\alpha_k \in A[f^{-1}]$ for $1 \leq k \leq n$ to generate the equation

$$1_{A[f^{-1}]} = \sum_{k=1}^{n} \alpha_k h_{i_k}$$

in $A[f^{-1}]$. We now define

$$b := \sum_{k=1}^{n} \alpha_k b_{i_k};$$

to show that this is our lift of the element

$$\left(\frac{b_{i_k}}{h_{i_k}}\right)_{1 \leq k \leq n} \in \prod_{k=1}^{n} A[h_{i_k}^{-1}]$$

it suffices to prove that the identity $bh_{i_\ell} = b_{i_\ell}$ holds in $A[f^{-1}]$ to deduce that $\lambda_{h_{i_\ell}}(b) = b_{i_\ell}/h_{i_\ell}$ for any $1 \leq \ell \leq n$. To this end we calculate that

$$bh_{i_\ell} = \left(\sum_{k=1}^{n} \alpha_k b_{i_k}\right) h_{i_\ell} = \sum_{k=1}^{n} \alpha_k (b_{i_k} h_{i_\ell}) = \sum_{k=1}^{n} \alpha_k b_{i_\ell} h_{i_k} = \sum_{k=1}^{n} \alpha_k h_{i_k} b_{i_\ell} = \left(\sum_{k=1}^{n} \alpha_k h_{i_k}\right) b_{i_\ell} = b_{i_\ell}$$

so that

$$\lambda_{h_{i_\ell}}(b) = \frac{b}{1} = \frac{bh_{i_\ell}}{h_{i_\ell}} = \frac{b_{i_\ell}}{h_{i_\ell}},$$

as desired.

We now must show that for any index $j \in I$ with $j \notin \{i_1, \cdots, i_n\}$, $\lambda_{g_j}(b) = a_j/g_j$. For this repeat the process of finding a $c \in A[f^{-1}]$ which restricts to each of the $A[h_i]$ by instead using the cover

$$D(f) = D(h_j) \cup \left(\bigcup_{k=1}^{n} D(h_{i_k})\right)$$

and producing an element

$$c = \gamma_j h_j + \sum_{k=1}^{n} \gamma_{i_k} h_{i_k}$$

for which $\lambda h_{i_k}(c) = b_{i_k}/h_{i_k}$ for all $1 \leq k \leq n$ and $\lambda_{h_j}(c) = b_j/h_j$. However, we then have by construction that

$$\langle \lambda_{h_{i_k}}(c) \rangle_{1 \leq k \leq n} = \langle \lambda_{h_{i_k}}(b) \rangle_{1 \leq k \leq n}$$

so it follows that $b = c$ which completes the proof. $\square$

## 3.3 The Locally Ringed Spaces $\operatorname{Spec} A$ and the Spectrum Functor

**Definition 3.3.1.** The structure sheaf on $\operatorname{Spec} A$ is the sheaf $\mathcal{O}_A$ which extends the $\mathcal{D}$-sheaf $\mathcal{O}_A$ of Lemma 3.2.5.

**Proposition 3.3.2.** *Let $A$ be a commutative ring with identity. Then:*

1. *$\mathcal{O}_A(\operatorname{Spec} A) \cong A$.*
2. *For any $f \in A$, $\mathcal{O}_A(D(f)) \cong A[f^{-1}]$.*
3. *For any $\mathfrak{p} \in \operatorname{Spec} A$, $\mathcal{O}_{A,\mathfrak{p}} = A_{\mathfrak{p}}$.*

*In particular, the pair $(\operatorname{Spec} A, \mathcal{O}_A)$ is a locally ringed space.*

*Proof.* (1): Recall that Proposition 3.1.14 gives $\operatorname{Spec} A = D(1)$. Thus Theorem 3.2.3 and Lemma 3.2.5 let us deduce

$$\mathcal{O}_A(D(1)) \cong A[1^{-1}] \cong A.$$

(2): This is immediate from Theorem 3.2.3 and Lemma 3.2.5.

(3): For this recall that the localization $A_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1}A$. This is a local ring whose unique maximal ideal is comprised of all the $a/f$ where $f \notin \mathfrak{p}$ and $a \in \mathfrak{p}$. Now consider that by construction

$$\mathcal{O}_{A,\mathfrak{p}} := \operatorname*{colim}_{\mathfrak{p} \in U} \mathcal{O}_A(U).$$

Because the colimit above is filtered and the category $\mathcal{D} = \{D(f) \mid f \in A\}$ is cofinal in $\mathbf{Open}(U)$, we also have

$$\mathcal{O}_{A,\mathfrak{p}} \cong \operatorname*{colim}_{\mathfrak{p} \in D(f) \subseteq D(g)} \mathcal{O}_A(D(f)) = \operatorname*{colim}_{\mathfrak{p} \in D(f) \subseteq D(g)} A[f^{-1}].$$

As such, we will have that $\mathcal{O}_{A,\mathfrak{p}} \cong A_{\mathfrak{p}}$ if we can show that $A_{\mathfrak{p}}$ is a colimit of the $A[f^{-1}]$. For this, note that because $\mathfrak{p} \in D(f)$ if and only if $f \notin \mathfrak{p}$, there is a canonical map $A[f^{-1}] \to A_{\mathfrak{p}}$ given by

$$\frac{a}{f^n} \mapsto \frac{a}{f^n}$$

and these necessarily commute with the maps $A[g^{-1}] \to A[f^{-1}]$ induced by $\mathcal{O}_A(D(f) \subseteq D(g))$. Thus let $B$ be a commutative ring with identity such that for any $f, g \in A$ with $D(f) \subseteq D(g)$ and $\mathfrak{p} \in D(f)$ there are morphisms $\varphi_f : A[f^{-1}] \to B$ for which we get a commuting diagram:

$$\begin{tikzcd}
 & B & \\
A[g^{-1}] \arrow[ur, "\varphi_g"] \arrow[rr, "\mathcal{O}_A(D(f)\subseteq D(g))"'] & & A[f^{-1}] \arrow[ul, "\varphi_f"']
\end{tikzcd}$$

By nature of the ring maps above, there is a unique morphism $\varphi_\infty : \operatorname{colim}_{\mathfrak{p} \in D(f) \subseteq D(g)} A[f^{-1}] \to B$ for which the diagrams

$$\begin{tikzcd}
A[g^{-1}] \arrow[dd, "\mathcal{O}_A(D(f)\subseteq D(g))"'] \arrow[dr, "\alpha_g"] \arrow[drr, bend left, "\varphi_g"] & & \\
 & \operatorname*{colim}_{\mathfrak{p} \in D(f) \subseteq D(g)} A[f^{-1}] \arrow[r, dashed, "\exists! \varphi_\infty"] & B \\
A[f^{-1}] \arrow[ur, "\alpha_f"'] \arrow[urr, bend right, "\varphi_f"'] & &
\end{tikzcd}$$

commute for all $f, g$ with $\mathfrak{p} \in D(f)$ and $D(f) \subseteq D(g)$. It also follows by construction that each of the diagrams

$$\begin{tikzcd}
 & A[g^{-1}] \arrow[dd, "\mathcal{O}_A(D(f)\subseteq D(g))"'] \arrow[dr, "\alpha_g"] & \\
A \arrow[ur, "\lambda_g"] \arrow[dr, "\lambda_f"'] & & \operatorname*{colim}_{\mathfrak{p} \in D(f) \subseteq D(g)} A[f^{-1}] \\
 & A[f^{-1}] \arrow[ur, "\alpha_f"'] &
\end{tikzcd}$$

commutes. Call any of the outer edge composites $\lambda : A \to \operatorname{colim}_{\mathfrak{p} \in D(f) \subseteq D(g)} A[f^{-1}]$. We thus define a morphism $\varphi : A \to B$ by $\varphi := \varphi_\infty \circ \lambda$.

We claim that for any $f \notin \mathfrak{p}$, $\varphi(f)$ is a unit in $B$. To this end, we find an inverse for $\varphi(f)$. Begin by considering

$$\varphi(f) = (\varphi_\infty \circ \lambda)(f) = (\varphi_\infty \circ \alpha_f \circ \lambda_f)(f) = (\varphi_f \circ \lambda_f)(f) = \varphi_f\left(\frac{f}{1}\right).$$

Now

$$1_B = \varphi_f(1_{A[f^{-1}]}) = \varphi_f\left(\frac{1}{1}\right) = \varphi_f\left(\frac{f}{f}\right) = \varphi_f\left(\frac{f}{1}\right)\varphi_f\left(\frac{1}{f}\right) = \varphi(f)\varphi_f\left(\frac{1}{f}\right),$$

so $\varphi(f)$ is indeed a unit in $B$ for all $f \notin \mathfrak{p}$.

Because we've shown that $\varphi : A \to B$ sends $f \notin \mathfrak{p}$ to units in $B$, there exists a unique morphism $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B$ making the diagram

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\lambda_{\mathfrak{p}}"'] & & B \\ & A_{\mathfrak{p}} \arrow[ur, dashed, "\exists! \varphi_{\mathfrak{p}}"'] & \end{tikzcd}$$

commute. However, because each $f \notin \mathfrak{p}$ is a unit in $A_{\mathfrak{p}}$ there are also unique morphisms $\lambda_{\mathfrak{p}}^f : A[f^{-1}] \to A_{\mathfrak{p}}$ making the diagrams

$$\begin{tikzcd} A \arrow[rr, "\lambda_{\mathfrak{p}}"] \arrow[dr, "\lambda_f"'] & & A_{\mathfrak{p}} \\ & A[f^{-1}] \arrow[ur, "\lambda_{\mathfrak{p}}^f"'] & \end{tikzcd}$$

commute for all $f \notin \mathfrak{p}$. A routine check shows also that the diagrams

$$\begin{tikzcd} A[g^{-1}] \arrow[dd, "\mathcal{O}_A(D(f)\subseteq D(g))"'] \arrow[dr, "\lambda_{\mathfrak{p}}^g"] & \\ & A_{\mathfrak{p}} \\ A[f^{-1}] \arrow[ur, "\lambda_{\mathfrak{p}}^f"'] & \end{tikzcd}$$

commute for all $f, g \notin \mathfrak{p}$ with $D(f) \subseteq D(g)$. Finally, we also derive using the universal properties of each localization at hand that

$$\varphi_f = \varphi_{\mathfrak{p}} \circ \lambda_{\mathfrak{p}}^f$$

for all $f \notin \mathfrak{p}$. This gives rise to a commuting diagram

$$\begin{tikzcd} A[g^{-1}] \arrow[dd, "\mathcal{O}_A(D(f)\subseteq D(g))"'] \arrow[dr, "\lambda_{\mathfrak{p}}^g"'] \arrow[drr, bend left, "\varphi_g"] & & \\ & A_{\mathfrak{p}} \arrow[r, dashed, "\exists! \varphi_{\mathfrak{p}}"] & B \\ A[f^{-1}] \arrow[ur, "\lambda_{\mathfrak{p}}^f"] \arrow[urr, bend right, "\varphi_f"'] & & \end{tikzcd}$$

and hence shows that

$$A_{\mathfrak{p}} \cong \operatorname*{colim}_{\mathfrak{p} \in D(f) \subseteq D(g)} A[f^{-1}].$$

For the final claim about $(\operatorname{Spec} A, \mathcal{O}_A)$ being a locally ringed space, we simply observe that the nonunits of $A_{\mathfrak{p}}$ take the form

$$A_{\mathfrak{p}} \setminus A_{\mathfrak{p}}^* = \left\{ \frac{a}{f} \;:\; f \notin \mathfrak{p}, a \in \mathfrak{p} \right\}$$

and because the elements in the numerator of each fraction come from the prime ideal $\mathfrak{p}$, the set $A_{\mathfrak{p}} \setminus A_{\mathfrak{p}}^*$ forms an ideal as well. Thus $A_{\mathfrak{p}}$ is a local ring and hence $(\operatorname{Spec} A, \mathcal{O}_A)$ is a locally ringed space. □

We now show that we have a spectrum functor $\operatorname{Spec} : \mathbf{Cring}^{\mathrm{op}} \to \mathbf{LRS}$ by describing how morphisms of rings $\varphi : A \to B$ give rise to morphisms $(\varphi^{-1}, \varphi^\sharp) : (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$ in the category $\mathbf{LRS}$ of locally ringed spaces.

**Proposition 3.3.3.** *Let $A, B$ be commutative rings with identity. Then there is a morphism of locally ringed spaces $(\varphi^{-1}, \varphi^\sharp) : (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$.*

*Proof.* We've already seen in Proposition 3.1.16 that $\varphi^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$ is a continuous morphism. Thus it suffices to define $\varphi^\sharp : \mathcal{O}_A \to (\varphi^{-1})_* \mathcal{O}_B$, and by Theorem 3.2.3 it even suffices to define $\varphi^\sharp$ on the distinguished base of opens $\mathcal{D} = \{D(f) \mid f \in A\}$. Begin by recalling the proof of Proposition 3.1.16 showed that $(\varphi^{-1})^{-1}(D(f)) = D(\varphi(f))$ for any $f \in A$. Thus we have that for any $f \in A$,

$$\left((\varphi^{-1})_* \mathcal{O}_B\right) \big(D(f)\big) = \mathcal{O}_B \left((\varphi^{-1})^{-1}\big(D(f)\big)\right) = \mathcal{O}_B(D(\varphi(f))) = B[\varphi(f)^{-1}].$$

Now fix an $f \in A$ and consider that the image of $f$ under the composite

$$A \xrightarrow{\varphi} B \xrightarrow{\lambda_{\varphi(f)}} B[\varphi(f)^{-1}]$$

is invertible. We then find that there is a unique morphism $\varphi_f : A[f^{-1}] \to B[\varphi(f)^{-1}]$ making the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\lambda_f"'] & B \arrow[d, "\lambda_{\varphi(f)}"] \\ A[f^{-1}] \arrow[r, dashed, "\exists! \varphi_f"'] & B[\varphi(f)^{-1}] \end{tikzcd}$$

commute. Moreover, for any $D(f) \subseteq D(g)$ it follows by from the universal property of localizations that for any $f, g \in A$ with $D(f) \subseteq D(g)$ the diagrams

$$\begin{tikzcd} A[g^{-1}] \arrow[r, "\varphi_g"] \arrow[d, "\mathcal{O}_A(D(f) \subseteq D(g))"'] & B[\varphi(g)^{-1}] \arrow[d, "\mathcal{O}_B(D(\varphi(f)) \subseteq D(\varphi(g)))"] \\ A[f^{-1}] \arrow[r, "\varphi_f"'] & B[\varphi(f)^{-1}] \end{tikzcd}$$

commute. We thus define the sheaf map $\varphi^\sharp : \mathcal{O}_A \to (\varphi^{-1})_* \mathcal{O}_B$ by

$$\varphi^\sharp_{D(f)} := \varphi_f : A[f^{-1}] \to B[\varphi(f)^{-1}]$$

for all $f \in A$.

To show that the pair $(\varphi^{-1}, \varphi^\sharp)$ is a map of locally ringed spaces we now only need to show that the map

$$\mathcal{O}_{A, \varphi^{-1} \mathfrak{p}} \to \mathcal{O}_{B, \mathfrak{p}}$$

is a local map of local rings for any $\mathfrak{p} \in \operatorname{Spec} B$. However, this map is equali to the canonical morphism

$$\varphi_{\mathfrak{p}} : A_{\varphi^{-1} \mathfrak{p}} \to B_{\mathfrak{p}}$$

and it is straightforward to show that $\varphi_{\mathfrak{p}}^{-1}(\mathfrak{m}_{B,\mathfrak{p}}) = \mathfrak{m}_{A, \varphi^{-1} \mathfrak{p}}$. Thus $(\varphi, \varphi^\sharp)$ is a morphism of locally ringed spaces. □

**Corollary 3.3.4.** *Taking spectra gives a functor* Spec : $\mathbf{Cring}^{\mathrm{op}} \to \mathbf{LRS}$ *where* Spec *is defined on objects by* $A \mapsto \operatorname{Spec} A$ *and on morphisms by sending a map* $\varphi \in \mathbf{Cring}(A, B)$ *to* $\operatorname{Spec}\varphi := (\varphi^{-1}, \varphi^{\sharp})$.

We now can define affine schemes, which we'll show are the essential image of the functor Spec described in Corollary 3.3.4. What this comes down to is ultimately showing that any map of locally ringed spaces $\operatorname{Spec} A \to \operatorname{Spec} B$ is the image of a unique morphism $\rho : B \to A$ of rings. From here on, however, we'll start to think of $\operatorname{Spec} A$ as a locally ringed space and not merely a topological space with a poorly behaved topology. As such, we'll start writing

$$\operatorname{Spec} A = (|\operatorname{Spec} A|, \mathcal{O}_A)$$

where $|\operatorname{Spec} A|$ is the underlying topological space and $\mathcal{O}_A$ is the structure sheaf.

**Theorem 3.3.5.** *Let $A$ and $B$ be commutative rings with identity and let* $\varphi = (|\varphi|, \varphi^{\sharp}) : \operatorname{Spec} B \to \operatorname{Spec} A$ *be a morphism of locally ringed spaces. Then there is a unique morphism* $\varphi^{\flat} \in \mathbf{Cring}(A, B)$ *for which* $\varphi = \operatorname{Spec}\alpha$. *In particular, the functor* Spec : $\mathbf{Cring}^{\mathrm{op}} \to \mathbf{LRS}$ *is fully faithful.*

*Proof.* Begin by observing that taking global sections of the morphism of sheaves $\varphi^{\sharp} : \mathcal{O}_A \to |\varphi|_* \mathcal{O}_B$ gives rise to a morphism of rings

$$\varphi^{\flat} := \varphi^{\sharp}_{|\operatorname{Spec} A|} : A \to B$$

(after noting $\mathcal{O}_A(|\operatorname{Spec} A|) \cong A, \mathcal{O}_B(|\operatorname{Spec} B|) \cong B$, and potentially composing along an isomorphism or two based on your taste for strictness with regards to identifying two isomorphic objects). Now, because $\varphi^{\flat}$ is given by the global sections of the sheaf map $\varphi^{\sharp}$ in **LRS**, for any $\mathfrak{p} \in |\operatorname{Spec} B|$ we have a commuting diagram

$$\begin{tikzcd} A \arrow[r, "\varphi^{\flat}"] \arrow[d, "\lambda_{|\varphi|\,\mathfrak{p}}"'] & B \arrow[d, "\lambda_{\mathfrak{p}}"] \\ A_{|\varphi|\,\mathfrak{p}} \arrow[r, "\varphi^{\sharp}_{\mathfrak{p}}"'] & B_{\mathfrak{p}} \end{tikzcd}$$

in **Cring**. Moreover, because the morphism of sheaves $\varphi^{\sharp}$ descends to local morphisms of local rings $\varphi^{\sharp}_{\mathfrak{p}} : \mathcal{O}_{A,|\varphi|\,\mathfrak{p}} \to \mathcal{O}_{B,\mathfrak{p}}$ at each point $\mathfrak{p} \in |\operatorname{Spec} B|$, we deduce that

$$|\varphi|(\mathfrak{p}) = (\varphi^{\flat})^{-1}(\mathfrak{p}).$$

Because this occurs for all $\mathfrak{p} \in |\operatorname{Spec} B|$, $|\varphi| = (\varphi^{\flat})^{-1}$. Now we observe that from the commutativity of the stalk diagrams, we also have induced commuting diagrams

$$\begin{tikzcd} A \arrow[r, "\varphi^{\flat}"] \arrow[d, "\lambda_f"'] & B \arrow[d, "\lambda_{\varphi^{\flat}(f)}"] \\ A_f \arrow[r, "\varphi^{\flat}_f"'] & B_{\varphi^{\flat}(f)} \end{tikzcd}$$

where $f \in A$, $\varphi^{\flat}(f) \in B$, and $A_f = \mathcal{O}_A(D(f))$ while

$$B_{\varphi^{\flat}(f)} = \mathcal{O}_B\left(D(\varphi^{\flat}(f))\right) = \mathcal{O}_B\left((\varphi^{\flat})^{-1}D(f)\right) = \mathcal{O}_B(|\varphi|^{-1}D(f)).$$

In particular, this shows that

$$\varphi^{\flat}_f = \varphi^{\sharp}_{D(f)},$$

and because this holds on the distinguished base $\mathcal{D} = \{D(f) \mid f \in A\}$ to $|\operatorname{Spec} A|$, it follows that it holds on every open of $|\operatorname{Spec} A|$. Thus $\varphi^{\sharp} = (\varphi^{\flat})^{\sharp}$ and so $\operatorname{Spec}\varphi^{\flat} = \varphi$. That this is the unique such morphism of rings making this true is straightforward, as any two such maps $\alpha, \varphi^{\flat}$ for which $\operatorname{Spec}\alpha = \varphi = \operatorname{Spec}\varphi^{\flat}$ must agree at every localization of $A$ and hence at all of $A$. $\square$

## 3.4 Affine Schemes, Schemes, and Examples

**Definition 3.4.1.** An affine scheme is a locally ringed space $X = (|X|, \mathcal{O}_X)$ for which there is a commutative ring with identity $A$ and an isomorphism of locally ringed spaces $\operatorname{Spec} A \cong X$ (cf. Definition 3.1.9 for the topological space and Definition 3.3.1 for the sheaf). We write **AffSch** for the category of affine schemes, i.e., the full subcategory of **LRS** generated by the essential image of the Spec functor.

**Corollary 3.4.2.** *There is an equivalence of categories* $\mathbf{Cring}^{\mathrm{op}} \simeq \mathbf{AffSch}$.

*Proof.* This is simply a restatement of Theorem 3.3.5 together with the fact that the category **AffSch** is defined to be the essential image of Spec in **LRS** (cf. Definition 3.4.1). ☐

We finally have the technology needed to define schemes! These are the "manifolds of commutative rings" that we've vaguely discussed earlier. While manifolds of open subsets of $\mathbb{R}^n$ are easy to describe, we have to be a little more careful what we mean by a manifold of commutative rings. However, using the equivalence of categories $\mathbf{Cring}^{\mathrm{op}} \simeq \mathbf{AffSch}$ allows us to treat each commutative ring as an "open patch" of some object by gluing[17] together affine schemes to build an object which locally looks like a commutative unital ring in the same way a manifold locally looks like $\mathbb{R}^n$.

**Definition 3.4.3.** Let $X = (|X|, \mathcal{O}_X)$ be a locally ringed space. We say that $X$ is a scheme if there is an open cover

$$|X| = \bigcup_{i \in I} |U_i|$$

for which each restricted locally ringed space $U_i := (|U_i|, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine scheme, i.e., for all $i \in I$ there is a commutative ring with identity $A_i$ for which there is a locally ringed space isomorphism

$$\operatorname{Spec} A_i \cong U_i.$$

**Definition 3.4.4.** The category **Sch** of schemes is defined to be the full subcategory of **LRS** generated by taking the objects to be schemes. In particular, if $S$ is a scheme we call the slice category $\mathbf{Sch}_{/S}$ the category of $S$-schemes or a category of relative schemes.

We give one lonely definition that will be useful later on (especially when we discuss finite type schemes).

**Definition 3.4.5.** A scheme $X$ is quasi-compact if and only if the underlying space $|X|$ is quasi-compact.

Before moving on to present myriad examples of schemes concretely, we'll give a description of how to build more schemes by gluing two (or more) known schemes along some open subschemes which are isomorphic to each other (in the case of having at least three schemes we glue along open patches, we also require these comparison isomorphisms to satisfy a cocycle condition; cf. Proposition 3.4.7 for the two scheme case and Proposition 3.4.8 for the arbitrary number of schemes case). This will give us tools to be able to do geometric constructions like build projective space or make constructions that are important in number theory and arithmetic geometry. For instance, if $K$ is a local field with ring of integers $A$ (and uniformizer $\pi$) then the Néron model of $\mathbb{G}_m = \operatorname{Spec} K[x, x^{-1}]$ over $\operatorname{Spec} A$, $\mathcal{N}_{\mathbb{G}_m}$ involves gluing countably many copies of $\operatorname{Spec} A[x, x^{-1}]$. Explicitly, we glue one copy of $\operatorname{Spec} A[x, x^{-1}]$ for each $n \in \mathbb{Z}$ along the open subschemes $\operatorname{Spec} A[x, x^{-1}] \to \operatorname{Spec} A[x, x^{-1}]$ given by $\operatorname{Spec}(x \mapsto \pi^n x)$ (cf. [16, Example 4.1], [11, Example 10.5]).

**Remark 3.4.6.** It is worth showing that if $X$ is a scheme and $|U| \subseteq |X|$ is open then the induced locally ringed space $U = (|U|, \mathcal{O}_X|_U)$ is a scheme as well. However, the scheme $U$ need not be affine, as the scheme $U = \mathbb{A}_K^{2,*}$ (the affine plane (cf. Example 3.4.14 below) over a field $K$ with a punctured origin) is an open subscheme of $\mathbb{A}_K^2$ which is not affine (and in fact is often listed as the standard example of a non-affine open subscheme of an affine scheme).

[17] I like to think of schemes as a papier-mâché of rings.

**Proposition 3.4.7** ([33, Example II.2.3.5]). *Let $X_1$ and $X_2$ be schemes with open subsets $|U_1| \subseteq |X_1|$ and $|U_2| \subseteq |X_2|$. Then if there is an isomorphism of locally ringed spaces $(U_1, \mathcal{O}_{X_1}|_{U_1}) \xrightarrow{(f,f^\sharp)} (U_2, \mathcal{O}_{X_2}|_{U_2})$ there is a scheme $(X, \mathcal{O}_X)$ which glues $X_1$ and $X_2$ along $U_1$ and $U_2$ in the sense that:*

- *$X$ has open subschemes isomorphic to both $X_1$ and $X_2$.*
- *$|X|$ is covered by (a homeomorphic image of) $|X_1| \cup |X_2|$.*
- *$\mathcal{O}_X|_{X_1} \cong \mathcal{O}_{X_1}$ and $\mathcal{O}_X|_{X_2} \cong \mathcal{O}_{X_2}$.*

*Sketch.* We do not prove this explicitly, as it is technical and tedious[18]. Instead, we'll describe how to build the space, its sheaf, and why the pair $(|X|, \mathcal{O}_X)$ is a scheme.

To define the space $|X|$, take the pushout

$$\begin{tikzcd}
& {|U_1|} \arrow[r] \arrow[dl, "|f|"'] \arrow[dd] & {|X_1|} \arrow[dd, "i_1"] \\
{|U_2|} \arrow[dr] & & \\
& {|X_2|} \arrow[r, "i_2"'] & {|X|} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

in the category **Top** of topological spaces. For a concrete description, we realize $|X|$ as the quotient space

$$|X| \cong \left(|X_1| \coprod |X_2|\right)_{/\simeq}$$

where the equivalence relation $\simeq$ is generated by saying that points $x_1 \in |X_1|$ and $x_2 \in |X_2|$ satisfy $x_1 \simeq x_2$ if and only if $x_2 = |f|(x_1)$. The sheaf $\mathcal{O}_X$ is generated by using the Gluing Lemma (cf. Proposition 3.4.7). That $(|X|, \mathcal{O}_X)$ is a scheme is routine by noting that if you have a point $x \in |X|$ which comes from $X_1$ you can argue by using an open affine $V$ in $X_1$ which covers $x_1$ and note that its image in $X$ must also be affine by the pushout condition and fact that the construction of the underlying space of $X$ keeps $V$ open; the same reasoning works for points from the $X_2$ component mutatis mutandis. Finally, that $X$ satisfies the three listed bullet points follows immediately from the construction and the fact that in this case the pushout maps $|i_1| : |X_1| \to |X|$ and $|i_2| : |X_2| \to |X|$ are monic while the topology on $|X|$ is chosen so that $|X_i|$ is open in $|X|$ for $i = 1, 2$. $\square$

This admits a nice generalization to arbitrary families of open subschemes which we present below. The argument is largely the same as the one above, but the index pushing is a little more technical and we omit even the sketch as a result.

**Proposition 3.4.8** ([33, Exercise II.2.12]). *Let $\{X_i \mid i \in I\}$ be a collection of schemes indexed by a set $I$ and assume that for each $i, j \in I$ there are open subschemes $U_{ij} \xrightarrow{\iota_{ij}} X_i$ for which there are isomorphisms of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ for which:*

1. *For every pair $i, j \in I$, $\varphi_{ji} = \varphi_{ij}^{-1}$;*
2. *If $i = j$ then $U_{ij} = X_i$ and $\varphi_{ij} = \mathrm{id}_{X_i}$;*

[18]In fact, it is technical and tedious but important enough I had to do it as an assignment question in my algebraic geometry class. Everyone knows the best place for technical, tedious, but important results is in assignments!

3. *For any triple $i, j, k \in I$,*

$$\varphi_{ij}\left(U_{ij} \cap U_{ik}\right) = U_{ji} \cap U_{jk}$$

*and*[19]

$$\varphi_{ik}|_{U_{ij}\cap U_{ik}} = \varphi_{jk}|_{U_{ij}\cap U_{jk}} \circ \varphi_{ij}|_{U_{ij}\cap U_{jk}},$$

*i.e., the diagram*

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[r, "\varphi_{ij}"] \arrow[d, "\varphi_{ik}"'] & U_{ji} \cap U_{jk} \arrow[d, "\varphi_{jk}"] \\ U_{ki} \cap U_{kj} \arrow[r, equal] & U_{kj} \cap U_{ki} \end{tikzcd}$$

*commutes.*

*Then there is a scheme $X$ with scheme morphisms $\psi_i : X_i \to X$ for which:*

1. *Each $\psi_i$ gives an isomorphism of $X_i$ with an open subscheme $U_i$ of $X$;*
2. *The images of the $X_i$ through the $\psi_i$ cover $X$, i.e.,*

$$|X| = \bigcup_{i\in I} |\psi_i|(|X_i|).$$

3. *For any $i, j \in I$ we have $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$.*
4. *For any pair of indices $i, j \in I$, $\psi_i|_{U_{ij}} = \psi_j|_{U_{ji}} \circ \varphi_{ij}$, i.e., the diagram*

$$\begin{tikzcd} U_{ij} \arrow[rr, "\iota_{ij}"] \arrow[d, "\varphi_{ij}"'] & & X_i \arrow[d, "\psi_i"] \\ U_{ji} \arrow[rr] \arrow[dr, "\iota_{ji}"'] & & X \\ & X_j \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

*commutes.*

**Remark 3.4.9.** The scheme $X$ constructed in Proposition 3.4.8 is the gluing of the $X_i$ along the opens $U_{ij}$. It is a colimit of the diagram induced by including the opens $U_{ij}$ into the schemes $X_i$, but sadly does not allow us to build all colimits in general. However, if we take an arbitrary family of schemes $\{X_i \mid i \in I\}$ but have each $U_{ij}$ and $\varphi_{ij}$ set to be the empty maps then this gives an alternative construction to the coproduct $X = \coprod_{i\in I} X_i$.

**Remark 3.4.10.** It's worth noting that this gluing condition/construction works generally for locally ringed spaces as well, save that we cannot deduce the glued object $X$ is a scheme. However, in this **LRS** context the gluing objects constitute a particular subclass of colimits that record all the "manifold-y" information of how locally ringed spaces may be papier mâché'd together. In fact, when studying functors between categories comprised of locally ringed spaces the class of functors which preserve this papier mâché data also preserves Grothendieck topological data (in the sense of giving isomorphic fundamental groups) under mild assumptions. For details see [76].

[19] Everywhere you see an intersection you should be thinking $U_{ij} \cap U_{jk} = U_{ij} \times_U U_{jk}$ for $U = X_i$. The idea here is that as we restrict our isomorphisms between path that come from three different schemes $X_i$, we should not be able to tell the difference and we should be able to move between the patches by restricting appropriately. This idea is very important in descent theory and is easier to see in effect, in my opinion, when studying pseudofunctors and stacks; cf. [75], for instance.

**Example 3.4.11.** Of course every affine scheme $X$ is a scheme, as $X \cong \operatorname{Spec} A$ is covered by itself.

**Example 3.4.12.** If $X = \{*\}$ is a singleton point and $\mathcal{O} : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Cring}$ is the sheaf

$$\mathcal{O}(U) = \begin{cases} \mathbb{Z}_p & \text{if } U = \{*\}; \\ 0 & \text{if } U = \varnothing; \end{cases}$$

with $\mathcal{O}(X \supseteq \varnothing)$ the unique map $\mathbb{Z}_p \to 0$, then the pair $(X, \mathcal{O})$ is a locally ringed space which is not a scheme.

**Example 3.4.13.** The affine line of a commutative ring $A$ is the scheme

$$\mathbb{A}^1_A := \operatorname{Spec} A[x].$$

When $A$ is a field, the points of $|\mathbb{A}^1_A|$ are given by irreducible polynomials $(f)$ with $\deg(f) \geq 2$ if $A$ is not algebraically closed, the linear polynomials $(x - a)$ for $a \in A$, and the zero ideal $A$. The only closed points are $(x-a)$ and it is in this way that the affine line records "analytic" information (defining "curves" on $A$ by looking at the closed points) as well as the "algebraic" information recorded by the irreducible polynomials $(f)$ with $\deg(f) \geq 2$.

**Example 3.4.14.** Let $A$ be a commutative ring. We then define affine $n$-space over $A$ to be the scheme

$$\mathbb{A}^n_A := \operatorname{Spec} A[x_1, \cdots, x_n].$$

This scheme is affine and, moreover, for any $n \in \mathbb{N}$ with $n \geq 1$

$$\mathbb{A}^n_A \cong \prod_{i=1}^{n} \operatorname{Spec} A[x_i].$$

Let's define projective 1-space of a commutative ring $A$. Classically, projective 1-space of $\mathbb{R}$ is determined by identifying antipodal points of the unit circle. The way we'd like to see this instead is by a more "stereographic" method: instead of identifying antipodal points, we roll up one copy of the real line along the unit circle and put the origin on the south pole, we roll up a second copy of the real line and put the origin on the north pole, and then we glue the resulting space together. It is in this way we'll build $\mathbb{P}^1_A$. First, the way in which we say that a scheme $A$ rolls out the line of $A$ is by considering the affine line $\mathbb{A}^1_A$. To say that we've oriented an origin, we localize at $(x - 0)$, as this determines all the functions which are invertible away from 0. That is, we consider the open subschemes

$$\operatorname{Spec} A[x]_{(x)} = \operatorname{Spec} A[x, x^{-1}] \to \operatorname{Spec} A[x] = \mathbb{A}^1_A\,.$$

To make $\mathbb{P}^1_A$, we need to glue the lines where the origins are placed at the antipodal poles of the circle. This corresponds to gluing the scheme $\mathbb{A}^1_A$ with $\mathbb{A}^1_A$ by pasting the open subschemes $\operatorname{Spec} A[x, x^{-1}]$ along the map

$$\operatorname{Spec}(x \mapsto x^{-1}) : \operatorname{Spec} A[x, x^{-1}] \to \operatorname{Spec} A[x, x^{-1}]$$

That is, $\mathbb{P}^1_A$ is the pushout at the top of the diagram

$$\begin{tikzcd} & \mathbb{P}^1_A & \\ \mathbb{A}^1_A \arrow[ur] & & \mathbb{A}^1_A \arrow[ul] \\ \operatorname{Spec} A[x, x^{-1}] \arrow[u] \arrow[rr, bend left, "\operatorname{Spec}(x \mapsto x^{-1})", "\cong"'] & & \operatorname{Spec} A[x, x^{-1}] \arrow[u] \arrow[ll, bend left, "\operatorname{Spec}(x \mapsto x^{-1})", "\cong"'] \end{tikzcd}$$

Note that the explicit decription of $\mathbb{P}^1_A$ is given as follows by using Proposition 3.4.7: the underlying space $|\mathbb{P}^1_A|$ is the pushout/gluing of $\mathbb{A}^1_A$ with $\mathbb{A}^1_A$ along the standard inclusion $|\operatorname{Spec} A[x, x^{-1}]|$ on one leg with the standard inclusion precomposed with the automorphism $\operatorname{Spec}(x \mapsto x^{-1})$ on the other leg. The sheaf $\mathcal{O}_{\mathbb{P}^1}$ is the sheaf on $|\mathbb{P}^1_A|$ by the Gluing Lemma (cf. Propositions 3.4.8 and 3.4.7). We can generalize this approach to define a projective $n$-space using Proposition 3.4.8, but we do not do that here; it is technical, but you can think of it as gluing $\operatorname{Spec} A[x_0, \cdots, x_n]$ along the isomorphisms

$$\operatorname{Spec}(x_i \mapsto x_j^{-1}) : \operatorname{Spec} A[x_0^{\pm 1}, \cdots, x_n^{\pm 1}] \to \operatorname{Spec} A[x_0^{\pm 1}, \cdots, x_n^{\pm 1}]$$

for $0 \leq i, j \leq n$ and $i \neq j$. Note also that none of the schemes $\mathbb{P}^n_A$ are affine for $n \geq 1$.

**Example 3.4.15** ("The" smallest non-affine scheme)**.** Let $p \in \mathbb{Z}$ be prime and consider the topological space $X = \{\eta, \zeta_1, \zeta_2\}$ equipped with topology

$$\mathbf{Open}(X) = \{\varnothing, \{\eta\}, \{\zeta_1, \eta\}, \{\zeta_2, \eta\}, X\}.$$

Define the structure sheaf $\mathcal{O}_X : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Cring}$ by

$$\mathcal{O}_X(U) := \begin{cases} \mathbb{Z}_p & \text{if } U = X, \{\zeta_1, \eta\}, \{\zeta_2, \eta\}; \\ \mathbb{Q}_p & \text{if } U = \{\eta\}; \\ 0 & \text{if } U = \varnothing; \end{cases}$$

with the restriction maps of the inclusions $\{\zeta_1, \eta\}, \{\zeta_2, \eta\} \subseteq X$ given by the identity, the restriction maps of the inclusions $\{\eta\} \subseteq \{\zeta_1, \eta\}, \{\zeta_2, \eta\}, X$ given by the injection $\mathbb{Z}_p \to \mathbb{Q}_p$ and the restrictions to the empty set the zero map. To see this is a scheme we simply note that $\zeta_1$ is covered by $U_1 = \{\zeta_1, \eta\}$ and $\zeta_2$ is covered by $U_2 = \{\zeta_2, \eta\}$. Both open subspaces induce locally ringed spaces $U_1 \cong \operatorname{Spec} \mathbb{Z}_p \cong U_2$. Finally, we can also cover $\eta$ by either of $U_1$ or $U_2$, but if you want to get fancy you can cover $\eta$ by the open $\{\eta\} = U_3$ and note the induced locally ring space satisfies $U_3 \cong \operatorname{Spec} \mathbb{Q}_p$.

That this is not affine follows by observing that because $\mathcal{O}_X(X) = \mathbb{Z}_p$, if $X$ were affine then $X \cong \operatorname{Spec} \mathbb{Z}_p$. However, since $|\operatorname{Spec} \mathbb{Z}_p| = \{s, \eta\}$ where the point $\eta$ is open and the point $\zeta$ is closed, $|X| \ncong |\operatorname{Spec} \mathbb{Z}_p|$. Because isomorphisms of locally ringed spaces are pairs $(|f|, f^\sharp)$ where $|f|$ is a homeomorphism and $f^\sharp$ is an isomorphism of sheaves, it follows that $X \ncong \operatorname{Spec} \mathbb{Z}_p$ and hence that $X$ is not affine.

As a final example to consider, we will construct an affine line with doubled origin. This will be a scheme $X$ which is simultaneously non-affine (for largely the same reason the scheme in Example 3.4.15 is not affine)

**Example 3.4.16** (The affine line with a doubled origin)**.** Let $K$ be a field (algebraically closed if this makes you more comfortable) and define $X_1, X_2 = \mathbb{A}^1_K$. Consider the open subscheme $U_1, U_2 = \operatorname{Spec} K[x]_{(x)} = \operatorname{Spec} K[x, x^{-1}]$ and the isomorphism $f = \operatorname{Spec}(x \mapsto x)$. We then glue $X_1$ to $X_2$ along the opens $U_1$ and $U_2$ to produce the scheme $X$ visualized in Figure 3.1. Note that the idea here is that any two pieces of $X_1$ and $X_2$ which agree off of the origin $(x)$ get identified, while the origins themselves are allowed to remain distinct. This is a non-affine scheme which represents an affine line with a doubled origin and is our second example of a non-separated scheme.

**Example 3.4.17.** Let $A$ be any commutative ring with identity and recall that if $(a_{ij})$ is an $n \times n$ matrix in $A$ and if $S_n$ is the permutation group on $n$ letters,

$$\det(a_{ij}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

The above equation is a polynomial equation in the entries $a_{ij}$, so we can actually make standard scheme-theoretic incarnations of matrix Lie groups based on these determinant conditions. For instance, the scheme $\mathrm{SL}_{n,A}$ which describes the special linear group of $n \times n$ matrices over $A$ is the scheme

$$\mathrm{SL}_{n,A} := \operatorname{Spec}\left( \frac{A[x_{ij} : 1 \leq i, j \leq n]}{(\det(x_{ij}) - 1)} \right)$$

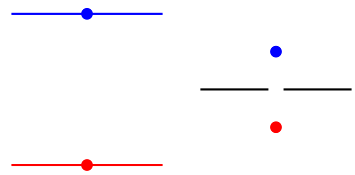

Figure 3.1: The schemes $X_1$ and $X_2$ on the left in red and blue, respectively. Note that the origins $U_1^c = \{(x)\}$ and $U_2^c = \{(x)\}$ are drawn as bold dots. The scheme $X$ is on the right where glued points are shown in black and the non-glued points remain in red and blue, respectively.

while the scheme $\mathrm{GL}_{n,A}$ which paramtrizes the general linear group of $n \times n$ matrices is the scheme

$$\mathrm{GL}_{n,A} := \operatorname{Spec}\left(\frac{A[y, x_{ij} : 1 \leq i,j \leq n]}{\det(x_{ij})y - 1}\right).$$

We now will describe how that every surjection of commutative rings $A \to A/\mathfrak{a}$ gives rise to a closed embedding of schemes. While we have seen (or have essentially seen) by now that $\operatorname{Spec} A[f^{-1}] \cong (D(f), \mathcal{O}_A|_{D(f)})$ for any $f \in A$, this is the flip side of things: the spectrum of the surjection $\operatorname{Spec}(\pi_{\mathfrak{a}}) : A \to A/\mathfrak{a}) : \operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$ picks out the subspace $V(\mathfrak{a})$ and induces a regular epimorphism of sheaves of rings $\mathcal{O}_A \to (\operatorname{Spec}\pi_{\mathfrak{a}})_* \mathcal{O}_{A/\mathfrak{a}}$.

**Definition 3.4.18.** A closed immersion of schemes $V \to X$ is a morphism $i \in \mathbf{Sch}(V, X)$ for which $|i|$ is a homeomorphism of $|V|$ to a closed subspace of $|X|$ and such that there is a surjective map of sheaves of rings

$$\mathcal{O}_X \to i_* \mathcal{O}_V.$$

**Proposition 3.4.19.** *Let $A$ be a commutative ring and let $\mathfrak{a}$ be an ideal of $A$ with canonical surjection $\pi_{\mathfrak{a}} : A \to A/\mathfrak{a}$. Then the map $\operatorname{Spec}(\pi_{\mathfrak{a}})$ is a closed immersion.*

*Proof.* A routine calculation shows that $|\operatorname{Spec}(\pi_{\mathfrak{a}})|$ is injective (and hence a homeomorphism on to its image) and that

$$\begin{aligned}
\operatorname{Im}(|\operatorname{Spec}\pi_{\mathfrak{a}}|) &= \left\{\mathfrak{p} \in |\operatorname{Spec} A| \;:\; \exists\, \mathfrak{q} \in \left|\operatorname{Spec}\left(\frac{A}{\mathfrak{a}}\right)\right| .\ \mathfrak{p} = |\operatorname{Spec}\pi_{\mathfrak{a}}|(\mathfrak{q})\right\} \\
&= \left\{\mathfrak{p} \in |\operatorname{Spec} A| \;:\; \exists\, \mathfrak{q} \in \left|\operatorname{Spec}\left(\frac{A}{\mathfrak{a}}\right)\right| .\ \mathfrak{p} = \pi_{\mathfrak{a}}^{-1}(\mathfrak{q})\right\} \\
&= \{\mathfrak{p} \in |\operatorname{Spec} A| : \mathfrak{a} \subseteq \mathfrak{p}\} = V(\mathfrak{a}).
\end{aligned}$$

Finally to check the surjectivity of the map $\pi_{\mathfrak{a}}^{\sharp}$ we use Proposition 2.1.9 and show that for any $\mathfrak{p} \in A$, the local map

$$\mathcal{O}_{A,\mathfrak{p}} \to \big((\pi_{\mathfrak{a}})_* \mathcal{O}_{A/\mathfrak{a}}\big)_{\mathfrak{p}}$$

is surjective. For this note that if $\mathfrak{p} \notin V(\mathfrak{a})$ there is nothing to show, as $((\pi_{\mathfrak{a}})_* \mathcal{O}_{A/\mathfrak{a}})_{\mathfrak{p}} \cong 0$ and the canonical map $A_{\mathfrak{p}} \to 0$ is always surjective.[20] Fix $\mathfrak{p} \in V(\mathfrak{a})$ and find $\mathfrak{q} \in |\operatorname{Spec} A/\mathfrak{a}|$ for which $\mathfrak{p} = \pi_{\mathfrak{a}}^{-1}(\mathfrak{q})$. We then have that the map

$$\mathcal{O}_{A,\mathfrak{p}} \to \mathcal{O}_{A/\mathfrak{a},\mathfrak{q}}$$

[20] This remark uses that $V(\mathfrak{a})$ is closed in potentially non-obvious ways, as this fails for open sets. If $\mathfrak{p} \notin D(\mathfrak{a})$ and if $s_{\mathfrak{p}} \in i_* \mathcal{O}_{V(\mathfrak{a}),\mathfrak{p}}$ then there is an open for which $s_{\mathfrak{p}}$ arises as the germ of some $s \in i_* \mathcal{O}_U$ for an open $U$ around $\mathfrak{p}$. However since $D(\mathfrak{a})$ is open by virtue of $V(\mathfrak{a})$ being closed, this germ $s_{\mathfrak{p}}$ is also the germ of $s$ when restricted to $U \cap D(\mathfrak{a})$. But then $i_* \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}}(D(\mathfrak{a}) \cap U) = \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}}(i^{-1}(D(\mathfrak{a}) \cap U)) = \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}}(\varnothing) = 0$. My apologies for putting this argument in the footnotes, as it is pretty important, but I want you to be warned as to why we could make this claim for closed sets and not for open sets.

is naturally isomorphic to the map

$$A_{\mathfrak{p}} \xrightarrow{(\pi_{\mathfrak{a}})_{\mathfrak{p}}} \left(\frac{A}{\mathfrak{a}}\right)_{\mathfrak{q}}$$

which is surjective. $\square$

**Example 3.4.20.** Let $K$ be a field and consider the scheme $\mathbb{A}^1_K$. Morphisms $\operatorname{Spec} K \to \mathbb{A}^1_K$ are all closed immersions realized from the fact that $K$ is a quotient field of $K[x]$. In fact, we have that the immersions $\operatorname{Spec} K \to \mathbb{A}^1_K$ are in natural bijection with the points of $K$, as each such immersion is the spectrum of an evaluation map $\mathrm{ev}_a : K[x] \to K$ given by $f(x) \mapsto f(a)$ for $a \in K$.[21] We also find that there are closed immersions $\operatorname{Spec} L \to \mathbb{A}^1_K$ induced by the quotients $K[x]/(f) \cong L$ for an irreducible polynomial $f$ over $K$.

We now introduce another important class of monics in **Sch** which is topologically dual to the closed immersions: open immersions. These immersions of schemes are to manifolds what inclusions of open patches are. They are those morphisms which involve on the topological side an inclusion of an open set but on the sheaf theoretic side involve recognizing that the sheaf on the included space is in essence a restriction of the structure sheaf on the larger space.

**Definition 3.4.21.** A morphism of schemes (respectively locally ringed spaces) $f : X \to Y$ is an open immersion if the map $|f| : |X| \to |Y|$ is a homeomorphism of $|X|$ onto an open subspace of $|Y|$ for which there is an isomorphism of sheaves $|f|^{-1}\mathcal{O}_Y \xrightarrow{\cong} \mathcal{O}_X$ on $|X|$.

**Remark 3.4.22.** By the adjunction calculus of the push-pull $|f|^{-1} \dashv |f|_*$, asking for $|f|^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ to be an isomorphism is equivalent to asking that for any open $|U| \subseteq |Y|$ contained in (the image of) $|X|$, the map of rings $\mathcal{O}_X(|U|) \to |f|_*\mathcal{O}_Y(|U|)$ is an isomorphism.

**Example 3.4.23.** If $K$ is any field then the open point $(0)$ of $\mathbb{A}^1_K$ corresponds to an open immersion $\operatorname{Spec} K(x) \to \operatorname{Spec} K[x]$ realized by the localization of $K[x]$ at the multiplicative set $K[x] \setminus \{0\}$.

We now give a quick proof of the fact that isomorphisms are open immersions. While potentially straightforward and obvious (isomorphisms are given by $f = (|f|, f^\sharp)$ where $|f|$ is a homeomorphism and $f^\sharp$ is an isomorphism of sheaves), but it is a surprisingly important result. It not only is crucial in proving structural properties of étale morphisms, but also allows us to deduce that the étale pretopology actually is a pretopology and that there is an inclusion of sites $(X, \mathrm{Zar}) \to (X, \acute{\mathrm{E}}\mathrm{t})$ for any scheme $X$. We will see this later on, however, so for the time being this is just a cute test of the definition.

**Proposition 3.4.24.** *Let $f : X \to Y$ be an isomorphism of locally ringed spaces. Then $f$ is an open immersion.*

*Proof.* First, since $|f|$ is a homeomorphism, it is an open morphism. Thus for each open set $U \subseteq |X|$, $f(U) \subseteq |f|(|X|) = |X|$ is an open set as well with $f(U) \cong U$. Moreover, we find that the costructure map $f^\flat : f^{-1}\mathcal{O}_X \to \mathcal{O}_X$ is an isomorphism as well because the structure map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_X$ is an isomorphism of sheaves. Thus $f$ is an open immersion. $\square$

We conclude this subsection with a convenient result on the stalks of both open and closed immersions that we'll find .

**Proposition 3.4.25.** *Let $f : X \to Y$ be an open immersion or a closed immersion of schemes. Then for any $x \in |X|$,*

$$\mathcal{O}_{X,x} \cong (f_*\mathcal{O}_X)_x.$$

[21] Another reason is that the identity functor on $K \downarrow$ **Cring** is represented by $K[x]$, so this is the equivalence $\mathbf{Cring}^{\mathrm{op}} \cong$ **AffSch** at work again.

*Proof.* In either case it suffices to assume that $|f|$ is the inclusion function of $|X|$ into $|Y|$.In the case that $f$ is an open immersion our proposition is immediate because the category $\mathbf{Open}(X)^{\mathrm{op}}$ is cofinal in the category $\mathbf{Open}(Y)^{\mathrm{op}}$ so

$$\mathcal{O}_{X,x} = \operatorname*{colim}_{\substack{x \in U \\ U \subseteq X \text{ open}}} \mathcal{O}_X(U) \cong \operatorname*{colim}_{\substack{x \in V \\ V \subseteq Y \text{ open}}} \mathcal{O}_X(f^{-1}(V)) = (f_* \mathcal{O}_X)_x.$$

Similarly, if $f$ is a closed immersion then

$$\operatorname*{colim}_{\substack{x \in V \\ V \subseteq Y \text{ open}}} (f_* \mathcal{O}_X)(V) = \operatorname*{colim}_{\substack{x \in V \\ V \subseteq Y \text{ open}}} \mathcal{O}_X(f^{-1}(V)) = \operatorname*{colim}_{\substack{x \in U \\ U \subseteq X \text{ open}}} \mathcal{O}_X(U) = \mathcal{O}_{X,x}\,.$$

□

**Proposition 3.4.26.** *Consider a pullback diagram of schemes:*

$$\begin{tikzcd} Z \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & S \end{tikzcd}$$

*Then if $f$ is an open (respectively closed) immersion, so is $p_2$.*

## 3.5 Properties of Schemes, Relative Schemes, and the Category of (Relative) Schemes

We now move to discuss some basic properties that schemes, their morphisms, and the category of schemes. A technical but important result is that the category **Sch** of schemes admits a terminal object and all pullbacks. To this first end we'll show that the category **Sch** has a terminal object which will make the number theorists in the crowd happy. For this we'll prove the existence of a terminal object by classifying the maps $X \to S$ for $S$ an affine scheme.

**Theorem 3.5.1** ([21, Theorem I-40])**.** *For any scheme $X$ and affine scheme* $\operatorname{Spec} A$ *there is a natural isomorphism*

$$\mathbf{Sch}(X, \operatorname{Spec} A) \cong \mathbf{Cring}(A, \mathcal{O}_X(|X|))$$

*given by*

$$(f, f^\sharp) \mapsto f^\sharp_{\operatorname{Spec} A}.$$

*Proof.* Proving that the assignment above is natural is straightforward, so we instead show that the assignment $(f, f^\sharp) \mapsto f^\sharp_{\operatorname{Spec} A}$ is an isomorphism by exhibit its inverse . Begin by fixing a map of rings $\varphi \in \mathbf{Cring}(A, \mathcal{O}_X(|X|))$ and a point $x \in |X|$. We then have that since the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_{X,x}$ is a prime ideal, writing $\lambda_x : \mathcal{O}_X(|X|) \to \mathcal{O}_{X,x}$ for the colimit map, the preimage $(\lambda_x \circ \varphi)^{-1}(\mathfrak{m}_x)$ is a prime ideal of $A$. Thus we define our function $|\varphi| : |X| \to |\operatorname{Spec} A|$ by

$$x \mapsto (\lambda_x \circ \varphi)^{-1}(\mathfrak{m}_x).$$

To show that $|\varphi|$ is continuous we will use Zariski descent.[22] Let $f \in A$ and consider the preimage $|\varphi|^{-1}D(f)$. We then calculate

$$\begin{aligned} |\varphi|^{-1}D(f) &= \{x \in |X| \ : \ |\varphi|(x) \in D(f)\} = \{x \in |X| \ : \ f \notin (\lambda_x \circ \varphi)^{-1}(\mathfrak{m}_x)\} \\ &= \{x \in |X| \ : \ \lambda_x(\varphi(f)) \notin \mathfrak{m}_x\}. \end{aligned}$$

[22]In [21] they just say that "it is easy to see" $|\varphi|$ is continuous. While it is not overly difficult, per se, I did think that it's technical enough and a good place to introduce how to do affine descent to show continuity in a scheme. Thus I've spelled this out completely in the interest of showing you how to work with schemes in a more day-to-day setting.

Now write

$$X = \bigcup_{x \in |X|} V_x$$

for each $V_x$ an affine open subscheme of $X$ with $x \in |V_x|$. Since we are interested in the stalks at $x$ in the calculation of $|\varphi|^{-1}D(f)$, if we can show that when restricted to each $V_y$ for which $x \in V_y$ the corresponding set is open affine-locally, then we will have our desired open preimage. For this note that if $x \in V_y$, because there is a commutative ring with identity $A_y$ for which $V_y \cong \operatorname{Spec} A_y$ we can associate $x$ to a prime ideal $\mathfrak{p}_x^y \in |\operatorname{Spec} A_y|$. However, we now find that the condition $\lambda_x(\varphi(f)) \notin \mathfrak{m}_x$ can be rephrased within $V_y$ as $\lambda_{\mathfrak{p}_x^y}(\varphi(f)) \notin \mathfrak{m}_{\mathfrak{p}_x^y}$. Because in $V_y$ this is a localization away from $\mathfrak{p}_x^y$, we see that $\mathfrak{m}_{\mathfrak{p}_x^y} = (A_y \setminus \mathfrak{p}_x^y)^{-1}\,\mathfrak{p}_x^y$. Moreover, using the definition of the local ring $\mathcal{O}_{X,x}$ implies the diagram

$$\begin{tikzcd}
A \arrow[r, "\lambda_x \circ \varphi"] \arrow[d, "\varphi"'] & \mathcal{O}_{X,x} \\
\mathcal{O}_X(|X|) \arrow[ur, "\lambda_x"] \arrow[r, "\mathcal{O}_X(X \supseteq V_y)"'] & \mathcal{O}_X(|V_y|) \arrow[u, "\lambda_x^y"']
\end{tikzcd}$$

commutes. Writing $\varphi|_{A_y} : A \to A_y$ for the composite

$$A \xrightarrow{\varphi} \mathcal{O}_X(|X|) \xrightarrow{\mathcal{O}_X(X \supseteq V_y)} \mathcal{O}_X(|V_y|) \to \cong A_y$$

we find that in $V_y$, $\mathfrak{p}_x^y \in D(\varphi(f)|_{A_y})$. In particular, it follows from this construction that in each $V_y$ our preimage takes the form

$$D(\varphi(f)|_{A_y}),$$

which is open in $V_y$. Doing this for all $x, y \in |X|$ we get that

$$\begin{aligned}
|\varphi|^{-1}D(f) &= \{x \in |X| \ : \ \lambda_x(\varphi(f)) \notin \mathfrak{m}_x\} = \{x \in |X| \ : \ \exists y \in |X|.\, x \in |V_y| \,\text{and}\, \lambda_{\mathfrak{p}_x^y}(\varphi(f)|_{A_y}) \notin \mathfrak{m}_x\} \\
&= \bigcup_{x \in |X|} \bigcup_{\substack{y \in |X| \\ x \in |V_y|}} D(\varphi(f)|_{A_y}),
\end{aligned}$$

which is a union (of a union, technically) of open sets in $|X|$ and hence open. Thus $|\varphi|$ is continuous.

We now define the sheaf map $\varphi^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to |\varphi|_* \mathcal{O}_X$ over $|\operatorname{Spec} A|$ by once again using Theorem 3.2.3. To this end fix $f \in A$ and observe that the image of $\varphi(f)$ in $\mathcal{O}_X(|\varphi|^{-1}D(f))$ is a unit[23]; as such there is a unique factorization of ring maps:

$$\begin{tikzcd}
\mathcal{O}_X(|X|) \arrow[rr, "\mathcal{O}_X(X \supseteq |\varphi|^{-1}D(f))"] \arrow[dr, "\lambda_{\varphi(f)}"'] & & \mathcal{O}_X(|\varphi|^{-1}D(f)) \\
& \mathcal{O}_X(|X|)[\varphi(f)^{-1}] \arrow[ur, dashed, "\exists! \alpha_{\varphi(f)}"'] &
\end{tikzcd}$$

It then follows that the image of $f \in A$ in the composition

$$A \xrightarrow{\varphi} \mathcal{O}_X(|X|) \xrightarrow{\lambda_{\varphi(f)}} \mathcal{O}_X(|X|)[\varphi(f)^{-1}] \xrightarrow{\alpha_{\varphi(f)}} \mathcal{O}_X(|\varphi|^{-1}D(f))$$

[23] To see this you can argue once again via Zariski descent and the observation that in any open affine of $X$ intersected with $|\varphi|^{-1}D(f)$, the image of $\varphi(f)$ restricted to that same affine piece is a unit.

is a unit. Thus we induce our map $A[f^{-1}] \to \mathcal{O}_X(|\varphi|^{-1}D(f))$ as the composite along the bottom edge of the commuting diagram below

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\lambda_f"'] & \mathcal{O}_X(|X|) \arrow[d, "\lambda_{\varphi(f)}"] \arrow[dr, "\mathcal{O}_X(X \supseteq |\varphi|^{-1}D(f))"] & \\
 & & \mathcal{O}_X(|\varphi|^{-1}D(f)) \\
A[f^{-1}] \arrow[r, "\varphi[f^{-1}]"'] & \mathcal{O}_X(|X|)[\varphi(f)^{-1}] \arrow[ur, "\alpha_f"'] &
\end{tikzcd}$$

i.e., we define

$$\varphi^\sharp_{D(f)} := \alpha_f \circ \varphi[f^{-1}].$$

That this is a natural transformation is a routine check using the universal properties of localizations, so we omit it here. By localizing all the way to any $\mathfrak{p} \in |\operatorname{Spec} A|$ where $\mathfrak{p} = |\varphi|(x)$ for $x \in X$, we find that the diagrams above extend to give local maps of local rings

$$\mathcal{O}_{A,\mathfrak{p}} \to \mathcal{O}_{X,x}$$

so $(|\varphi|, \varphi^\sharp)$ is indeed a morphism of schemes.

We now show that the assignments $\Phi : \mathbf{Cring}(A, \mathcal{O}_X(|X|)) \to \mathbf{Sch}(X, \operatorname{Spec} A)$ given by $\varphi \mapsto (|\varphi|, \varphi^\sharp)$ and $\Psi : \mathbf{Sch}(X, \operatorname{Spec} A) \to \mathbf{Cring}(A, \mathcal{O}_X(|X|))$ given by $(|f|, f^\sharp) \mapsto f^\sharp_{\operatorname{Spec} A}$ are mutually inverse. Not that it follows by construction that for any $\varphi \in \mathbf{Cring}(A, \mathcal{O}_X(|X|))$,

$$(\Psi \circ \Phi(\varphi)) = \varphi^\sharp_{\operatorname{Spec} A} = \varphi.$$

Alternatively, if $f = (|f|, f^\sharp) : X \to \operatorname{Spec} A$ is a morphism of schemes we get

$$(\Phi \circ \Psi)(f) = (|f^\sharp_{\operatorname{Spec} A}|, (f^\sharp_{\operatorname{Spec} A})^\sharp).$$

Because the local maps of local rings induced by $f^\sharp_{\operatorname{Spec} A}$ take the form

$$\mathcal{O}_{A,(\lambda_x \circ f^\sharp_{\operatorname{Spec} A})^{-1}(\mathfrak{m}_x)} \to \mathcal{O}_{X,x}$$

and $(|f|, f^\sharp)$ is a morphism of locally ringed spaces, it follows from the fact that $(f^\sharp_x)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{|f|(x)}$ and the commuting diagram

$$\begin{tikzcd}
A \arrow[r, "f^\sharp_{\operatorname{Spec} A}"] \arrow[d, "\lambda_{|f|(x)}"'] & \mathcal{O}_X(|X|) \arrow[d, "\lambda_x"] \\
A_{|f|(x)} \arrow[r, "f^\sharp_x"'] & \mathcal{O}_{X,x}
\end{tikzcd}$$

that $\lambda^{-1}_{|f|(x)}(\mathfrak{m}_{|f|(x)}) = (\lambda_x \circ f^\sharp_{\operatorname{Spec} A})^{-1}(\mathfrak{m}_x)$. Thus we conclude for all $x \in X$ that

$$|f|(x) = (\lambda_x \circ f^\sharp_{\operatorname{Spec} A})^{-1}(\mathfrak{m}_x)$$

and hence that $|f| = |f^\sharp_{\operatorname{Spec} A}|$. It follows similarly by checking against localizations at the basic opens $D(f)$ that $f^\sharp_{D(f)} = (f^\sharp_{\operatorname{Spec} A})^\sharp_{D(f)}$. This allows us to deduce that

$$\left(|f^\sharp_{\operatorname{Spec} A}|, (f^\sharp_{\operatorname{Spec} A})^\sharp\right) = (|f|, f^\sharp),$$

which proves the theorem. $\square$

The isomorphism

$$\mathbf{Sch}(X, \operatorname{Spec} A) \cong \mathbf{Cring}(A, \mathcal{O}_X(|X|))$$

of Theorem 3.5.1 looks suspiciously similar to an adjunction isomorphism. If we rephrase this correctly we can see that not only is this exactly the case, but also that the category **AffSch** is a reflective subcategory of the category **Sch**.

**Corollary 3.5.2.** *The category of affine schemes is a reflective subcategory of the category of schemes.*

*Proof.* Let $A$ be a commutative ring with identity and let $X$ be a scheme. Note that from the equivalence $\mathbf{Cring}^{\mathrm{op}} \simeq \mathbf{AffSch}$ we find that from Theorem 3.5.1

$$\mathbf{Sch}(X, \operatorname{Spec} A) \cong \mathbf{Cring}(A, \mathcal{O}_X(|X|)) = \mathbf{Cring}^{\mathrm{op}}(\mathcal{O}_X(|X|), A) \cong \mathbf{AffSch}(\operatorname{Spec} \mathcal{O}_X(|X|), \operatorname{Spec} A).$$

From this we see that the inclusion functor $\mathbf{AffSch} \to \mathbf{Sch}$ is right adjoint to the functor $\operatorname{Spec} \mathcal{O}_X(-)$ which sends a scheme to its affinization. $\square$

Another corollary of Theorem 3.5.1 is that we can deduce the existence of the terminal object in **Sch**.

**Corollary 3.5.3.** *The affine scheme* $\operatorname{Spec} \mathbb{Z}$ *is a terminal object in* **Sch**.

*Proof.* Because the integers $\mathbb{Z}$ are the initial object in the category of unital rings we find for any scheme $X$

$$\mathbf{Sch}(X, \operatorname{Spec} \mathbb{Z}) \cong \mathbf{Cring}(\mathbb{Z}, \mathcal{O}_X(|X|)) \cong \{*\}.$$

$\square$

We now will discuss the limits and colimits that exist within the categories of schemes. We won't focus too much on colimits in **AffSch**, because they all exist as a formal consequence of limits and colimits existing in **Cring** and the equivalence of categories $\mathbf{Cring}^{\mathrm{op}} \simeq \mathbf{AffSch}$. Instead we'll show that the category **Sch** of schemes is both not cocomplete and not infinitely complete (although we'll see it does have all finite limits) before going on to talk about pullbacks in **Sch**.

**Proposition 3.5.4.** *The category* **Sch** *of schemes admits arbitrary coproducts.*

*Proof.* Let $\{X_i \mid i \in I\}$ be a family of schemes indexed by an index set $I$. Define the coproduct by

$$\left|\coprod_{i \in I} X_i\right| := \coprod_{i \in I} |X_i|,$$

i.e., the disjoint union of the spaces $|X_i|$. We then define the structure sheaf $\mathcal{O}_{\coprod X_i}$ by asserting, for any $U \subseteq \coprod_{i \in I} X_i$ open,

$$\mathcal{O}_{i \in I}(U) := \prod_{i \in I} \mathcal{O}_{X_i}\left(\iota_i^{-1}(U)\right)$$

where $\iota_i : |X_i| \to \coprod_{i \in I} |X_i|$ is the $i$-th inclusion into the disjoint union. That this determines a coproduct follows from the definitions and is straightforward to check from here. $\square$

While the category of schemes admits all coproducts, it is the coequalizers that are much more troublesome.[24] The problem is that coequalizers

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\operatorname{coeq}(f,g)} \operatorname{Coeq}(f, g)$$

[24] As per usual, we're using the standard categorical fact that if a category has all coproducts and all coequalizers then it has all colimits. Because exotic colimits are effectively impossible to describe and are really coproducts and coequalizers in disguise (like how transformers are robots in disguise) it suffices to describe the coproducts and coequalizers in the first place.

are "quotients" which glue $Y$ by folding together the images of $f$ and $g$. Because the scheme maps $f$ and $g$ have complete freedom in how they are selected, we could do some crazy pastings which need not respect the fact that whatever object we get out must have locally affine patches. In fact, this is exactly what happens in general (as we'll show in an example/proposition below).

**Proposition 3.5.5.** *The category* **Sch** *of schemes is not cocomplete.*

*Sketch.* We show a pair of parallel morphisms in **Sch** which do not admit a coequalizer in **Sch**. Fix a field $K$ and consider the affine line $\mathbb{A}^1_K = \operatorname{Spec} K[x]$ and note that its generic point (dense point) $(0)$ corresponds to a map of schemes $\operatorname{Spec} K(x) \to \operatorname{Spec} K[x]$ which arises as the spectrum of the localization $K[x] \to K(x)$. Let $\eta = \operatorname{Spec} K(x)$ and consider the two maps $\ell, r : \eta \to \mathbb{A}^1_K \coprod \mathbb{A}^1_K$ which embed $\eta$ into the first and second copy of $\mathbb{A}^1_K$, respectively. This coequalizer does not exist because the gluing of $\mathbb{A}^1 \coprod \mathbb{A}^1$ at only the generic point would ask for a scheme $X$ which has every closed point doubled but with exactly one dense point. However, this cannot happen because if we glue two dense points together and double every maximal point then no closed point would have an affine neighborhood. $\square$

I've presented the next remark/example as a warning that the colimits that arise in **Sch** are in general different than the colimits in **AffSch**, so even if a colimit arises in **AffSch** you need to be careful when checking if this is what you have when working in the entire category of schemes. I've also made some comments towards the end regarding modern arithmetic geometry and how to use this example as a stepping stone towards adic, perfectoid, and general $p$-adic geometry for experts or for the interested. Readers who would like to continue to study the category of schemes and its limits should click on the hyperline to Lemma 3.5.8, which will bypass the remark and number-theoretic geometry entirely.

**Remark 3.5.6** (Achtung!)**.** It is worth remarking here that the category of schemes is the canonical[25] example of a situation where computing colimits in a full (even reflective) subcategory can be different than computing the colimit in the full category (where the colimit may even fail to exist). For an explicit example of this phenomenon, recall from commutative algebra/number theory that the $p$-adic integers are the completion of the integers around an infinitesimal neighborhood of a prime $p$ equipped with the adic topology, i.e.,

$$\mathbb{Z}_p \cong \varprojlim_{n \in \mathbb{N}} \frac{\mathbb{Z}}{p^n \mathbb{Z}} \cong \varprojlim_{n \in \mathbb{N}} \frac{(\mathbb{Z} \setminus p\,\mathbb{Z})^{-1}\,\mathbb{Z}}{(\mathbb{Z} \setminus p\,\mathbb{Z})^{-1} p\,\mathbb{Z}}.$$

Because of this construction and the fact that **AffSch** turns limits of rings into colimits of affine schemes, we find

$$\underset{n \in \mathbb{N}, \mathbf{AffSch}}{\operatorname{colim}} \operatorname{Spec}\left(\frac{\mathbb{Z}}{p^n \mathbb{Z}}\right) \cong \operatorname{Spec}\left(\varprojlim_{n \in \mathbb{N}} \frac{\mathbb{Z}}{p^n \mathbb{Z}}\right) \cong \operatorname{Spec} \mathbb{Z}_p\,.$$

Note that in particular $|\operatorname{colim}_{n \in \mathbb{N}, \mathbf{AffSch}} \operatorname{Spec}(\mathbb{Z} / p^n \mathbb{Z})| \cong |\operatorname{Spec} \mathbb{Z}_p| = \{(0), (p)\}$ equipped with the Sierpinski topology (the point $(0)$ is open and the point $(p)$ is closed).

Alternatively, note that for any $n \geq 1$ (so that $\mathbb{Z} / p^n \mathbb{Z}$ is nonzero) we have that $\operatorname{Spec} \mathbb{Z} / p^n \mathbb{Z}$ has an underlying space consisting of a single point and structure sheaf determined by saying the global sections are given by $\operatorname{Spec} \mathbb{Z} / p^n \mathbb{Z}$. Consider the maps of spectra

$$\operatorname{Spec}(\pi_n) : \operatorname{Spec} \frac{\mathbb{Z}}{p^n \mathbb{Z}} \to \operatorname{Spec} \frac{\mathbb{Z}}{p^{n+1} \mathbb{Z}}$$

induced spatially by the identity (so by sending the unique point underlying $\operatorname{Spec} \mathbb{Z} / p^n \mathbb{Z}$ to the unique point underlying $\mathbb{Z} / p^{n+1} \mathbb{Z}$) and induced on sheaves by saying the comorphism

$$\pi_n^\sharp : \mathcal{O}_{\mathbb{Z} / p^{n+1} \mathbb{Z}} \to \mathcal{O}_{\mathbb{Z} / p^n \mathbb{Z}}$$

[25]To algebraic geometers or other sheafy people, at least

is given by the canoncial surjection with kernel $(p)$:

$$\frac{\mathbb{Z}}{p^{n+1}\,\mathbb{Z}} \xrightarrow{\pi_n} \frac{\mathbb{Z}}{p^n\,\mathbb{Z}}.$$

In **LRS** (and in **Sch**) the colimit of these maps is calculated by first noting that $|\operatorname{colim}_{n\in\mathbb{N},\mathbf{LRS}} \operatorname{Spec}\pi_n| \cong \{*\}$; because we do not force these maps to have affine images, upon limiting we are left with a single point in **LRS** as opposed to the "thicker" two points in **AffSch**. The sheaf is calculated by asserting

$$\mathcal{O}_{\operatorname{colim}_{n\in\mathbb{N},\mathbf{LRS}} \operatorname{Spec}\pi_n}(\{*\}) \cong \mathbb{Z}_p$$

Another way to think of this strange phenomenon is that the category of affine schemes introduces an invisible generic point to the above system that recovers $\operatorname{Spec}\mathbb{Z}_p$ while in **LRS** and **Sch** we cannot introduce this point in any natural way. In particular, the locally ringed space satisfying

$$\operatorname*{colim}_{\mathbf{LRS}} \operatorname{Spec} \frac{\mathbb{Z}}{p^n\,\mathbb{Z}} =: \operatorname{Spf}\mathbb{Z}_p,$$

is what is called the formal scheme of $\mathbb{Z}_p$ around $p$. It witnesses the completion of $\mathbb{Z}_p$ around the special fibre (closed point) $(p)$ *only* and is not a scheme. It is, however, an ind-scheme and a filtered colimit of schemes. The theory of formal schemes is extremely important in $p$-adic and arithmetic geometry, as formal schemes not only are the ind-(co)completion of the category of schemes but also allow us to take the more delicate approach of "what if my sheaf of rings was a sheaf of *topological* rings whose stalks were all complete?" Additionally, they are a helpful stepping stone for considering and understanding rigid analytic spaces, adic spaces (in the sense of Huber), perfectoid spaces, valuation theory in ultrametric analysis/geometry, and more.

For what follows we'll need to have pullbacks on hand, so we recall them quickly here. Recall that a pullback in a category $\mathscr{C}$ over a cospan $X \xrightarrow{f} S \xleftarrow{g} Y$ is an object $P$ equipped with morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ for which the diagram

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & S \end{tikzcd}$$

commutes; moreover, given any maps $h : Z \to X$ and $k : Z \to Y$ for which $f \circ h = g \circ k$ then there is a unique morphism $\theta : Z \to P$ making

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists!\theta"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\ & P \arrow[r, "p_1"'] \arrow[d, "p_2"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & S \end{tikzcd}$$

commute. Usually we'll write pullbacks with the notation $X \times_S Y$ and assume the structure morphisms $f : X \to S, g : Y \to S$ are clear from context.

We now move to discuss limits in **Sch**. Because we already know the terminal object in **Sch** (cf. Corollary 3.5.3) we'll begin this in earnest by first studying the pullbacks in **AffSch** and then gluing these together to construct pullbacks in **Sch**. First, however, we need a constructive lemma.

**Lemma 3.5.7.** *Let $X, Y$, and $Z$ be affine schemes with morphisms $f : X \to Z$ and $g : Y \to Z$. Then the pullback $X \times_Z Y$ exists in* **AffSch** *and satisfies, if $X \cong \operatorname{Spec} A, Y \cong \operatorname{Spec} B, Z \cong \operatorname{Spec} C$,*

$$X \times_Z Y \cong \operatorname{Spec}(A \otimes_C B).$$

*In particular, pullbacks of affine schemes exist in the category* **Sch**.

*Proof.* Recall from commutative algebra that the pushout in **Cring** of a span $A \leftarrow C \rightarrow B$ is the tensor product $A \otimes_C B$. Thus in $\mathbf{Cring}^{\mathrm{op}}$ pullbacks are given on objects by tensor products and the projections are the opposite of the tensor inclusions. Finally using the equivalence $\mathbf{Cring}^{\mathrm{op}} \simeq \mathbf{AffSch}$ gives

$$X \times_Z Y \cong \operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes_C B).$$

The final claim of the lemma follows from the fact that since **AffSch** is a reflective subcategory of **Sch**, the inclusion $\mathbf{AffSch} \rightarrow \mathbf{Sch}$ preserves and reflects limits. □

Because affine schemes admit pullbacks in **Sch** (and these pullbacks are also affine) which are given by the spectrum of the tensor product of the corresponding rings, we'll proceed to show how we can glue these tensor products together to build up a pullback of schemes $X \times_S Y$ even when $X, Y$, and $S$ need not be affine. For this we'll largely follow the approach sketched in [33], save with a little more declaration of the facts we'll use. Of particular interest will be how to pullback against opens and how to glue morphisms together via Zariski descent (which we only state but not prove; the proof is an exercise in checking sheafy things which we'll omit), as these are the techniques which really let us hit the ground running. Finally, for the reader who is willing to take the construction of pullbacks as a given and simply see how they are used before treading through the technical details of the proof of existence, please feel free to click on the reference to Theorem 3.5.12 here.

**Lemma 3.5.8.** *Let $X$ and $Y$ be schemes. Then to give a morphism $f : X \rightarrow Y$ is equivalent to giving an open cover $\{|U_i| \subseteq X \mid i \in I\}$ of $|X|$ and morphisms of schemes $f_i : U_i \rightarrow Y$ for which given any $i, j \in I$*

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

The next lemma we show will be the main tool we use to build our pullbacks, as it allows us to deduce the existence of pullbacks of open coverages.

**Lemma 3.5.9.** *Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a cospan of schemes and assume that the pullback $X \times_S Y$ exists. Then if $U$ is an open subscheme of $X$ and if the pullback $X \times_S Y$ exists in* **Sch***, the pullback $U \times_S Y$ exists.*

*Proof.* Begin by defining the scheme $p_1^{-1}(U)$ by setting

$$p_1^{-1}(U) := \left(|p_1|^{-1}(|U|), p_1^{-1}\,\mathcal{O}_U\right).$$

We now will show that $p_1^{-1}(U) \cong U \times_S Y$. Begin by noting that $U$ has a map $U \rightarrow S$ via the composition

$$U \xrightarrow{i} X \xrightarrow{f} S.$$

Let $Z$ be a scheme with maps $h : Z \rightarrow U$ and $k : Z \rightarrow Y$ for which the diagram

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "k"'] & U \arrow[d, "f\circ i"] \\ Y \arrow[r, "g"'] & S \end{tikzcd}$$

commutes. Note that appending the immersion $U \rightarrow X$ to the right of the square above gives the diagram

$$\begin{tikzcd} Z \arrow[r, "h"'] \arrow[rr, bend left, "i\circ h"] \arrow[d, "k"'] & U \arrow[r, "i"'] \arrow[d, "f\circ i"] & X \arrow[dl, "f"] \\ Y \arrow[r, "g"'] & S & \end{tikzcd}$$

which commutes because, by assumption.

$$f \circ i \circ h = g \circ k.$$

Now using the induced commuting square with vertices $Z, X, Y, S$ we find that there is a unique morphism $\theta : Z \to X \times_S Y$ for which

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!\theta"] \arrow[drr, bend left, "i \circ h"] \arrow[ddr, bend right, "k"'] & & \\
& X \times_S Y \arrow[r, "p_1"'] \arrow[d, "p_2"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & S
\end{tikzcd}$$

commutes. However, as $p_1 \circ \theta = i \circ h$ factors through $U$, it follows that $\theta$ must factor through $p_1^{-1}(U)$ as the image of $\theta$ lies in $U$. This implies in particular that the diagram

$$\begin{tikzcd}
Z \arrow[dr, "\theta"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\
& p_1^{-1}(U) \arrow[r, "p_1"] \arrow[d, "p_2|_{p_1^{-1}(U)}"] & U \arrow[d, "f \circ i"] \\
& Y \arrow[r, "g"'] & S
\end{tikzcd}$$

commutes. Finally a straightforward argument shows that $\theta$ is unique[26] and hence proves that $p_1^{-1}(U) \cong U \times_S Y$. $\square$

The next lemma shows how to use the existence of pullbacks described above to deduce the existence of pullbacks along a cover of $X$.

**Lemma 3.5.10.** *Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a cospan of schemes and let $\{U_i \xrightarrow{\iota_i} X \mid i \in I\}$ be a cover of $X$ by open subschemes $U_i$. If each pullback $U_i \times_S Y$ exists then $X \times_S Y$ exists.*

*Proof.* Our strategy for constructing the scheme $X \times_S Y$ is to use our hypotheses together with Lemma 3.5.9 to show that there are open patch pullbacks $U_{ij}$ of the $U_i \times_S Y$ and set ourselves up to use Proposition 3.4.8 in order to glue together the $U_i \times_S Y$ along the $U_{ij}$ and get $X \times_S Y$. Afterwards we'll use Lemma 3.5.8 to construct the projections and show the universal properties.

Let us build our open patches. For each $i, j \in I$ define $X_{ij} = U_i \cap U_j$ and note that this is an open subscheme of $U_i$. Write $p_{1,i} : U_i \times_S Y \to U_i$ for the first projection. Because each pullback $U_i \times_S S$ exists, by Lemma 3.5.9 the schemes $X_{ij} \times_S Y$ exist and are given by

$$p_{1,i}^{-1}(X_{ij}) = p_{1,i}^{-1}(X_{ij}) \cong X_{ij} \times_S Y.$$

We thus define $X_i := U_i \times_S Y$ and

$$U_{ij} := X_{ij} \times_S Y$$

for all $i, j \in I$. Because we have $X_{ij} = U_i \cap U_j \cong X_j \cap X_i = X_{ji}$ we find that there are unique isomorphisms

$$\varphi_{ij} : U_{ij} \xrightarrow{\cong} U_{ji}$$

[26]Argue by taking two maps $\psi, \theta : Z \to p_1^{-1}(U)$ making the diagram commute and then consider the extension $i : p_1^{-1}(U) \to X \times_S Y$. Now show that $i \circ \theta = i \circ \psi$ and use that the immersion $i$ is monic in **Sch**.

induced by the fact that pullbacks are unique up to unique isomorphisms. Furthermore, the uniqueness of these isomorphisms then give rise to the cocycle conditions

$$\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$$

and

$$\varphi_{ik}|_{U_{ij} \cap U_{jk}} = \varphi_{jk}|_{U_{ij} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{jk}}.$$

Thus we apply Proposition 3.4.8 and define the scheme

$$Z = \bigcup_{i \in I} (U_i \times_S Y)$$

to be the gluing of the $U_i \times_S Y$ along the open subschemes $U_{ij}$. We will write $\mathrm{incl}_i : U_i \times_S Y \to Z$ for the inclusion immersion at $i \in I$.

We now prove that $Z \cong X \times_S Y$ is a pullback over $X$ and $Y$ over $S$. For this we first define the projections $p_1 : Z \to X$ and $p_2 : Z \to Y$; in both cases we'll be gluing maps, so we only describe how to build the first projection as the second follows mutatis mutandis. Begin by considering the family of maps $p_{1,i} : U_i \times_S Y \to U_i$ and, for all $i, j \in I$, $p_{1,ij} : X_{ij} \times_S Y \to X_{ij}$. Note that by construction $p_{1,ij} : X_{ij} \times_S Y$ fits into the commuting diagram

$$\begin{tikzcd}
U_i \times_S Y \arrow[r, "p_{1,i}"] & U_i \\
X_{ij} \times_S Y \arrow[u] \arrow[d] \arrow[r, "p_{1,ij}"] & U_i \cap U_j \arrow[u] \arrow[d] \\
U_j \times_S Y \arrow[r, "p_{1,j}"'] & U_j
\end{tikzcd}$$

of schemes where each of the vertical maps are open immersions. Consequently we have that $p_{1,i}|_{U_{ij}} = p_{1,ij} = p_{1,j}|_{U_{ij}}$. Apply Lemma 3.5.8 glue the $p_{1,i}$ to form the map $p_1 : Z \to X$. Similarly, $p_2 : Z \to Y$ is defined by gluing the $p_{2,i} : U_i \times_S Y \to Y$ along the $U_{ij} = X_{ij} \times_S Y$.

Let us now show the universal property of $Z$. Fix a scheme $W$ with morphisms $h : W \to X$ and $h : W \to Y$ for which the square

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "k"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & S
\end{tikzcd}$$

commutes. Consider the open subschemes $W_i := h^{-1}(U_i)$ and note that because the $U_i$ cover $X$ we find

$$\bigcup_{i \in I} W_i = \bigcup_{i \in I} h^{-1}(U_i) = h^{-1}\left(\bigcup_{i \in I} U_i\right) = h^{-1}(X) = Z.$$

Moreover, upon restricting $h$ to $W_i$ we find that $h|_{W_i} =: h_i$ lands in the open $U_i$ by construction. Alternatively, we also write $k_i := k|_{W_i} : W_i \to Y$ for the restriction of $k$ to $W_i$. These maps fit into a commuting diagram

$$\begin{tikzcd}
W_i \arrow[r, "h_i"] \arrow[d, "k_i"'] & U_i \arrow[d, "f \circ \iota_i"] \\
Y \arrow[r, "g"'] & S
\end{tikzcd}$$

as for all $i \in I$,

$$f \circ \iota_i \circ h_i = f \circ \iota_i \circ h|_{W_i} = (f \circ h)|_{W_i} = (g \circ k)|_{W_i} = g \circ k|_{W_i} = g \circ k_i.$$

Because each scheme $U_i \times_S Y$ exists we get factorizations:

$$\begin{tikzcd}
W_i \arrow[dr, dashed, "\exists!\theta_i"] \arrow[drr, bend left, "h_i"] \arrow[ddr, bend right, "k_i"'] & & \\
& U_i \times_S Y \arrow[r, "p_{1,i}"'] \arrow[d, "p_{2,i}"] & U_i \arrow[d, "f \circ \iota_i"] \\
& Y \arrow[r, "g"'] & S
\end{tikzcd}$$

Composing each map $\theta_i : W_i \to U_i \times_S Y$ with the inclusion $\mathrm{inlc}_i : U_i \times_S Y \to Z$ we obtain a family of morphisms $\{\mathrm{incl}_i \circ \theta_i : W_i \to Z \mid i \in I\}$. It is also straightforward to check that for any pair $i, j \in I$ we have

$$Z_i \cap Z_j = h^{-1}(U_i) \cap h^{-1}(U_j) = h^{-1}(U_i \cap U_j)$$

which allows us to deduce that

$$(\mathrm{incl}_i \circ \theta_i)|_{Z_i \cap Z_j} = (\mathrm{incl}_j \circ \theta_j)|_{Z_i \cap Z_j}.$$

Applying Lemma 3.5.8 yet again gives that there is a gluing of the $\mathrm{incl}_i \circ \theta_i$ to a map $\theta : W \to Z$ which makes the diagram

$$\begin{tikzcd}
W \arrow[dr, "\theta"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\
& Z \arrow[r, "p_1"'] \arrow[d, "p_2"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & S
\end{tikzcd}$$

commute. Finally, we can check if any two maps $\theta, \psi : W \to Z$ which make the diagram commute agree by checking locally in terms of the gluings $\{\mathrm{incl}_i \circ \theta_i : W_i \to Z_i \mid i \in I\}$ and $\{\mathrm{incl}_i \circ \psi_i : W_i \to Z_i \mid i \in I\}$. Over these local neighborhoods the maps agree by the universal property of the $U_i \times_S Y$ and so we deduce that the diagram above takes the form

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!\theta"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\
& Z \arrow[r, "p_1"'] \arrow[d, "p_2"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & S
\end{tikzcd}$$

which shows that $Z \cong X \times_S Y$, as was desired. $\square$

The next proposition gives a partial result: pullbacks exist over any affine base scheme $S$.

**Proposition 3.5.11.** *Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a cospan of schemes for which $S$ is affine. Then the pullback $X \times_S Y$ exists. Moreover, if $\{X_i \mid i \in I\}$ is an affine open cover of $X$ and $\{Y_j \mid j \in J\}$ is an affine open cover of $Y$ then $\{X_i \times_S Y_j \mid i \in I, j \in J\}$ is an affine open cover of $X \times_S Y$*

*Proof.* Begin by recalling that for any of the schemes $X_i$ and $Y_j$, the pullback $X_i \times_S Y_j$ exists by Lemma 3.5.7. This allows us to deduce using Lemma 3.5.10 that the schemes $X \times_S Y_j$ exists for any $j \in J$ and that $X_i \times_S Y$ exists for any $i \in I, j \in J$. Putting these together allows us to deduce that $X \times_S Y$ exists for any affine scheme $S$. Finally, the statement regarding the open affine cover follows from the fact that the schemes $X_i \times_S Y_j$ are all affine and the fact that we can first glue $X$ and then glue $Y$ to form $X \times_S Y$ or we can first glue $Y$ and then glue $X$ as suggested by the diagram below:

$$\begin{tikzcd} X_i \times_S Y_j \arrow[r] \arrow[d] & X \times_S Y_j \arrow[d] \\ X_i \times_S Y \arrow[r] & X \times_S Y \end{tikzcd}$$

□

**Theorem 3.5.12.** *The category of affine schemes admits finite pullbacks.*

*Proof.* We already know that **Sch** admits empty pullbacks because $\operatorname{Spec}\mathbb{Z}$ is the terminal object by Corollary 3.5.3. As such, it suffices to show that binary pullbacks exist by a standard induction[27]. Fix a cospan of schemes $X \xrightarrow{f} S \xleftarrow{g} Y$ and let $\{S_i \xrightarrow{\iota_i} S \mid i \in I\}$ be an affine cover of $S$. Let $X_i := f^{-1}(S_i)$ and $Y_i := g^{-1}(S_i)$ and note that by Proposition 3.5.11 the scheme $X_i \times_{S_i} Y_i$ exists. We claim that $X_i \times_{S_i} Y_i \cong X_i \times_S Y$ so to so show this assume we have a commuting square:

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "k"'] & X_i \arrow[d, "\iota_i \circ f"] \\ Y \arrow[r, "g"'] & S \end{tikzcd}$$

To show that $X_i \times_{S_i} Y_i \cong X_i \times_S Y$ it suffices to prove that $k$ factors through $Y_i$ (and hence through $S_i$ as well)[28]. Using the commutativity hypothesis we find the equation

$$g \circ k = \iota_i \circ f \circ h$$

tells us that $g \circ k$ factors through $S_i$. This in turn implies, since $Y_i = g^{-1}(S_i)$, that $k$ must factor through $Y_i$ as in the diagram:

$$\begin{tikzcd} Z \arrow[d, "k"'] \arrow[dr, "k"] & \\ Y_i \arrow[r] & Y \end{tikzcd}$$

However, we then deduce the existence of a natural map $X_i \times_S Y \to X_i \times_{S_i} Y_i$, which must be an isomorphism by using the universal property each object satisfies. Now, because $X_i \times_{S_i} Y_i \cong X_i \times_S Y$ for all $i \in I$ and $\{X_i \mid i \in I\}$ cover $X$,[29] we apply Lemma 3.5.10 and conclude that $X \times_S Y$ exists. □

**Corollary 3.5.13.** *For any base scheme $S$ the slice category $\mathbf{Sch}_{/S}$ is finitely complete.*

*Proof.* Because **Sch** has a terminal object by Corollary 3.5.3 and finite pullbacks by Theorem 3.5.12, the result follows from a standard category-theoretic fact (cf. [9]). For the relative case $\mathbf{Sch}_{/S}$ simply note that $S$ is the terminal object in $\mathbf{Sch}_{/S}$, products are given by pullbacks $X \times_S Y$, and pullbacks of a cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$ of schemes over $S$ are given by $X \times_Z Y$. □

[27] Singleton pullbacks are the terminal object in the slice category and three-or-more-but-finitely-many pullbacks can always be computed by iterating the terms pairwise.

[28] A straightforward argument involving the fact that limits commute with limits (and that immersions are monic) shows that $X_i \times_{S_i} Y_i \to X_i \times_S Y$; because both objects have universal properties, it suffices to give a map $X_i \times_S Y$ in order to conclude the subobject map $X_i \times_{S_i} Y_i \to X_i \times_S Y$ is an isomorphism.

[29] As before the argument is $\cup_{i \in I} X_i = \cup_{i \in I} f^{-1}(S_i) = f^{-1}(\cup_{i \in I} S_i) = f^{-1}(S) = X$.

**Proposition 3.5.14.** *Let $X \to S \to Y$ be a cospan of schemes and let $Z$ be a scheme. Then there is an isomorphism*

$$(X \times_S Y) \coprod W \cong \left(X \coprod W\right) \times_S \left(Y \coprod W\right).$$

*Proof.* Because coproducts and products in $\mathbf{Sch}_{/S}$ are computed affine-locally, it suffices to assume that $X, S, Y$, and $W$ are affine. Write $X \cong \operatorname{Spec} A$, $Y \cong \operatorname{Spec} B$, $Z \cong \operatorname{Spec} C$, and $W \cong \operatorname{Spec} D$ for commutative unital rings $A, B, C, D$. Then

$$\begin{aligned}(X \times_S Y) \coprod W &\cong (\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B) \coprod \operatorname{Spec} D \cong \operatorname{Spec}(A \otimes_C B) \coprod \operatorname{Spec} D \\ &\cong \operatorname{Spec}((A \otimes_C B) \times D) \cong \operatorname{Spec}((A \times D) \otimes_C (B \times D)) \\ &\cong \left(\operatorname{Spec} A \coprod \operatorname{Spec} D\right) \times_{\operatorname{Spec} C} \left(\operatorname{Spec} B \coprod \operatorname{Spec} D\right) \cong \left(X \coprod W\right) \times_S \left(Y \coprod W\right).\end{aligned}$$

□

**Proposition 3.5.15.** *The category $\mathbf{Sch}_{/S}$ does not admit infinite products for any base scheme $S$ with $|S| \ncong \varnothing$.*

*Proof.* It can be shown/checked that the locally ringed space

$$\prod_{n \in \mathbb{N}} \mathbb{P}^1_{\mathbb{Z}}$$

is not a scheme[30]. From here the result follows by noting that for any scheme $S$ with nonempty underlying space,

$$\left(\prod_{n \in \mathbb{N}} \mathbb{P}^1_{\mathbb{Z}}\right) \times_{\operatorname{Spec} \mathbb{Z}} S \cong \prod_{n \in \mathbb{N}} \left(\mathbb{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S\right) \cong \prod_{n \in \mathbb{N}} \mathbb{P}^1_S$$

in **LRS**. This is also not a scheme for the same reasons the $\mathbb{N}$-indexed product of $\mathbb{P}^1_{\mathbb{Z}}$ is not a scheme. Finally, the assumption that $|S| \ncong \varnothing$ ensures that $\mathbb{P}^1_S \ncong S$, which is the only requirement for the above to not collapse. □

We now close this section by showing some use of pullbacks as they arise in nature. Because the pullback in **Sch** locally looks like taking the spectrum of tensor products $A \otimes_C B$, the pullback in **Sch** is a very structured pullback. On one hand this means that, as usual with pullbacks, we can think of the product $X \times_S Y$ as taking a scheme $Y$ over $S$

$$\begin{tikzcd} Y \arrow[d, "g"] \\ S \end{tikzcd}$$

and changing bases along the map $X \xrightarrow{f} S$ to make a scheme over $X$:

$$\begin{tikzcd} X \times_S Y \arrow[d, "p_1"] \\ X \end{tikzcd}$$

However, on the other hand, this means for a cospan of schemes $X \xrightarrow{f} S \xleftarrow{g} Y$ that taking this base change imposes whatever local ring-theoretic information that $X$ contains gets combined and mixed algebraically with whatever ring-theoretic information that $Y$ contains. This is particularly helpful for defining "compactifications" of curves, taking algebraic information over one field and translating it to another, and more.

[30] This is actually a little bit difficult and we omit it here.

**Example 3.5.16.** If $S$ is any affine scheme (say $S \cong \operatorname{Spec} A$) then

$$\begin{aligned}\mathbb{A}_S^n \cong \mathbb{A}_A^n \cong \operatorname{Spec} A[x_1, \cdots, x_n] &\cong \operatorname{Spec}\left(\bigotimes_{i=1}^{n} A[x_i]\right) \\ &\cong \operatorname{Spec} A[x_1] \times_{\operatorname{Spec} A} \operatorname{Spec} A[x_2] \times_{\operatorname{Spec} A} \cdots \times_{\operatorname{Spec} A} \operatorname{Spec} A[x_n] \\ &\cong \mathbb{A}_S^1 \times_S \mathbb{A}_S^1 \times_S \cdots \times_S \mathbb{A}_S^1.\end{aligned}$$

In particular, in $\mathbf{Sch}_{/S}$ we find

$$\mathbb{A}_S^n \cong \prod_{i=1}^{n} \mathbb{A}_S^1$$

so as expected affine $n$-space is the $n$-fold product of affine 1-space.

**Example 3.5.17.** If $S$ is any affine scheme (say $S \cong \operatorname{Spec} A$) then affine $n$-space over $S$ can be seen via

$$\mathbb{A}_S^n = \operatorname{Spec} A[x_1, \cdots, x_n] \cong \operatorname{Spec}(\mathbb{Z}[x_1, \cdots, x_n] \otimes_{\mathbb{Z}} A) \cong \mathbb{A}_{\mathbb{Z}}^n \times_{\operatorname{Spec} \mathbb{Z}} S.$$

This allows us to define affine $n$-space for non-affine schemes $S$ by setting $\mathbb{A}_S^n := \mathbb{A}_{\mathbb{Z}}^n \times_{\operatorname{Spec} \mathbb{Z}} S$.

**Example 3.5.18.** The coordinate ring of an elliptic curve over a field $K$ is the scheme

$$C := \operatorname{Spec} \frac{K[x,y]}{(y^2 + ay - x^3 - bx^2 - cxy - dx - e)}$$

for some choice of coefficients $a, b, c, d, e \in K$. Equivalently, if $p \in K[x,y]$ is the polynomial $p(x,y) = y^2 + ay - x^3 - bx^2 - cxy - dx - e$ then $C$ is

$$C := \operatorname{Spec} \frac{K[x,y]}{(p)}.$$

If $L/K$ is any field extension of $K$ then there is a map $\operatorname{Spec} L \to \operatorname{Spec} K$. The corresponding coordinate ring of the elliptic curve over $L$ is

$$C_L := C \times_{\operatorname{Spec} K} \operatorname{Spec} L = \operatorname{Spec} \frac{K[x,y]}{(p)} \times_{\operatorname{Spec} K} \operatorname{Spec} L \cong \operatorname{Spec} \frac{L[x,y]}{(p)}.$$

**Example 3.5.19.** Let $X$ be any scheme over $\operatorname{Spec} K$ for which there is an open affine cover $\{U_i \mid i \in I\}$ of the form

$$U_i \cong \operatorname{Spec} \frac{K[x_1, \cdots, x_{n_i}]}{(f_1, \cdots, f_{m_i})}.$$

Then $X \times_K X$[31] is a scheme, but $\mathcal{O}_{X \times_A X}(U)$ can have zero divisors even if every ring $\mathcal{O}_X(V)$ is an integral domain! For example, consider $X = \operatorname{Spec} \mathbb{C}$ as a scheme over $\operatorname{Spec} \mathbb{R}$ (the structure map is the field extension $\operatorname{Spec}(\mathbb{R} \to \mathbb{C})$). Then $\mathcal{O}_X$ is an integral domain for every open of $|\operatorname{Spec} \mathbb{R}|$ (its values are $\mathbb{C}$ at the point and 0 at the empty set) but the global sections of $X \times_{\mathbb{R}} X \cong \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ satisfy

$$\mathcal{O}_{X \times_{\mathbb{R}} X}(|X \times_{\mathbb{R}} X|) \cong \mathcal{O}_{\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}}(|\operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2,$$

which admits zero divisors.

We now close this subsection by introducing fibres of morphisms and describing what their underlying spaces are for use later. While we will not discuss these extensively here, the notion of fibres of morphisms are one of the most important and crucial in algebraic geometry and really in modern mathematics.[32]

[31] I officially entered lazy mode. From here on, if $X, Y$ are schemes over $\operatorname{Spec} A$ we'll write $X \times_A Y$ to denote the pullback $X \times_{\operatorname{Spec} A} Y$ for readability.

[32] I am really doing you a disservice by not spending more time on fibres, but I am afraid that doing so will take too much room and time in these already far too long notes. I just hope that the examples I give below will help you see why they are helpful and useful in practice, as they allow us to see curves as they sit and vary above other spaces.

**Definition 3.5.20.** Let $f : X \to Y$ be a morphism of schemes and let $y \in |Y|$. The scheme-theoretic fibre of $f$ above $y$ is the scheme

$$f^{-1}(y) := X \times_Y \operatorname{Spec} \kappa(y)$$

where the map $\operatorname{Spec} \kappa(y) \to Y$ is induced by the ring map $\mathcal{O}_Y(Y) \to \mathcal{O}_{Y,y} \to \kappa(y)$.

**Lemma 3.5.21.** *The underlying space of the fibre of $f$ above $y$ is*

$$|f^{-1}(y)| = \{x \in |X| \ : \ f(x) = y\}.$$

**Example 3.5.22.** Let $K$ be an algebraically closed field of characteristic 0 and consider the map $\mathbb{A}^1_K \to \mathbb{A}^1_K$ given by $\operatorname{Spec}(t \mapsto t^n)$ for $n \geq 2$ (this is visualized in Figure 3.2). The non-generic points of $\mathbb{A}^1_K$ all take

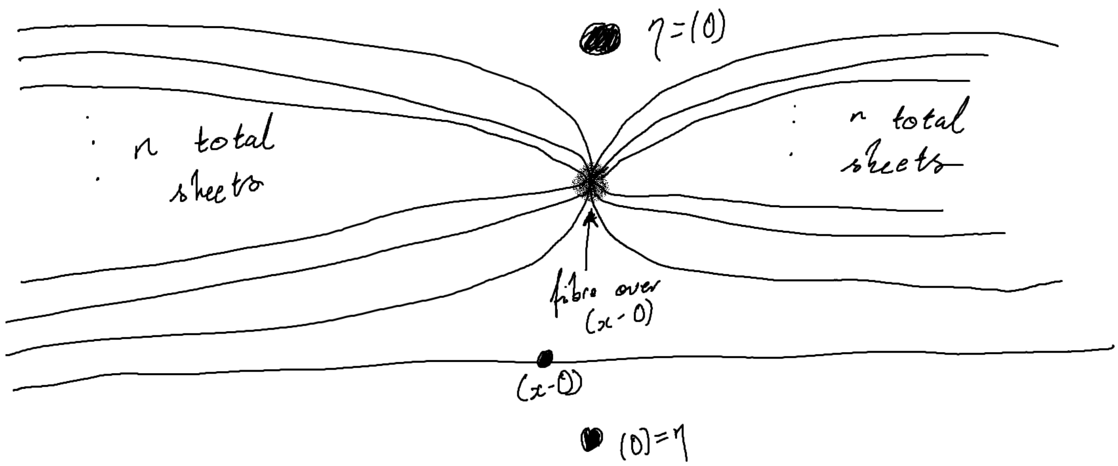


Figure 3.2: This is a visualization of the fibres above each point of $\mathbb{A}^1_K$ under the mapping $\operatorname{Spec}(t \mapsto t^n)$.

the form $y = (t - a)$ for some $a \in K$. For any $a \neq 0$ we find that the fibre above $(x - a)$ is the scheme $f^{-1}(t - a) = \coprod_{i=1}^{n} \operatorname{Spec} K$ (there are $n$ points that sit above $a$ based on the $n$ roots of $a$ in $K$) while the fibre above $(x - 0)$ is the scheme

$$f^{-1}(x - 0) = \operatorname{Spec}\left(\frac{K[x]}{(x^n)}\right),$$

which has one point of "infinitesimal degree $n$"[33]

## 3.6 Separated Morphisms, Finite Type Morphisms, and Varieites: Why?

It is with minimal overstatement that we can say varieties occupy *the* central study of algebraic geometry. Many theorems and constructions in arithmec geometry and pure algebraic geometry involve studying and understanding varieties over various different fields (in fact, while reductionist, we can say the entire study of elliptic curves over a scheme $S$ is the study of proper smooth 1-dimensional curves $E$ with geometrically connected fibres of genus one and a section $0 : S \to E$ of the structure map $E \to S$). This also means that varieties give us a more reasonable characterization of what it means to be a complex manifold/analytic space in terms of only ring theoretic data. Because we capture elliptic curves in terms of scheme theory, we also capture complex tori and compact complex Lie groups as well! In particular, varities must carry

[33]This is entirely informal and meant only as an analogy. However, you can see this scheme as giving us an infinitesimal neighborhood of dimension/degree $n$ around the variable $x$ and this intuition is leveraged frequently in $p$-adic algebraic geometry and perfectoid geometry.

a scheme-theoretic notion of Hausorff-ness that the Zariski topology does not detect, as well as some finite generation properties.

Let us motivate some of the study of varities in number theory and representation theory that are not based on elliptic curves overtly. The study of $L$-functions and (parts of) the Langlands Programme can be stated in algebro-geometric terms and be seen to try to find different ways to study the $L$-functions of certain varieties over (global or local) fields by way of representation theory. In fact, the classical analytic number theorertic expression that allows us to write the Riemann zeta function in terms of its Euler product (recall here that $s \in \mathbb{C}$ with $\Re(s) > 1$)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\substack{p \in \mathbb{Z} \\ p \text{ prime}, p>0}} \frac{1}{1-p^{-s}}$$

can be seen as the beginning of this idea. We can rewrite this as

$$\prod_{\substack{p \in \mathbb{Z} \\ p \text{ prime}, p>0}} \frac{1}{1-p^{-s}} = \prod_{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z} \setminus \{(0)\}} \frac{1}{1-|\kappa(\mathfrak{p})|^{-s}}.$$

After noting that $|\kappa(0)| = |\mathbb{Q}| = \infty$ and defining $|\kappa(0)|^{-s} := 0$ for $\Re(s) > 1$ (as a definition of convenience justified by calculating the modulus of $z^{-s}$ as $|z| \to \infty$ for any fixed $s \in \mathbb{C}$ with $\Re(s) > 1$) we can further rewrite

$$\zeta(s) = \prod_{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}} \frac{1}{1-|\kappa(\mathfrak{p})|^{-s}}.$$

However, we should view each field $\kappa(\mathfrak{p})$ as the trivial variety over $\kappa(\mathfrak{p})$ and treat the above Euler product as a product of local $L$-functions. In this way we get

$$\zeta(s) = \prod_{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}} \frac{1}{1-|\kappa(\mathfrak{p})|^{-s}} = \prod_{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}} L_{\mathfrak{p}}(\operatorname{Spec} \kappa(\mathfrak{p}))$$

so that not only is the Riemann zeta function a product indexed by the points of the scheme $\operatorname{Spec} \mathbb{Z}$, it is also the product of the local $L$-functions at $\mathfrak{p}$ of the variety $\operatorname{Spec} \kappa(\mathfrak{p}) \cong \operatorname{Spec} \mathbb{F}_p \cong \operatorname{Spec} \mathbb{Z}_p/(p)$ over $\operatorname{Spec} \kappa(\mathfrak{p})$. While this discussion doesn't generalize quite as smoothly as one would hope (and I have done it a great disservice by not mentioning the Adelic theory that really describes what's going on in that last equality), the point is just that the study of varities not only gives us great powers in complex geometry but also gives us powerful tools with which to study number-theoretic phenomenon.

As we begin diving into the study of varieties, it can be helpful to keep in mind some complex geometric intuition in mind. Much of the theory of varities begain in Serre's paper *Faiseaux Algeébriques Cohérents* which entrenched sheaf-theoretic and sheaf cohomological techniques into algebraic geoemtry and the study of varieties. These techniques were initially highly motivated by cohomology in several complex variables and classical algebraic geometry. In these cases varieties were taken as certain locally closed subspaces of $\mathbb{C}^n$ or $\mathbb{P}\,\mathbb{C}^n$ equipped with their sheaves of regular functions. These classical varities then gave models/algebratizations of algebraic complex manifolds and their sheaves of holomorphic functions. Motivated further by Hilbert's Nullstellensatz and other techniques from commutative algebra, these classical varities allowed an algebraic approach to studying finitely generated, reduced $\mathbb{C}$-algebras and the complex manifolds whose rings of functions look like finitely generated reduced $\mathbb{C}$-algebras (in fact, Hilbert's Nullstellensatz has been known for a long time now to give an equivalence of categories between affine $\mathbb{C}$-varities and finitely generated reduced $\mathbb{C}$-algebras). Because of this complex analytic intuition and technology, we need to introduce three notions for detecting the following properties:

- Algebraic complex manifolds are all Hausdorff, so we need a scheme-theoretic way of recording this (especially considering the underlying spaces of schemes are almost never Hausdorff).
- The rings of regular functions on these locally closed pieces of $\mathbb{C}^n$ or $\mathbb{P}\,\mathbb{C}^n$ are all reduced.
- The complex algebraic manifolds are all "finitely generated" in the sense that they are locally solution spaces of finitely many polynomials in finitely many variables.

## 3.7 Separated Schemes/Morphisms

To study varities we'll first focus our attention on how to see whether a scheme has the right notion of being sufficiently separated. For this we recall an equivalent characterization of Hausdorff spaces from point set topology.

**Proposition 3.7.1.** *A topological space $X$ is Hausdorff if and only if the diagonal subset $\Delta_X = \{(x,x) \mid x \in X\} \subseteq X \times X$ is a closed subspace of $X \times X$ (equipped with the product topology).*

We use this as our jumping off point to capture the separation strucutre of a scheme, as we can get a diagonal morphism for every object in any category with (binary) products.

**Definition 3.7.2.** Let $\mathscr{C}$ be a category with binary products. For any object $X \in \mathscr{C}_0$, the diagonal morphism $\Delta_X : X \to X \times X$ is the unique morphism making the diagram

$$\begin{tikzcd} & X \arrow[dl, equal, bend right] \arrow[dr, equal, bend left] \arrow[d, dashed, "\exists! \Delta_X"] & \\ X & X \times X \arrow[l, "\pi_2"'] \arrow[r, "\pi_1"'] & X \end{tikzcd}$$

commute in $\mathscr{C}$.

**Remark 3.7.3.** If $\mathscr{C}$ is a category with finite pullbacks and if $S \in \mathscr{C}_0$ then the product in the category $\mathscr{C}_{/S}$ is the pullback $X \times_S X$ against the structure morphisms $\nu_X : X \to S$ as in the diagram:

$$\begin{tikzcd} X \times_S X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\nu_X"] \\ X \arrow[r, "\nu_X"'] & S \end{tikzcd}$$

In this case the diagonal is a map:

$$\begin{tikzcd} & X \arrow[dl, equal, bend right] \arrow[dr, equal, bend left] \arrow[d, dashed, "\exists! \Delta_X|_Y"] & \\ X & X \times_S X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X \end{tikzcd}$$

In general the diagonal morphism is an immersion of schemes (it sends $|X|$ to a subspace of $|X \times_Y X)$ but it isn't necessarily open or closed). However, we can say that the diagonal is always *locally* closed, i.e., it arises as the intersection of an open and closed subscheme in $X \times_Y X$. This will motivate our definition of an immersion of schemes (which is also often called a locally closed immersion in some references).

**Definition 3.7.4.** An morphism of schemes $k : V \to X$ is said to be an immersion if there is a closed immersion $i : V \to U$ and an open immersion $j : U \to X$ for which the diagram

$$\begin{tikzcd} V \arrow[rr, "k"] \arrow[dr, "i"'] & & X \\ & U \arrow[ur, "j"'] & \end{tikzcd}$$

commutes.

**Example 3.7.5.** Any open immersion $j : U \to X$ and closed immersion $i : V \to X$ is locally closed. In the case of the open subscheme define the closed map to be $\mathrm{id}_U : U \to U$ in order to write $j = j \circ \mathrm{id}_U$, while in the case of the closed subscheme define the open immersion to be $\mathrm{id}_X : X \to X$ in order to write $i = \mathrm{id}_X \circ i$.

**Example 3.7.6.** This may put the cart before the horse a little (cf. Lemma 3.7.13), but if $X$ is the line with a doubled origin defined over a base ring $A$ from Example 3.4.15 then the map $X \to X \times_A X$ is a locally closed immersion which is not closed nor open.

**Proposition 3.7.7.** *Any open, closed, or locally closed immersion $k : V \to X$ is a monomorphism in* **Sch**.

*Sketch.* See [27, Section 4.2] and/or [27, Proposition 5.3.8]. For more detail and a complete characterization of monics in **Sch**, see [31, Proposition 17.2.6]. In particular, that proof characterizes monics as those whose diagonal morphisms are isomorphisms, but we will not discuss this here (as it relies on the sheaf of relative differentials $\Omega^1_{X/Y}$). $\square$

**Definition 3.7.8.** If $X$ is a scheme over $S$ with structure map $f : X \to S$ then we say that $f$ is separated over $S$ (or $X$ is a separated $S$-scheme) if the image of the diagonal $\Delta_X : X \to X \times_S X$ is a closed subspace of $|X \times_S X|$. If $X$ is separated over $\operatorname{Spec} \mathbb{Z}$ then we say that $X$ is separated.[34]

**Remark 3.7.9.** We will frequently abuse notation/terminology and say that an $S$-scheme $X$ is separated either without reference to the structure map $f : X \to S$ or by just saying that $X$ is a separated scheme over $S$.

We now present some examples of separated and non-separated schemes. Of particular interest is Proposition 3.7.10 which says that affine schemes are separated (and so all of our patches are scheme-theoretically Hausdorff). What is interesting about this is that it tells us the way that schemes may fail to be able to separate information lies in the way we glue our open patches together, analogously to how quotient spaces of Hausdorff spaces need not be Hausdorff.

**Proposition 3.7.10.** *Every affine scheme* $\operatorname{Spec} A$ *is separated over* $\operatorname{Spec} \mathbb{Z}$.

*Proof.* For any ring $A$ the diagonal map $\Delta_X : \operatorname{Spec} A \to \operatorname{Spec} A \times_{\mathbb{Z}} \operatorname{Spec} A$ is the spectrum of the map $\nabla : A \otimes_{\mathbb{Z}} A \to A$ given on pure tensors by $a \otimes b \mapsto ab$. Thus, because $\nabla$ is a surjection of rings, we invoke Proposition 3.4.19 to conclude that $\Delta_X$ is a closed immersion. $\square$

**Example 3.7.11.** The schemes $\mathbb{A}^n_A$ and $\mathbb{P}^n_A$ are separated over $\operatorname{Spec} A$ for any $n \in \mathbb{N}$ and any ring $A$.

**Example 3.7.12.** The line with doubled origin $X$ of Example 3.4.15 is not separated if $A \ncong 0$.

**Lemma 3.7.13** ([27, Proposition 5.3.9], [73, Proposition 10.1.3])**.** *If $X$ is any $S$-scheme then the diagonal $\Delta_X|_S : X \times_S X$ is a locally closed immersion.*

*Proof.* We follow the proof of [73], as it is more concrete and hands on than that of [27]. We will find a series of open sets in $X \times_S X$ which cover $\Delta_X(X)$ and then exhibit $X$ as a closed subscheme of the union of these opens. First let $\{\iota_i : V_i \to S \mid i \in I\}$ be an affine open cover of $S$ and let $\{\iota_{ij} : U_{ij} \to f^{-1}(V_i) \mid j \in J_i\}$ be an open affine cover of the pullback scheme $f^{-1}(V_i)$. As per usual we find that

$$X = f^{-1}(S) = f^{-1}\left(\bigcup_{i \in I} V_i\right) = \bigcup_{i \in I} f^{-1}(V_i) = \bigcup_{i \in I}\bigcup_{j \in J_i} U_{ij}$$

so $\{\iota_{ij} : U_{ij} \to X \mid i \in I, j \in J_i\}$ is an affine open cover of $X$.

Fix an $i \in I$ and $j \in J_i$ and consider the scheme $U_{ij} \times_{V_i} U_{ij}$. We claim now that the collection $\{U_{ij} \times_{V_i} U_{ij} \mid i \in I, j \in J_i\}$ is an open cover of $\Delta(X)$ in $X \times_S X$. To see this we first note that for each scheme $U_{ij} \times_{V_i} U_{ij}$,

$$\Delta_X^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij}$$

[34]Strictly speak we should be saying that the map $\Delta_X : X \to X \times_S X$ is a closed immersion of $S$-schemes, but it is a classical theorem that $\Delta_X$ is a closed immersion if and only if the image $|\Delta_X|(|X|)$ is a closed subspace of $|X \times_S X|$ (cf. [33, Corollary II.4.2] or [27, Proposition 5.3.9]), so we have made that our definition for the scope of these notes.

so

$$\Delta_X^{-1}\left(\bigcup_{i\in I}\bigcup_{j\in J_i} U_{ij}\times_{V_i} U_{ij}\right) = \bigcup_{i\in I}\bigcup_{j\in J_i} U_{ij}\times_{V_i} U_{ij}\Delta_X^{-1}(U_{ij}\times_{V_i} U_{ij}) = \bigcup_{i\in I}\bigcup_{j\in J_i} U_{ij} = X.$$

Because each $U_{ij}\times_{V_i} U_{ij} \hookrightarrow X\times_S X$, it follows from the above calculation of the preimage through $\Delta_X$ that we have a gluing

$$\Delta_X(X) \hookrightarrow \bigcup_{i\in I}\bigcup_{j\in J_i} U_{ij}\times_{V_i} U_{ij}.$$

In particular, there is a covering of the diagonal by $\{U_{ij}\times_{V_i} U_{ij} \hookrightarrow \Delta_X(X) \mid i\in I, j\in J_i\}$.

We now show that the covering of $\Delta_X(X)$ we constructed is affine open in $X\times_S X$. Because each scheme $V_i$ and $U_{ij}$ are affine, write $V_i \cong \operatorname{Spec} A_i$ and $U_{ij}\cong \operatorname{Spec} B_{ij}$. Then we have that

$$U_{ij}\times_{V_i} U_{ij} \cong \operatorname{Spec} B_{ij}\times_{A_i}\operatorname{Spec} B_{ij} \cong \operatorname{Spec}\left(B_{ij}\otimes_{A_i} B_{ij}\right)$$

by Lemma 3.5.7 so the scheme $U_{ij}\times_{V_i} U_{ij}$ is affine as well. Finally each scheme $U_{ij}\times_{V_i} U_{ij}$ is open in $X\times_S X$, as in Theorem 3.5.12 we defined the affine open base of $X\times_S X$ to be given by

$$\mathcal{B} = \{U_{ij}\times_{V_i} U_{ik} \mid i\in I; j,k\in J_i\}.$$

It remains to be shown that $\Delta_X(X)$ is a closed subscheme of the open subscheme

$$U := \bigcup_{i\in I}\bigcup_{j\in J_i} U_{ij}\times_{V_i} U_{ij}.$$

Write $j : U \to X\times_S X$ for the open immersion. For this it suffices to prove that each map $\Delta_{U_{ij}} : U_{ij} \to U_{ij}\times_{V_i} U_{ij}$ is a closed immersion, as these morphisms glue to give the diagonal $\Delta_X$. However, we can rewrite the diagonal $\Delta_{U_{ij}}$ as the map

$$\operatorname{Spec} B_{ij} \xrightarrow{\operatorname{Spec}(a\otimes b\mapsto ab)} \operatorname{Spec}\left(B_{ij}\otimes_{A_i} Bij\right),$$

and each of these maps are closed immersions by Proposition 3.7.10. Thus we conclude that the map $\Delta_X(X) \to U$ is a closed immersion. Writing the closed immersion $i : X \xrightarrow{\cong} \Delta_X(X) \to U$, this proves that the diagonal $\Delta_X : X \to X\times_S X$ factors as

$$\begin{tikzcd} X \arrow[rr, "\Delta_X"] \arrow[dr, "i"'] & & X\times_S X \\ & U \arrow[ur, "j"'] & \end{tikzcd}$$

and hence is a locally closed immersion. $\square$

Separated morphisms enjoy many convenient categorical properties involving being stable under pullback, being stable under composition, and other such things. We present a list of many of these properties in general, but do not prove all of them. Instead we refer the reader to [27, Section 5.4] for the general/original proofs of these facts, [33, Corollary II.4.6] for the Noetherian case, and [73, Section 10.1] for a more modern approach.

**Proposition 3.7.14.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of $S$-schemes. Then*

1. *If $f$ and $g$ are both open immersions, so is $g\circ f$.*
2. *If $f$ and $g$ are both closed immersions, so is $g\circ f$.*
3. *If $f$ is a monomorphism then $f$ is separated. In particular, open, closed immersions are separated.*

4. *If $f$ and $g$ are separated, so is $g \circ f$.*

5. *If $f$ is separated and $h : W \to Y$ is any morphism of schemes then in the pullback*

$$\begin{tikzcd} W \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ W \arrow[r, "h"'] & Y \end{tikzcd}$$

*the projection $p_1$ is separated as well. That is, separated morphisms are stable under base change.*

6. *Being separated is local on the target, i.e., $f$ is separated if and only if for any open cover $\{U_i \xrightarrow{\iota_i} Y \mid i \in I\}$ of $Y$ the morphism $f : f^{-1}(U_i) \to U_i$ is separated.*

*Sketch.* (1): It follows immediately from the fact that homeomorphisms are open maps and that $|f|$ and $|g|$ are both homeomorphisms onto open subspaces of each respective scheme that $|g| \circ |f|$ is a homeomorphism of $|X|$ onto an open subspace of $|Z|$. As for the isomorphism of sheaves, the adjunction calculus and definition of composition of comorphisms in **LRS** allows us to calculate that the map $(g \circ f)^\flat : (g \circ f)^{-1} \mathcal{O}_Z \to \mathcal{O}_X$ takes the form:

$$\begin{array}{c} \mathcal{O}_Z \xrightarrow{(g\circ f)^\sharp} (g \circ f)_* \mathcal{O}_X \\ \hline\hline \mathcal{O}_Z \xrightarrow{g^\sharp} g_* \mathcal{O}_Y \xrightarrow{g_*(f^\sharp)} g_* \left(f_* \mathcal{O}_X\right) \\ \hline g^{-1} \mathcal{O}_Z \xrightarrow{g^\flat} \mathcal{O}_Y \xrightarrow{f^\sharp} f_* \mathcal{O}_X \\ \hline f^{-1} \left(g^{-1} \mathcal{O}_Z\right) \xrightarrow{f^{-1}(g^\flat)} f^{-1} \mathcal{O}_Y \xrightarrow{f^\flat} \mathcal{O}_X \\ \hline\hline (g \circ f)^{-1} \mathcal{O}_Z \xrightarrow{(g\circ f)^\flat} \mathcal{O}_X \end{array}$$

Because $f^{-1}$ is a functor and each of $f^\flat$ and $g^\flat$ are isomorphisms, it follows that $(g \circ f)^\flat = f^\flat \circ f^{-1}(g^\flat)$ is an isomorphism of sheaves over $|X|$.

(2): That $|g \circ f| = |g| \circ |f|$ is a homoemorphism of $|X|$ onto a closed subspace of $|Z|$ follows mutatis mutandis to (1) (in this case use that homeomorphisms are closed morphisms). To see that the map $\mathcal{O}_Z \to (g \circ f)_* \mathcal{O}_X$ is surjective we once again only need to show this for points in $|X|$. We calculate that for any $x \in |X|$ the map $(g \circ f)^\sharp_x : \mathcal{O}_{Z,(g\circ f)(x)} \to \mathcal{O}_{X,x}$ can be written as:

$$\begin{array}{c} \mathcal{O}_{Z,(g\circ f)(x)} \xrightarrow{(g\circ f)^\sharp_x} \mathcal{O}_{X,x} \\ \hline \mathcal{O}_{Z,(g\circ f)(x)} \xrightarrow{g^\sharp_{f(x)}} g_*(\mathcal{O}_Y)_{f(x)} \xrightarrow{(g_* f^\sharp)_x} \left(g_*(f_* \mathcal{O}_X)\right)_x \\ \hline \mathcal{O}_{Z,(g\circ f)(x)} \xrightarrow{g^\sharp_{f(x)}} \mathcal{O}_{Y,f(x)} \xrightarrow{f^\sharp_x} \mathcal{O}_{X,x} \end{array}$$

Note that we used Proposition 3.4.25 a few times without comment in the above derivation and in particular used this to write $g_* f^\sharp_x = f^\sharp_*$. Because the maps $g^\sharp : \mathcal{O}_Z \to g_* \mathcal{O}_Y$ and $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$ are surjective, we find that the maps $g^\sharp_{f(x)} : \mathcal{O}_{Z,(g\circ f)(x)} \to \mathcal{O}_{Y,f(x)}$ and $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are surjective as well. However, this implies that the composition $\mathcal{O}_{Z,(g\circ f)(x)} \to \mathcal{O}_{X,x}$ is surjective for any $x \in |X|$ and hence that $(g \circ f)^\sharp : \mathcal{O}_Z \to (g \circ f)_* \mathcal{O}_X$ is as well. Thus we conclude that $g \circ f$ is a closed immersion, as desired.

(3): This is found in [27, Proposition 5.5.1].

(4): Assume that $f$ and $g$ are both separated and consider the commuting diagram:

$$\begin{tikzcd} X \arrow[r, "\Delta_X|_Y"] & X \times_Y X \arrow[r, "\alpha"] \arrow[d, "f \circ p_1"'] & X \times_Z X \arrow[d, "f \times_Z f"] \\ & Y \arrow[r, "\Delta_Y|_Z"'] & Y \times_Z Y \end{tikzcd}$$

Note that $f \circ p_1 = f \circ p_2 : X \times_Y X \to Y$ and the map $\alpha$ is induced via the universal property of the pullback $X \times_Z X$ as in the diagram:

$$\begin{tikzcd} X \times_Y X \arrow[dr, dashed, "\exists!\alpha"] \arrow[drr, bend left, "p_1"] \arrow[ddr, bend right, "p_2"'] & & \\ & X \times_Z X \arrow[r, "p_1"'] \arrow[d, "p_2"] & X \arrow[d, "g \circ f"] \\ & X \arrow[r, "g \circ f"'] & Z \end{tikzcd}$$

Moreover, because

$$p_1 \circ \alpha \circ \Delta_X|_Y = p_1 \circ \Delta_X|_Y = \mathrm{id}_X = p_2 \circ \Delta_X|_Y = p_2 \circ \alpha \circ \Delta_X|_Y$$

it follows that $\alpha \circ \Delta_X|Y = \Delta_X|_Z : X \to X \times_Z X$.

We now note that an elementary argument also shows that the square

$$\begin{tikzcd} X \times_Y X \arrow[r, "\alpha"] \arrow[d, "f \circ p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_Z X \arrow[d, "f \times_Z f"] \\ Y \arrow[r, "\Delta_Y|_Z"'] & Y \times_Z Y \end{tikzcd}$$

is a pullback in **Sch**[35]. Because $g : Y \to Z$ is separated, $\Delta_Y|_Z$ is a closed immersion so by from Proposition 3.4.26 we have that $\alpha$ is a closed immersion as well. Thus from the fact that $\Delta_X|_Y$ is also a closed immersion by virtue of $f$ being separated, applying Part (2) gives that $\Delta_X|_Z = \alpha \circ \Delta_X|_Y$ is a closed immersion as well.

(5): Consider a pullback square

$$\begin{tikzcd} S \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ W \arrow[r, "h"'] & Y \end{tikzcd}$$

and use it to form the pullback:

$$\begin{tikzcd} S \arrow[r, "p_1"] \arrow[d, "\Delta_S|_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\Delta_X|_Y"] \\ S \times_Y S \arrow[r, "p_1 \times p_1"'] & X \times_Y X \end{tikzcd}$$

Because $f$ is separated, $\Delta_X|_Y$ is a closed immersion. Invoking Proposition 3.4.26 allows us to deduce that $\Delta_S|_Y$ is a closed immersion as well and hence allows us to deduce that $p_2 : S \to Y$ is separated.

(6): This is [73, Proposition 10.1.11]. □

[35]This is a purely categorical argument. Any such square in a category $\mathscr{C}$ in which all relevant pullbacks exist is always a pullback.

**Proposition 3.7.15.** *Let $f : X \to S$ and $g : Y \to S$ be $S$-schemes and let $t : S \to T$ be a separated morphism of schemes. Then the canoncial map $X \times_S Y \to X \times_T Y$ is a closed immersion.*

*Proof.* Begin by writing $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ for the first and second projection and note that we induce maps to $T$ via the commuting diagram:

$$\begin{tikzcd}
X \times_S Y \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] & \\
Y \arrow[r, "g"'] & S \arrow[r, "t"'] & T
\end{tikzcd}$$

We write $\langle p, q\rangle_T : X \times_S Y \to X \times_T Y$ for the map induced by the universal property. Let $\pi : X \times_S Y \to S$. We then find that the diagram

$$\begin{tikzcd}
X \times_S Y \arrow[r, "\langle p,q\rangle_T"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_T Y \arrow[d, "f \times_T g"] \\
S \arrow[r, "\Delta_S|_T"'] & S \times_T S
\end{tikzcd}$$

is a pullback square. Because $t : S \to T$ is separated, the diagonal $\Delta_S|_T$ is a closed immersion. Finally using Proposition 3.4.26 gives that $\langle p, q\rangle_T$ is a closed immersion, as desired. $\square$

**Corollary 3.7.16.** *Let $f : X \to S$ and $g : Y \to S$ be scheme maps where $g$ is separated and and let $f : X \to Y$ be a morphism of $S$-schemes. Then the map $\langle \mathrm{id}_X, f\rangle : X \to X \times_S Y$ is a closed immersion.*

*Proof.* Apply Proposition 3.7.15 to the isomorphism $\theta : X \to X \times_S S$ and the map $\langle p, q\rangle_S : X \times_S S \to X \times_S Y$ after checking that $\langle \mathrm{id}_X, f\rangle = \langle p, q\rangle_S \circ \theta$. $\square$

**Remark 3.7.17.** In general when given $S$-schemes $f : X \to S$ and $g : Y \to S$ we always have that the morphism $X \times_S Y \to X \times_T Y$ described in Proposition 3.7.15 is a locally closed immersion (cf. [27, Corollary 5.3.10]). Similarly, for $S$-schemes $X$ and $Y$ together with a morphism $f : X \to Y$, the map $\langle \mathrm{id}_X, f\rangle : X \to X \times_S Y$ is always a locally closed immersion (cf. [27, Corollary 5.3.11]). What this tells us is that separated morphisms allow us to separate the pieces of the diagonal $\Delta_X(X)$ far enough apart that the complement of $\Delta_X(X)$ is actually open in all of $X \times_S X$ and not just in the cover we constructed in Lemma 3.7.13.

**Proposition 3.7.18.** *Let $f : X \to S$ be a separated morphism with $S$ an affine scheme. Then if $U$ and $V$ are affine open subschemes of $X$, $U \cap V$ is an affine subscheme of $X$ as well.*

*Proof.* We write $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$, and $S \cong \operatorname{Spec} R$ for commutative unital rings $R$. It is then routine to show that there is an isomorphism[36]

$$U \cap V = U \times_X V \cong (U \times_S V) \times_{X \times_S X} X$$

so that the diagram

$$\begin{tikzcd}
U \times_X V \arrow[r] \arrow[d] & U \times_S V \arrow[d] \\
X \arrow[r, "\Delta_X|_S"'] & X \times_S X
\end{tikzcd}$$

[36] We've already used this many times in our discussion of separated schemes and proofs of their propoerties thus far.

is a pullback diagram. However, since $f$ is separated the diagonal $\Delta_X|_S$ is a closed immersion. Thus by Proposition 3.4.26 the map $U \cap V = U \times_X V \to U \times_S V$ is a closed immersion as well. It thus follows that $U \cap V$ is a closed subscheme of $U \times_S V \cong \operatorname{Spec}(A \otimes_R B)$ and hence has the form

$$U \cap V \cong \operatorname{Spec}\left(\frac{A \otimes_R B}{\mathfrak{a}}\right)$$

for some ideal $\mathfrak{a} \trianglelefteq A \otimes_R B$ by Proposition 3.4.19. $\square$

**Remark 3.7.19.** This proposition fails to hold when $f$ is non-separated; for an example, let $X$ be the affine plane $\mathbb{A}^2_K$ with a doubled origin over a field $K$ (so glue two copies of $\mathbb{A}^2_K$ at the closed point $(x, y)$ describing the origin in $\mathbb{A}^2_K$). Let $U$ and $V$ be distinct copies of the affine plane $\mathbb{A}^2_K$ in $X$ (so each of $U$ and $V$ contain distinct origins). Then the intersection $U \cap V \cong \mathbb{A}^{2;*}_K$ is isomorphic to the punctured affine plane, which is not affine (it is the union of $\operatorname{Spec} K[x, y^{\pm 1}]$ and $\operatorname{Spec} K[x^{\pm 1}, y]$ — it is a nice exercise to show that the union of these affine schemes is not affine[37]).

We now show a proposition that allows us to deduce in a composition $g \circ f$ when the pre-composite $f$ is separated. For this we'll need to use the unproven result discussed above in Remark 3.7.17 that the map $\langle \mathrm{id}_X, f\rangle : X \to X \times_S Y$ is a locally closed immersion. The reason we need this is primarily because immersions are all separated, so in any case the map $\langle \mathrm{id}_X, f\rangle$ is always separated.

**Proposition 3.7.20.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of $S$-schemes. If $g \circ f$ is separated then $f$ is separated as well.*

*Proof.* Our strategy is to use Proposition 3.7.14 to show that $f$ is separated. Begin by factoring the map $f$ as:

$$\begin{tikzcd} X \arrow[r, "\langle \mathrm{id}_X, f\rangle"] \arrow[dr, "f"'] & X \times_Z Y \arrow[d, "p_2"] \\ & Y \end{tikzcd}$$

Note that because $X \times_Z Y$ arises as the pullback

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "g \circ f"'] & Z \end{tikzcd}$$

so because $g \circ f$ is separated it follows via Part (3) of Proposition 3.7.14 that $p_2$ is separated as well. Moreover, by Remark 3.7.17 the map $\langle \mathrm{id}_X, f\rangle$ is a locally closed immersion and hence separated by Part (3) of 3.7.14. From this we deduce that

$$f = p_2 \circ \langle \mathrm{id}_X, f\rangle$$

is separated by Part (4) of 3.7.14. $\square$

The big take-away from this section is the following definition/proposition pair which states that in the category of separated schemes over a base $S$, every morphism is separated. We'll define the category of separated schemes over $S$ first, however, so that we're all on the same page and then show that the category $\mathbf{SepSch}_{/S}$ has all finite limits.

**Definition 3.7.21.** Let $S$ be a base scheme. Define the category $\mathbf{SepSch}_{/S}$ of separated $S$-schemes as follows:

[37] And in fact was a question on the final exam for the algebraic geometry class I took as a graduate student.

- Objects: $S$-schemes $X$ for which the structure map $\nu_X : X \to S$ is a separated morphism.
- Morphisms: Morphisms of $S$-schemes.
- Composition and Identities: As in $\mathbf{Sch}_{/S}$.

**Proposition 3.7.22.** *Let $f \in \mathbf{SepSch}_{/S}(X, Y)$. Then $f$ is separated.*

*Proof.* Consider the commuting triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\nu_X"'] & & Y \arrow[dl, "\nu_Y"] \\ & S & \end{tikzcd}$$

induced by the fact that $f$ is a morphism of $S$-schemes. Because $\nu_X = \nu_Y \circ f$ is separated, it follows from Proposition 3.7.20 that $f$ is separated. $\square$

**Corollary 3.7.23.** *The category $\mathbf{SepSch}_{/S}$ is a full subcategory of $\mathbf{Sch}_{/S}$.*

**Proposition 3.7.24.** *The category $\mathbf{SepSch}_{/S}$ is finitely complete.*

*Proof.* Note that because the identity morphism is separated, $\mathbf{SepSch}_{/S}$ has a terminal object and it is $\mathrm{id}_S : S \to S$. Thus we only need to show the existence of binary pullbacks in $\mathbf{SepSch}_{/S}$ to prove the proposition. For this fix a cospan of separated $S$-schemes $X \xrightarrow{f} Z \xleftarrow{g} Y$. Then in the pullback diagram in $\mathbf{Sch}_{/S}$

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

we see that both $p_1$ and $p_2$ are separated by Part (5) of Proposition 3.7.14 and the fact that $f$ and $g$ are separated (this is Proposition 3.7.22). Then by Part (4) of Proposition 3.7.14 the composite $X \times_Z Y \to S$ is separated, as each morphism in the composition

$$X \times_Z Y \xrightarrow{p_1} X \xrightarrow{f} Z \xrightarrow{\nu_Z} S$$

is separated. Thus the pullback $(X \times_Z Y, p_1, p_2)$ in $\mathbf{Sch}_{/S}$ remains a pullback in $\mathbf{SepSch}_{/S}$. $\square$

## 3.8 Reduced Schemes

We now move on to give a short discussion on reduced schemes. Intuitively these are schemes whose structure sheaves $\mathcal{O}_X$ are free of nilpotents. There is much we could and should say about reduced schemes and the reduction of schemes, but we will take a lighter touch here and instead say that being reduced is not a property of schemes that is particularly well behaved. For instance, we will give a straightforward example (cf. Proposition 3.8.10) that shows even when we have nice reduced schemes, depending on the morphisms between these schemes the pullback may not be reduced. Our most fundamental result of this digression is Theorem 3.8.18 which describes how to build the full subcategory of reduced schemes as a coreflective subcategory of schemes, as well as how to construct pullbacks of reduced schemes.

**Definition 3.8.1.** A scheme $X$ is reduced if for every open $U \subseteq |X|$ the ring $\mathcal{O}_X(U)$ is reduced, i.e., $\mathcal{O}_X(U)$ has no nontrivial nilpotents.

**Remark 3.8.2.** It is worth keeping in mind that even if a ring $A$ is not reduced, the quotient ring $A/\sqrt{(0)}$ is always reduced. When referring to the reduction of a stalk of a sheaf, we'll write $\mathfrak{nil}(\mathcal{O}_{X,x})$ for the ideal

$$\mathfrak{nil}(\mathcal{O}_{X,x}) := \sqrt{(0)_{\mathcal{O}_{X,x}}} = \{f \in \mathcal{O}_{X,x} \mid \exists n \in \mathbb{N} \,.\, f^n = 0\}.$$

In general the ideal $\sqrt{(0)}$ is called the nilradical of a ring $A$ because it contains all the nilpotents in $A$.[38]

The next few examples show that we can easily build reduced and non-reduced affine schemes by starting either with reduced rings or introducing nilpotents to polynomial rings.

**Example 3.8.3.** If $A$ is any ring then the scheme $\operatorname{Spec} A$ is reduced if and only if $A$ is reduced.

**Example 3.8.4.** If $A$ is any nonzero ring then the scheme $\operatorname{Spec} A[x]/(x^2)$ is never reduced.

**Example 3.8.5.** If $A$ is a reduced ring then the scheme $\mathbb{P}^n_A$ is reduced.

**Remark 3.8.6** (Achtung!)**.** If $X$ is non-reduced it can be the case that the global sections $\mathcal{O}_X(|X|)$ is a reduced ring. This only, however, happens for non-affine schemes, so keep in mind whenever you're working with schemes that the global sections do not determin the scheme.

**Definition 3.8.7.** A scheme $X$ is integral if and only if every for every open $U \subseteq |X|$ the ring $\mathcal{O}_X(U)$ is an integral domain.

**Remark 3.8.8.** For the careful/astute/pathology-minded reader: you may be worndering about the empty open $\varnothing \subseteq |X|$. For any scheme we have $\mathcal{O}_X(\varnothing) = 0$ (this is a consequence of $\mathcal{O}_X$ being a sheaf), and you may be wondering if we have forgotten to claim that we need to assert that $U \neq \varnothing$ in Definition 3.8.7. However, we have not. A ring $R$ is an integral domain if and only if $R$ is commutative and $ab = 0$ implies that $a = 0$ or $b = 0$. In the zero ring $0 = \{*\}$, we always have that $ab = 0$ implies $a = b = 0$, so the zero ring is indeed an integral domain.[39] Be aware that some references (cf. [73, Definition 5.2.4], for instance) only ask for the condition in Definition 3.8.7 to hold for $U \neq \varnothing$, but this is not a necessary worry by the observation here.

**Proposition 3.8.9.** *Let $X$ be a scheme. Then $X$ is reduced if and only if for all $x \in |X|$ the local ring $\mathcal{O}_{X,x}$ is reduced.*

*Proof.* $\Longrightarrow$ : Assume that $X$ is reduced. For each $x \in |X|$ find an $f_x \in \mathcal{O}_{X,x}$ and an $n \in \mathbb{N}$ for which $f_x^n = 0$. Now, find opens $U \subseteq |X|$ with $x \in U$ and sections $f_U \in \mathcal{O}_X(U)$ for which $\alpha_U(f_U) = f_x$ where $\alpha_U : \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is the colimit map; note that this implies that $\alpha_U(f_U^m) = f_x^m$ for all $m \in \mathbb{N}$. Now because the colimit constructing $\mathcal{O}_{X,x}$ is a filtered colimit we can find an open $U_0 \subseteq X$ with $x \in U_0$ and $f_{U_0} \in \mathcal{O}_X(U_0)$ such that $\alpha_{U_0}(f_{U_0}) = f_x$ with the following property: if $V \subseteq U_0$ is open in $|X|$ with $x \in V$ and if $f_V \in \mathcal{O}_X(V)$ with $\alpha_V(f_v) = f_x$ then $\mathcal{O}_X(U_0 \supseteq V)(f_{U_0}) = f_V$. Because $f_x^n = 0$ and the each ring $\mathcal{O}_X(U)$ is reduced it follows that $f_V = 0$ for all $V \subseteq U_0$ with $x \in U_0$. This implies, however, that $f_x = \alpha_{U_0}(f_{U_0}) = \alpha_{U_0}(0) = 0$ so we conclude that $\mathcal{O}_{X,x}$ is reduced.

$\Longleftarrow$ : Assume that each local ring $\mathcal{O}_{X,x}$ is reduced and let $U \subseteq |X|$ be open. If $U = \varnothing$ we are done so assume $U \neq \varnothing$ and let $f \in \mathcal{O}_X(U)$ with $f^n = 0$ for some $n \in \mathbb{N}$. Then the image of $f$ in each local ring $\mathcal{O}_{X,u}$ is zero so we conclude that $f = 0$ by Proposition 2.1.10. $\square$

Being reduced is a strong property, as it says there is no infinitesimal information in the rings of the structure sheaf. However, just because the algebras of reduced schemes do not contain infinitesimal information does not mean that morphisms between such schemes are unable to introduce and record infinitesimal

[38] This is one of the places where it is absolutely crucial that we work with commutative rings instead of noncommutative rings. In the noncommutative case you end up working with radical theory and have to deal with the fact that there are uppoer and lower nilradicals of a ring (which in turn are properly distinct from the Jacobson radical of the ring: it is possible to have a chain of the form $\mathfrak{nil}(R)_{\text{lower}} \subsetneq \mathfrak{nil}(R)_{\text{upper}} \subsetneq J(R)$.

[39] Perhaps the stupidest integral domain of all time, but c'est la vie.

information; in fact, ramified maps (such as the double cover of the line $t$ by a parabola $t^2$) introduce nilpotent information into the fibres of the maps, and this is an obstruction to reduced schemes being stable under pullback. This, while inconvenient, is not a bug; it allows us to introduce a theory of deformations and use singularities and changes in fibre dimension to study schemes and the actions/symmetries upon them. Note that in the proposition below our example is essentially[40] a special case of Example 3.5.22.

**Proposition 3.8.10.** *Reduced schemes are not stable under pullback, i.e., there is a cospan of schemes $X \times_Z Y$ with $X, Y, Z$ reduced for which $X \times_Z Y$ is not reduced.*

*Proof.* Let $K$ be an algebraically closed field and let $X = Z = \mathbb{A}^1_K$ while $Y = \operatorname{Spec} K$. Define the map $X \to Z$ to be $\operatorname{Spec}(t \mapsto t^2)$ and let $Y \to Z$ be the map $\operatorname{Spec}(x \mapsto 0)$. Then we find that in the pullback

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & \mathbb{A}^1_K \arrow[d, "\operatorname{Spec}(t \mapsto t^2)"] \\ \operatorname{Spec} K \arrow[r, "\operatorname{Spec}(x \mapsto 0)"'] & \mathbb{A}^1_K \end{tikzcd}$$

we have

$$X \times_Z Y = \mathbb{A}^1_K \times_{\mathbb{A}^1_K} \operatorname{Spec} K \cong f^{-1}(x - 0) \cong \operatorname{Spec}\left(\frac{K[x]}{(x^2)}\right)$$

which is evidently non-reduced. $\square$

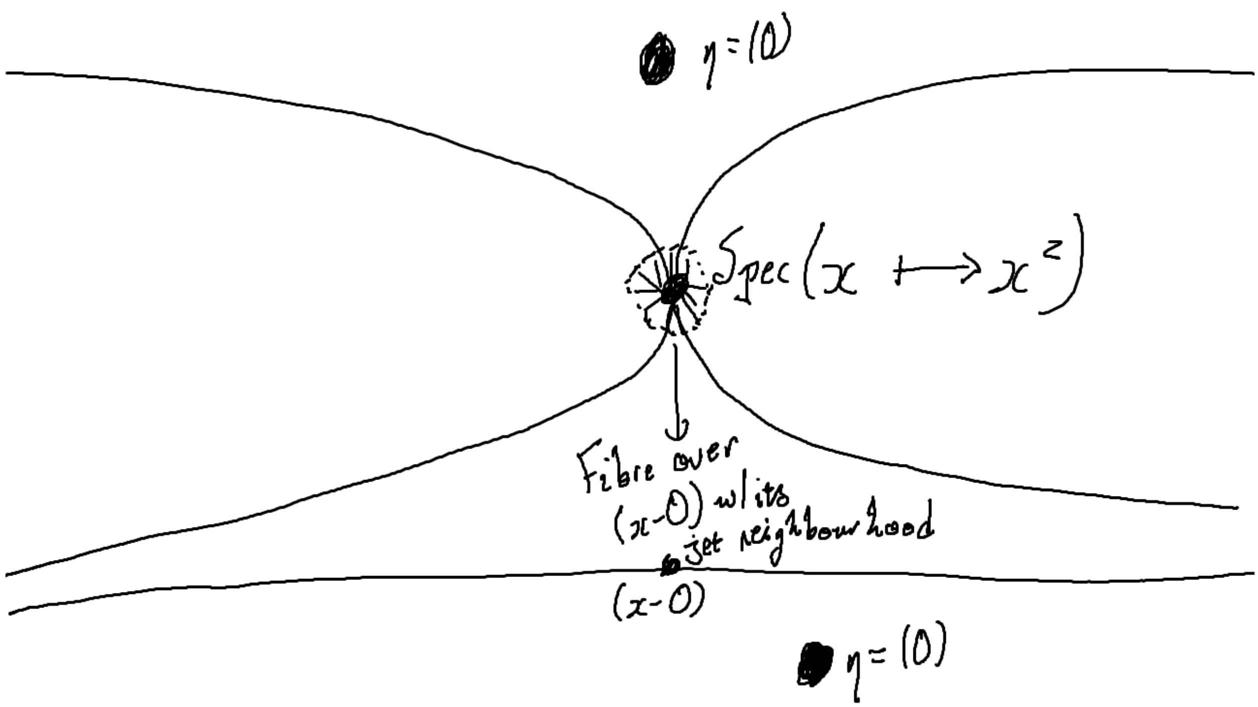


Figure 3.3: This is a visualization of how the map $\operatorname{Spec}(x \mapsto x^2)$ is non-reduced at the origin.

[40]But not because technical reasons like characteristic zero versus arbitrary characteristic.

The good news is that all is not lost. This example simply shows our deformations can record ramification in fibres, which is desired at a scheme-theoretic level. If we want to stay in a completely reduced world, we simply have to show how to build canonical reductions of our schemes and how they differ from our original schemes. Our first result in this vein is to show *what* the reduction is as well as its functorial properties. As for how to intuit the canonical scheme, affine-locally the reduction $X_{\mathrm{red}}$ looks like $\operatorname{Spec} A/\sqrt{(0)} \to \operatorname{Spec} A$.

**Proposition 3.8.11.** *Let $X$ be a scheme. There is a unique closed reduced subscheme $X_{\mathrm{red}}$ with $|X_{\mathrm{red}}| = |X|$.*

*Proof.* This is [27, Proposition 5.1.1] and we present that argument. To build the scheme $X_{\mathrm{red}}$ it suffices to construct the scheme affine locally and glue using Proposition 3.4.8. It thus suffices to construct $(\operatorname{Spec} A)_{\mathrm{red}}$ for an affine scheme $\operatorname{Spec} A$, as from there the result follows via gluing.

Define the ring $A_{\mathrm{red}} := A/\sqrt{(0)}$. We will show that $|\operatorname{Spec} A_{\mathrm{red}}| \cong |\operatorname{Spec} A|$ and that there is no other closed subscheme $Z$ of $\operatorname{Spec} A$ with $|Z| \cong |\operatorname{Spec} A|$. First let us verify the stated homeomorphism. Since the ring map $A \to A/\sqrt{(0)}$ is a surjection of rings we know that the embedding $\operatorname{Spec} A/\sqrt{(0)} \to \operatorname{Spec} A$ is a closed immersion. As such, we just need to verify that this map is surjective with a continuous inverse. However, this is routine by noting that the prime ideals in $A/\sqrt{(0)}$ correspond to prime ideals modulo nilpotents as

$$\sqrt{(0)} = \bigcap_{\mathfrak{p} \in |\operatorname{Spec} A|} \mathfrak{p}\,.$$

We now show the second claim, i.e., that $\operatorname{Spec} A_{\mathrm{red}}$ is unique among closed subschemes $Z$ of $\operatorname{Spec} A$ with the property that $|Z| \cong |X|$. For this fix such a closed subscheme and note that there is a surjection of sheaves of rings $\mathcal{O}_X \to (i_Z)_* \mathcal{O}_Z$. This implies that there is a sheaf $\mathcal{I}$ where each $\mathcal{I}(U)$ is an ideal of $\mathcal{O}_X(U)$. In particular, this induces a short exact sequence of sheaves of $\mathcal{O}_X$-modules[41]

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow (i_Z)_* \mathcal{O}_Z \longrightarrow 0$$

Using the same strategy as above, we reduce to the case where $X = \operatorname{Spec} A$ and $Z$ is a closed subscheme $\operatorname{Spec} A/\mathfrak{a}$ for some ideal $\mathfrak{a}$. In particular, the sheaf of ideals $\mathcal{I}$ is given on global sections by $\mathcal{I}(X) = \mathfrak{a}$ and on stalks by $\mathcal{I}_x = \mathfrak{a}_x$. Moreover, because $Z$ is reduced it follows that for each $x \in X$ the stalk $\mathcal{I}_x$ is an ideal of $\mathcal{O}_{X,x}$ for which is contained in the prime ideal $\mathfrak{p}_x$. Because this happens for each $x \in |X|$ we find that the ideal $\mathfrak{a}$ must be contained in each prime ideal $\mathfrak{p}$ of $A$ and hence in the nilradical $\sqrt{(0)}$ of $A$. Moreover if $Z \ncong \operatorname{Spec}(A/\sqrt{(0)})$ then there is at least one point $x \in |X|$ for which $\mathcal{I}_x \neq \mathfrak{nil}(A_{\mathfrak{p}_x})$. However this implies that

$$(i_Z)_* \mathcal{O}_{Z,x} \cong A_{\mathfrak{p}_x}/\,\mathfrak{a}_x \ncong A_{\mathfrak{p}_x}/(\sqrt{(0)})_x$$

so $\mathcal{O}_{Z,x}$ is not reduced. This implies in turn that $Z$ cannot be reduced. ☐

**Remark 3.8.12.** In what follows we'll be giving an identification of spaces $|X_{\mathrm{red}}| = |X|$. In particular, this greatly simpilfies writing down the scheme-theoretic morphism $i : X_{\mathrm{red}} \to X$. Because $|i| = \mathrm{id}_{|X|}$ the comorphism of structure sheaves $\mathcal{O}_X \to i_* \mathcal{O}_{X_{\mathrm{red}}}$ is simply a morphism of sheaves $\mathcal{O}_X \to \mathcal{O}_{X_{\mathrm{red}}}$[42].

**Remark 3.8.13.** To make it explicit and clear, if $X$ is a scheme then the sheaf of rings $\mathcal{O}_{X_{\mathrm{red}}}$ is the unique (up to unique isomorphism) sheaf of rings on $|X|$ defined by the property that stalkwise $\mathcal{O}_{X_{\mathrm{red}},x} = \mathcal{O}_{X,x}\,/\,\mathfrak{nil}(\mathcal{O}_{X,x})$.[43]

[41] For the experts: this is in fact a short exact sequence of quasi-coherent sheaves.

[42] A nice calculation and use of the definitions shows that the pushforward of the identity morphism is the identity functor.

[43] In my opinion the right way to formalize and describe this sheaf of rings is by using the 2-categorical phenomenon regarding what we call quasi-coherent sheaves (which is 2-categorical because it involves a certain gluing of sheaves at a functorial level), but we will not provide that perspective explicitly here. More information on this can be found (sadly without the 2-categorical language) in the [27] proof of Proposition 3.8.11 (namely [27, Proposition 5.1.1]) and in [27, Section 1.4]. These techniques are not necessary to construct this scheme, however; inniether [33] nor [21] take this approach.

This assignment is functorial in schemes $X$, as we'll show below. In particular, it gives us a functor to the category of reduced schemes which we'll show is a right adjoint to the inculsion of reduced schemes into schemes (so the category of reduced schemes over $S$ is a coreflective subcategory of $\mathbf{Sch}_{/S}$). In particular write $\mathbf{RedSch}_{/S}$ for the subcategory of $\mathbf{Sch}_{/S}$ whose objects are all reduced schemes over a base scheme $S$. For this we'll need one lemma describing how to build $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ from an $S$-scheme map $f : X \to Y$.

**Definition 3.8.14.** The category $\mathbf{RedSch}_{/S}$ of reduced $S$-schemes is defined by:

- Objects: Reduced $S$-schemes.
- Morphisms: $f : X \to Y$ is a morphism in $\mathbf{RedSch}_{/S}$ if and only if $f$ is a morphisms of $S$-schemes.
- Composition and Identities: As in $\mathbf{Sch}_{/S}$.

**Lemma 3.8.15.** *Let $f : X \to Y$ be a morphism of $S$-schemes. Then there is a morphism $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ which satisfies, if $f : X \to Y$ and $g : Y \to Z$ are scheme morphisms, $(g \circ f)_{\mathrm{red}} = g_{\mathrm{red}} \circ f_{\mathrm{red}}$. In particular there is a reduction functor* $\mathrm{red} : \mathbf{Sch}_{/S} \to \mathbf{RedSch}_{/S}$.

*Proof.* We first define $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$. Note that since $|X_{\mathrm{red}}| = |X|$, we define $|f_{\mathrm{red}}| = |f|$. To define $f^{\sharp}_{\mathrm{red}}$, we note that by construction of each of $X_{\mathrm{red}}$ and $Y_{\mathrm{red}}$ there is a short exact sequence of sheaves of $\mathcal{O}_X$ and $\mathcal{O}_Y$ modules

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_{\mathrm{red}}} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{Y^{\mathrm{red}}} \longrightarrow 0$$

where the sheaves $\mathcal{I}_X$ and $\mathcal{I}_Y$ are sheaves of $\mathcal{O}_X$-ideals and $\mathcal{O}_Y$-ideals with the defining property that $\mathcal{I}_{X,x} = \mathfrak{nil}(\mathcal{O}_{X,x})$ and $\mathcal{I}_{Y,y} = \mathfrak{nil}(\mathcal{O}_{Y,y})$ for all $x \in |X|$ and $y \in |Y|$, respectively. Moreover, for each $x \in |X|$ we find that the stalk map $f^{\sharp}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ sends nilpotents to nilpotents; this implies that $f^{\sharp}$ restricts from $\mathcal{O}_Y \to f_*\,\mathcal{O}_X$ to a map $f^{\sharp}_{\mathcal{I}} : \mathcal{I}_Y \to f_*\,\mathcal{I}_X$. In particular, there is a commuting diagram of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_Y \arrow[r] \arrow[d, "f^{\sharp}_{\mathcal{I}}"'] & \mathcal{O}_Y \arrow[r] \arrow[d, "f^{\sharp}"'] & \mathcal{O}_{Y_{\mathrm{red}}} \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & f_*\,\mathcal{I}_X \arrow[r] & f_*\,\mathcal{O}_X \arrow[r] & f_*\,\mathcal{O}_{X_{\mathrm{red}}} \arrow[r] & 0
\end{tikzcd}$$

with each row a short exact sequence of $\mathcal{O}_Y$ modules and the existence of $\rho$ coming from the fact that the category of modules with respect to a ring object in a topos is an Abelian category[44].

We now need to show that $\rho$ is a morphism between sheaves of local rings. Taking stalks, for any $x \in |X|$, gives a commuting diagram of $\mathcal{O}_{Y,f(x)}$ modules

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{Y,f(x)} \arrow[r] \arrow[d, "(f^{\sharp}|_{\mathcal{I}})_x"'] & \mathcal{O}_{Y,f(x)} \arrow[r] \arrow[d, "f^{\sharp}_x"'] & \mathcal{O}_{Y_{\mathrm{red}},f(x)} \arrow[r] \arrow[d, dashed, "\exists!\rho_x"] & 0 \\
0 \arrow[r] & \mathcal{I}_{X,x} \arrow[r] & \mathcal{O}_{X,x} \arrow[r] & \mathcal{O}_{X_{\mathrm{red}},x} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Note that the map $\rho_x$ is induced by the universal property of the quotient ring/module

$$\mathcal{O}_{Y_{\mathrm{red}},f(x)} = \left(\mathcal{O}_{Y,f(x)}\right)_{\mathrm{red}} = \frac{\mathcal{O}_{Y,f(x)}}{\mathfrak{nil}(\mathcal{O}_{Y,f(x)})} = \frac{\mathcal{O}_{Y,f(x)}}{\mathcal{I}_{Y,f(x)}}.$$

[44] This is a high tech fact saying that we can take quotients of sheaves and have them work structurally how we desire. I did not want to show it explicitly due to the time it would take (although future me may write it up in an appendix), but if you want to verify how this works the trick is to take the quotient as presheaves and then sheafify. Ring objects in sheaf toposes are just sheaves of rings (this is straightforward to show) and since the inclusion to presheaves preserves ring objects, you only need to argue sheafification preserves diagrams like above. This follows, however, by using thaat sheafifcation preserves finite limits (cf. [54]).

A straightforward argument shows that the maps $\rho_x$ are local morphisms of local rings by using that $\rho_x$ take the form
$$\rho_x(\overline{a}_{\mathcal{O}_{X,x}}) = \overline{f_x^\sharp(a)}_{\mathcal{O}_{Y,y}},$$
where $\overline{\alpha}_R$ means taking the image of the element $\alpha \in R$ under the quotient $R \to R_{\text{red}}$. Thus the map $\rho : \mathcal{O}_{Y_{\text{red}}} \to f_* \mathcal{O}_{X_{\text{red}}}$ is a comorphism of sheaves. We thus define $f_{\text{red}}$ to be the map
$$f_{\text{red}} := (|f|, \rho),$$
which is a morphism of locally ringed spaces by construction.

Consider any pair of composable morphisms of $S$-schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$. By construction we have that
$$|g \circ f|_{\text{red}} = |g|_{\text{red}} \circ |f|_{\text{red}}.$$
To check that $(g \circ f)^\sharp_{\text{red}} = g^\sharp \circ f^\sharp$ simply use the fact that the map $\mathcal{O}_Z \to (g \circ f)_{\text{red}} \mathcal{O}_X$ is the unique map induced by a universal property. Thus $(g \circ f)_{\text{red}} = g_{\text{red}} \circ f_{\text{red}}$; the fact that $(\text{id}_X)_{\text{red}} = \text{id}_{X_{\text{red}}}$ is trivial to check. Putting this all together we get the existence of the functor $\text{red} : \mathbf{Sch}_{/S} \to \mathbf{RedSch}_{/S}$, as desired. $\square$

**Lemma 3.8.16.** *For any $S$-scheme $X$ the inclusion $\varepsilon_X : X_{\text{red}} \to X$ forms the object assignment of a natural transformation $\varepsilon : \text{incl} \circ \text{red} \Rightarrow \text{id}_{\mathbf{Sch}_{/S}}$.*

*Proof.* Let $f : X \to Y$ be a morphism of schemes. We must show that the diagram
$$\begin{tikzcd} X_{\text{red}} \arrow[r, "\varepsilon_X"] \arrow[d, "f_{\text{red}}"'] & X \arrow[d, "f"] \\ Y_{\text{red}} \arrow[r, "\varepsilon_Y"'] & Y \end{tikzcd}$$
commutes. However, this was done in the proof of Lemma 3.8.15. Topologically there is nothing to show as $|\varepsilon_X| = \text{id}_{|X|}, |\varepsilon_Y| = \text{id}_{|Y|}$, and $|f_{\text{red}}| = |f|$. Alternatively, the commutativity of the diagram
$$\begin{tikzcd} \mathcal{O}_Y \arrow[r, "\varepsilon_Y^\sharp"] \arrow[d, "f^\sharp"'] & \mathcal{O}_{Y_{\text{red}}} \arrow[d, "f_{\text{red}}^\sharp"] \\ f_* \mathcal{O}_X \arrow[r, "\varepsilon_X^\sharp"'] & f_* \mathcal{O}_{X_{\text{red}}} \end{tikzcd}$$
is exactly the rightmost square in the diagram:
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_Y \arrow[r] \arrow[d, "f_{\mathcal{I}}^\sharp"'] & \mathcal{O}_Y \arrow[r] \arrow[d, "f^\sharp"'] & \mathcal{O}_{Y_{\text{red}}} \arrow[r] \arrow[d, "\rho"] & 0 \\ 0 \arrow[r] & f_* \mathcal{I}_X \arrow[r] & f_* \mathcal{O}_X \arrow[r] & f_* \mathcal{O}_{X_{\text{red}}} \arrow[r] & 0 \end{tikzcd}$$
$\square$

**Lemma 3.8.17.** *If $X$ is any reduced $S$-scheme then there is a unique isomorphism $X_{\text{red}} \cong X$.*

*Proof.* By Proposition 3.8.9 since $X$ is reduced, each ring $\mathcal{O}_{X,x}$ is recduced (for all $x \in |X|$). Now by Remark 3.8.13 and the fact that the rings $\mathcal{O}_{X,x}$ are reduced (and hence have trivial nilradical) we have that the stalks of $\varepsilon_X$ take the form
$$\mathcal{O}_{X,x} \xrightarrow{\varepsilon_{X,x}} \mathcal{O}_{X_{\text{red}},x} = \frac{\mathcal{O}_{X,x}}{\mathfrak{nil}(\mathcal{O}_{X,x})} = \frac{\mathcal{O}_{X,x}}{(0)}.$$
Thus each $\varepsilon_{X,x}$ is an isomorphism; by Proposition 2.1.9 it follows that $\varepsilon_X$ is as well. Finally that this isomorphism is unique is trivial to verify from the fact that the stalks are given by quotienting at the zero ideal. $\square$

**Theorem 3.8.18.** *For any base scheme $S$ the category $\mathbf{RedSch}_{/S}$ of reduced $S$-schemes is a coreflective subcategory of the category of $S$-schemes $\mathbf{Sch}_{/S}$. That is, if* $\mathrm{incl} : \mathbf{RedSch}_{/S} \to \mathbf{Sch}_{/S}$ *is the inclusion there is an adjunction:*

$$\begin{tikzcd} \mathbf{Sch}_{/S} \arrow[rr, bend right, "\mathrm{red}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{RedSch}_{/S} \arrow[ll, bend right, "\mathrm{incl}"'] \end{tikzcd}$$

*Proof.* We will prove the adjunction by showing that $\varepsilon$ is the counit of adjunction and hence satisfies the couniversal property. That is, we will show that if $Y$ is any reduced scheme and if $X$ is any scheme then any map $f : \mathrm{incl}(Y) \to X$ factors uniquely as

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, dashed, "\exists!\,\mathrm{incl}(f^\flat)"'] & & X \\ & \mathrm{incl}(X_{\mathrm{red}}) \arrow[ur, "\varepsilon_X"'] & \end{tikzcd}$$

where there is a unique morphism $g^\flat : Y \to X_{\mathrm{red}}$ making the diagram commute. First define $f^\flat : Y \to X_{\mathrm{red}}$ as the composite:

$$Y \xrightarrow{\varepsilon_Y^{-1}} Y_{\mathrm{red}} \xrightarrow{f_{\mathrm{red}}} X_{\mathrm{red}}$$

It then follows that

$$\varepsilon_X \circ \mathrm{incl}(f^\flat) = \varepsilon_X \circ f^\flat = \varepsilon_X \circ f_{\mathrm{red}} \circ \varepsilon_Y^{-1} = f \circ \varepsilon_Y \circ \varepsilon_Y^{-1} = f,$$

so the diagram does indeed factor as desired. Finally for the uniqueness of $f^\flat$ we note that this follows from the uniqueness of the isomorphism $\varepsilon_Y^{-1}$ implied by Lemma 3.8.17 and the fact that $f_{\mathrm{red}}$ is constructed uniquely by universal properties in the proof of Lemma 3.8.15. Thus the diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, dashed, "\exists!\,\mathrm{incl}(f^\flat)"'] & & X \\ & \mathrm{incl}(X_{\mathrm{red}}) \arrow[ur, "\varepsilon_X"'] & \end{tikzcd}$$

commutes, proving that $\mathrm{incl} \dashv \mathrm{red} : \mathbf{RedSch}_{/S} \to \mathbf{Sch}_{/S}$. ∎

**Remark 3.8.19.** It follows from the the fact that $\mathbf{RedSch}_{/S}$ is a coreflective subcategory of $\mathbf{Sch}_{/S}$ that $\mathbf{RedSch}_{/S}$ is a full subcategory. Note that this is because the unit of adjunction $\eta : \mathrm{id}_{\mathbf{RedSch}_{/S}} \to \mathrm{red} \circ \mathrm{incl}$ is an isomorphism.

**Remark 3.8.20.** For the adventurous: If you can show that the adjunction diagram above exists without using the isomorphism $\varepsilon_Y^{-1}$, you can give an extremely slick proof of the fact that $\varepsilon_X : X_{\mathrm{red}} \to X$ is an isomorphism for reduced schemes $X$.[45] While this is really quite slick, I did not take this approach because of how straightforward the adjunction is to prove when we have access to $\varepsilon_Y^{-1}$.

Our final remarks about this theorem before moving on to discuss finite type schemes is a construction of the pullback in $\mathbf{RedSch}_{/S}$ and a compoarison of this pullback with the pullback in $\mathbf{Sch}_{/S}$. While the two are different, as we will see, the pullback of reduced schemes does at least exist and is given in a canonical way.

[45] You can characterize coreflective subcategories $\mathrm{incl} : \mathscr{C} \to \mathscr{D}$ as those adjunctions $\mathrm{incl} \dashv R : \mathscr{C} \to \mathscr{D}$ whose counit $\varepsilon : \mathrm{incl} \circ R \Rightarrow \mathrm{id}_{\mathscr{D}}$ is an isomorphism on the objects of $\mathscr{C}$.

**Proposition 3.8.21.** *The category* $\mathbf{RedSch}_{/S}$ *is finitely complete for any base scheme* $S$*. Moreover, the pullback is given by, for any cospan of reduced schemes* $X \xrightarrow{f} Z \xleftarrow{g} Y$*,*

$$X \overset{\mathbf{RedSch}}{\times}_Z Y \cong \left(X \overset{\mathbf{Sch}}{\times}_Z Y\right)_{\mathrm{red}}.$$

*Proof.* The existence of all finite limits in $\mathbf{RedSch}_{/S}$ follows from the fact that $\mathbf{Sch}_{/S}$ is fintiely complete and the right adjoint $\mathrm{red} : \mathbf{Sch}_{/S} \to \mathbf{RedSch}_{/S}$ creates limits. Finally the fact that the right adjoint creates limits gives the desired equation

$$X \overset{\mathbf{RedSch}}{\times}_Z Y \cong \left(X \overset{\mathbf{Sch}}{\times}_Z Y\right)_{\mathrm{red}}.$$

∎

**Example 3.8.22.** This is an example of a pullback of reduced schemes which is not a pullback of schemes (we'll be reusing the example given in Proposition 3.8.10). Let $K$ be a field, let $S = \operatorname{Spec} K$, and let $X = Z = \mathbb{A}^1_K$ in the category $\mathbf{RedSch}_{/S}$ equipped with the map $\operatorname{Spec}(t \mapsto t^2) : \mathbb{A}^1_K \to \mathbb{A}^1_K$. Consider the section $0 : \operatorname{Spec} K \to \mathbb{A}^1_K$ given by $\operatorname{Spec}(x \mapsto 0)$ and note that

$$\mathbb{A}^1_K \overset{\mathbf{Sch}_{/K}}{\times}_{\mathbb{A}^1_K} \operatorname{Spec} K \cong \operatorname{Spec}\left(\frac{K[x]}{(x^2)}\right).$$

It then follows that

$$\mathbb{A}^1_K \overset{\mathbf{RedSch}_{/K}}{\times}_{\mathbb{A}^1_K} \operatorname{Spec} K \cong \left(\mathbb{A}^1_K \overset{\mathbf{Sch}_{/K}}{\times}_{\mathbb{A}^1_K} \operatorname{Spec} K\right)_{\mathrm{red}} \cong \operatorname{Spec}\left(\frac{K[x]}{(x^2)}\right)_{\mathrm{red}} \cong \operatorname{Spec} K.$$

**Proposition 3.8.23.** *Let* $f : X \to Y$ *be a morphism of schemes. Then* $f$ *is separated if and only if* $f_{\mathrm{red}}$ *is separated.*

*Proof.* $\Longrightarrow$ : Assume that $f$ is separated and consider the naturality square of $\varepsilon$:

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "\varepsilon_X"] \arrow[d, "f_{\mathrm{red}}"'] & X \arrow[d, "f"] \\ Y_{\mathrm{red}} \arrow[r, "\varepsilon_Y"'] & Y \end{tikzcd}$$

Because $\varepsilon_X$ is a closed immersion it is separated by Part (3) of Proposition 3.7.14 so the total composite $f \circ \varepsilon_X = \varepsilon_Y \circ f_{\mathrm{red}}$ is separated by Part (4) of Proposition 3.7.14. Applying Proposition 3.7.20 to $\varepsilon_Y \circ f_{\mathrm{red}}$ allows us to deduce that $f_{\mathrm{red}}$ is separated.

$\Longleftarrow$ : Assume that $f_{\mathrm{red}}$ is separated and consider the commuting diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "\Delta_{X_{\mathrm{red}}|Y_{\mathrm{red}}}"] \arrow[d, "\varepsilon_X"'] & X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} X_{\mathrm{red}} \arrow[d, "\langle \varepsilon_X, \varepsilon_X \rangle_Y"] \\ X \arrow[r, "\Delta_{X|Y}"'] & X \times_Y X \end{tikzcd}$$

It follows from the isomorphism $X_{\mathrm{red}} \times_Y X_{\mathrm{red}} \cong X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} X_{\mathrm{red}}$ that we can rewrite this diagram as:

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "\Delta_{X_{\mathrm{red}}|Y}"] \arrow[d, "\varepsilon_X"'] & X_{\mathrm{red}} \times_Y X_{\mathrm{red}} \arrow[d, "\varepsilon_X \times \varepsilon_X"] \\ X \arrow[r, "\Delta_{X|Y}"'] & X \times_Y X \end{tikzcd}$$

It suffices to prove that the image of $|\Delta_X|_Y|$ is closed (cf. Definition 3.7.8). For this we note that upon restricting to the underlying spaces in the diagram

$$\begin{tikzcd} |X_{\mathrm{red}}| \arrow[r, "|\Delta_{X_{\mathrm{red}}}|_Y|"] \arrow[d, "|\varepsilon_X|"'] & |X_{\mathrm{red}} \times_Y X_{\mathrm{red}}| \arrow[d, "|\varepsilon_X \times \varepsilon_X|"] \\ |X| \arrow[r, "|\Delta_X|_Y|"'] & |X \times_Y X| \end{tikzcd}$$

both vertical arrows $|\varepsilon_X|$ and $|\varepsilon_X \times \varepsilon_X|$ are homeomorphisms. Thus, because the diagram commutes, because homeomorphisms are closed morphisms, and the image of $|\Delta_{X_{\mathrm{red}}}|_Y|$ is closed in $|X_{\mathrm{red}} \times_Y X_{\mathrm{red}}|$, we conclude that the image of $|\Delta_X|_Y|$ is closed in $|X \times_Y X|$. ☐

**Corollary 3.8.24.** *The category* $\mathbf{RedSepSch}_{/S}$ *of separated reduced* $S$*-schemes is a coreflective subcategory of the category* $\mathbf{SepSch}_{/S}$ *of separated* $S$*-schemes.*

*Proof.* To show this it suffices to prove that every map used in the proof of Theorem 3.8.18 is separated when the given $S$-schemes are themselves separated. However, this follows immediately from the fact that closed immersions are separated by Part (3) of Proposition 3.7.14 (in particular every component of $\varepsilon$ is a separated morphism) and by applying Proposition 3.8.23. ☐

## 3.9 Finite Type Schemes/Morphisms

We now move on to discuss the last technical terminology we need before introducing varieties: finite type schemes/morphisms. This is largely a finiteness condition on schemes[46]. However, because schemes should be seen as *relative* objects (always over a base scheme $S$), we want to see an $S$-scheme $X$ as being finite type *over the base* $S$ so that we can have a sensible theory that allows us to have a "relatively finite" perspective to things. While this may seem strange to say out loud, it is something at which that we're very well practiced as mathematicians[47]. For instance, as a ring the field $\mathbb{C}$ is horrifically infinitely generated: it is an uncountably infinite transcendental (algebraically closed) field extension of $\mathbb{Q}$, and $\mathbb{Q}$ itself is an infinitely generated ring over $\mathbb{Z}$.[48] These conditions should not worry you overmuch, however[49], as what matters is some relative notion of finite. For instance, in real analytic geometry, we often take $\mathbb{R}$ (and finite real vector spaces) as "sufficiently finite" for our applications and for good reason. As such, because we're interested in manifolds of rings over some base manifold $S$, we'll be introducing things as finite type over $S$, which is saying that we're working in a world that is finitely generated with the assumption that our base world $S$ is "finite enough" for our applications/situations. An example of this in practice is working with the categories $\mathbb{Q}\,\mathbf{Vect}$ and $\mathbb{C}\,\mathbf{Vect}$ of rational or complex vector spaces. In each case we have a notion of finite spaces, and which is "more natural" or "more correct" is a deeply subjective and personal choice. Instead, what matters for us is developing a relative theory that is adaptable to all situations.

We'll begin our study of finite type spaces by looking at a specific example or two. If $K$ is a field, we think of $S = \operatorname{Spec} K$ as a single point and $\mathbb{A}^1_K = \operatorname{Spec} K[x]$ as a line over $K$. The line $\mathbb{A}^1_K$ is "finitely generated" over $K$ in the sense of Figure 3.4: Similarly, the plane $\mathbb{A}^2_K = \operatorname{Spec} K[x,y]$ is "finitely generated" over $\operatorname{Spec} K$ in the sense of Figure 3.5 and so each of the curves through $\mathbb{A}^2_K$ cut out by polynomial equations must also

[46]One of stupidly many different finiteness conditions, really. There are many, many finiteness conditions in practice that get used in practice (such as finite type, locally of finite type, finite, quasi-compact, and more) for schemes, but we'll focus solely on finite type in these notes with at most minor dirgressions on some of these other conditions.

[47]Likely without consciously realizing it, unless you're really into number theory, algebraic geometry, Galois theory, the transcendence theory of fields, or other areas of algebra where you often change fields/coefficients all over the place.

[48]The easiest way I know to show this is to consider the ring $\mathbb{Z}[1/2]$. This is a proper subring of $\mathbb{Q}$ which is not finitely generated as a $\mathbb{Z}$-algebra: it admits a presentation as $\mathbb{Z}[x_n : n \in \mathbb{N}]/\mathfrak{a}$ where $\mathfrak{a}$ is the ideal $\mathfrak{a} = (2x_0 = -1, 2x_{n+1} - x_n : n \in \mathbb{N})$.

[49]Unless you want to do constructive mathematics or you're an ultrafinitist. In either case I respect your beliefs and choices (although I probably shouldn't be saying "choice" here), but I'll be using the Axiom of Choice (while flagging it) and infinite things (while maybe flagging them) as needed.


Figure 3.4: This is a visualization of $\mathbb{A}^1_K$ as it sits over $\operatorname{Spec} K$ for a field $K$.

be finite generate over $S$. Generally the only way we get something truly finite over $K$ is by taking finite field extensions (as in Figure 3.6) but the lines should be finite enough for doing geometry. In order to get (an easily described) space which is truly infinite in its geometric nature over $\operatorname{Spec} K$, we need to construct a space $\mathbb{A}^\infty_K$ which contains each possible $\mathbb{A}^n_K$ and curve for any $n \in \mathbb{N}$. Perhaps unsurprisingly we get $\mathbb{A}^\infty_K = \operatorname{Spec} K[x_n : n \in \mathbb{N}]$. What makes each of these spaces $\mathbb{A}^n_K$ "finite enough" and $\mathbb{A}^\infty_K$ infinite is how we can generate the algebras $K[x_1, \cdots, x_n]$ which induces/generates $\mathbb{A}^n_K$ versus the algebra $K[x_n : n \in \mathbb{N}]$ that generates/induces $\mathbb{A}^\infty_K$. The difference here is finite generation of algebras. Each curve or hypersurface in $\mathbb{A}^n_K$ is realized as a quotient algebra $K[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$ while our infinite spaces cannot be expressed as such. Using this we find that the schemes we need to work with need to be "finitely generated" manifolds over $S$, which we will explain below. First, we'll give a definition of what it means to be locally finitely generated as a scheme over $S$ and then describe how to realize finite generation in complete generality.

**Definition 3.9.1** ([33, Page 84]). A morphism $f : X \to Y$ of schemes is locally of finite type if there is an open affine cover $\{\iota_i : V_i \to Y\} i \in I\}$ of $Y$ with $\operatorname{Spec} B_i \cong V_i$ such that for each $i \in I$ the preimage scheme $f_i^{-1}(V_i)$ admits an open affine cover $\{\iota_{ij} : U_{ij} \to f^{-1}(V_i) \mid j \in J_i\}$ for which $U_{ij} \cong \operatorname{Spec} A_{ij}$ and each ring $A_{ij}$ is a finitely generated $B_i$ algebra.

**Definition 3.9.2** ([33, Page 84]). A morphism $f : X \to Y$ of schemes is finite type if it is locally of fintie type and each pullback scheme $f^{-1}(V_i)$ can be covered by finitely many open affine subschemes $U_{ij}$.

An important, but technical and tedious, fact we'll need is the proposition below that shows if a morphism which is (locally of) finite type for some affine open cover, it is (locally of) finite type for any affine open cover.[50]

**Proposition 3.9.3.** *A morphism $f : X \to Y$ is locally of finite type if and only if for every affine cover $\{\iota_i : V_i \to Y \mid i \in I\}$ of $Y$ there is an open affine cover $\{\iota_{ij} : U_{ij} \to f^{-1}(V_i) \mid j \in J_i\}$ for which if $V_i \cong \operatorname{Spec} B_i$ and $U_{ij} \cong \operatorname{Spec} A_{ij}$ then each $A_{ij}$ is a finitely generated $B_i$-algebra.*

**Proposition 3.9.4.** *A morphism $f : X \to Y$ is locally of finite type if and only if for every affine cover $\{\iota_i : V_i \to Y \mid i \in I\}$ of $Y$ there is an open affine cover $\{\iota_{ij} : U_{ij} \to f^{-1}(V_i) \mid j \in J_i\}$ for which finitely many $U_{ij}$ cover $f^{-1}(V_i)$ and if $V_i \cong \operatorname{Spec} B_i$ and $U_{ij} \cong \operatorname{Spec} A_{ij}$ then each $A_{ij}$ is a finitely generated $B_i$-algebra.*

[50] In fact, while we will not present the proof here, it is often regarded as a rite of passage in algebraic geometry to do the exercise and show how this works out; I highly recommend it. The trick is to use intersections of affine opens and find affine subscovers of those affine intersections. These extend to affine covers of everything in sight, and now taking pullbacks looks like taking tensor products of finitely generated algebras, so gluing gives your desired results. That being said, there is a lot of book-keeping in doing this correctly and carefully, and this is something that everyone should do *exactly once,* just like proving the Lebesgue measure is a Haar measure for $\mathbb{R}$. You can also see this material worked out using what is called the Affine Communication Lemma or in [27, Section 6.3], should you be interested in looking it up.

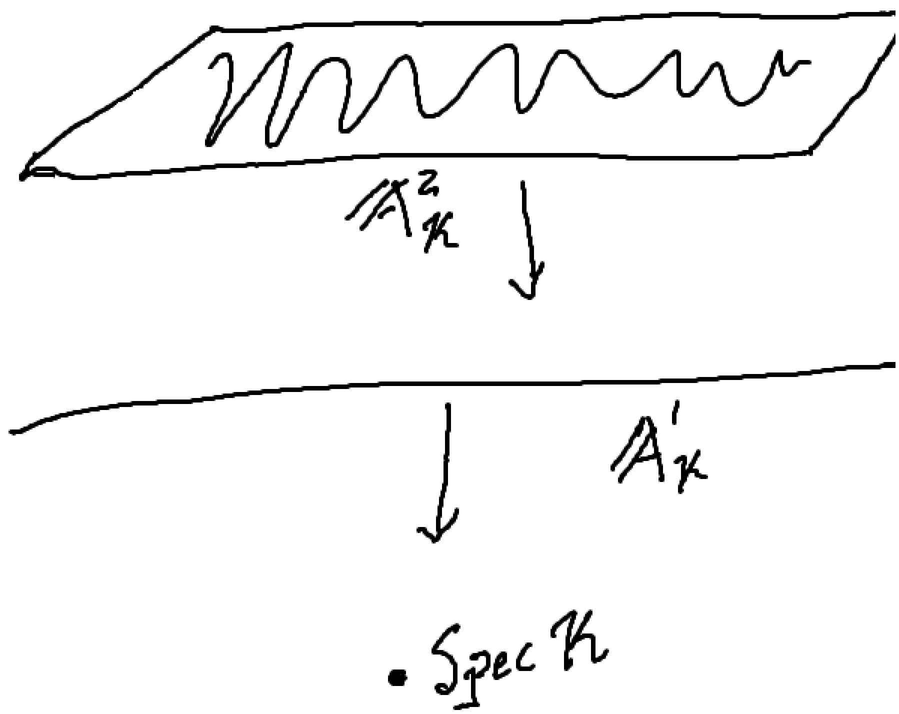


Figure 3.5: This is a visualization of $\mathbb{A}^2_K$ and $\mathbb{A}^1_K$ as they sit over $\operatorname{Spec} K$ for a field $K$.

A more restrictive version of finitely generated/locally finitely generated is simply a finite morphism. We describe these below for the sake of completeness but will not spend too much time with them, as it is too restrictive for our desired applications.

**Definition 3.9.5** ([33, Page 84])**.** A morphism of schemes $f : X \to Y$ is said to be finite if there exists an affine cover $\{\iota_i : V_i \to Y \mid i \in I\}$ with $V_i \cong \operatorname{Spec} B_i$ such that for each $i \in I$ the preimage scheme $f^{-1}(U_i) \cong \operatorname{Spec} A_i$ for some ring $A_i$ and each ring $A_i$ is a finite $B_i$-algebra, i.e., $A_i$ is finitely generated as a $B_i$-**module**.

Here is a handy table I use to mentally organize/distinguish between the three finiteness conditions on morphisms we described above.

| Morphism Type/Class | Finiteness Condition in Words |
|---|---|
| Finite | $X$ looks like a finite module over $Y$ |
| Finite Type | $X$ looks like a finitely generated $Y$-algebra and you only need finitely many such algebras to cover $X$ per open patch of $Y$ |
| Locally of Finite Type | Locally $X$ looks like a finitely generated $Y$-algebra, but you could need infinitely many such algebras per open patch of $Y$ |

**Example 3.9.6.** If $S = \operatorname{Spec} A$ is an affine scheme and $X = \operatorname{Spec} B$ is an affine $S$-scheme then the structure map $f : X \to S$ is:

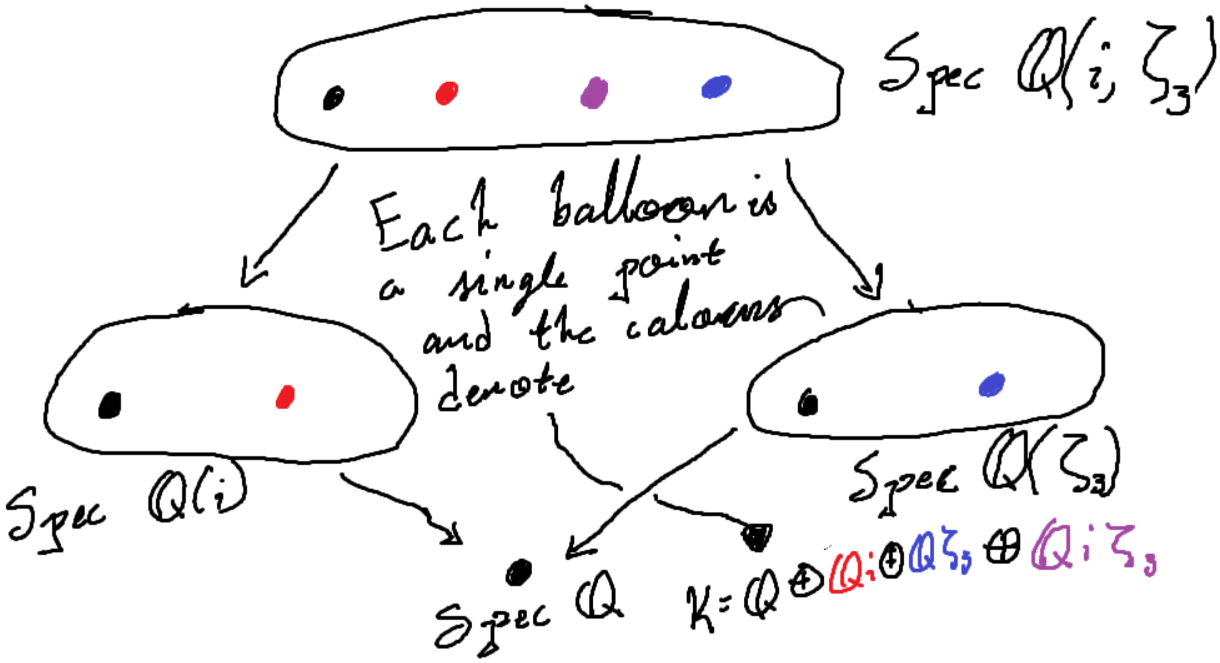


Figure 3.6: A drawing of the field extensions $\operatorname{Spec}\mathbb{Q}(\zeta_3, i), \operatorname{Spec}\mathbb{Q}(\zeta_3), \operatorname{Spec}\mathbb{Q}(i)$ over $\operatorname{Spec}\mathbb{Q}$ where $i^2 = -1$ and $\zeta_3$ is a primitive cube root of unity.

- Finite if and only if $B$ is a finite $A$-algebra, i.e., there is an epimorphism of $A$-modules $A^n \to B$ for some $n \in \mathbb{N}$.

- Finite type if and only if $B$ is a finitely generated $A$-algebra, i.e., there is a surjection of $A$-algebras $A[x_1, \cdots, x_n] \to B$ for some $n \in \mathbb{N}$.

**Remark 3.9.7.** Just because a scheme $X$ is finite type over a scheme $S$ does not mean that $X \times_Y X$ is finite type over all schemes $Y$! For instance, if $X = \operatorname{Spec}\mathbb{C} = S$ then $X \times_S X \cong \operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}) \cong \operatorname{Spec}\mathbb{C}$, which is trivially finite type over $\mathbb{C}$. However, if $Y = \operatorname{Spec}\mathbb{Z}$ then $X \times_Y X \cong \operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{C})$ which is some horridly uncountably infinitely generated ring.

Before proceeding, I'd like to give a characterization of finite type morphisms in terms of locally finite type morphisms with additional strength. In particular, we'll see (perhaps without proof) that finite type maps are locally of finite type and quasi-compact; however, we first need to know what quasi-compact maps are.

**Definition 3.9.8** ([33, Exercise II.3.2])**.** A morphism $f : X \to Y$ of schemes is quasi-compact if there is an affine open cover $\{\iota_i : V_i \to Y \mid i \in I\}$ of $Y$ shuch that $f^{-1}(V_i)$ is quasi-compact for eac $i \in I$.

**Proposition 3.9.9.** *A morphism $f : X \to Y$ of schemes is quasi-compact if and only if for every affine open cover $\{V_i \to Y \mid i \in I\}$ the pullback scheme $f^{-1}(V_i)$ is quasi-compact.*

**Proposition 3.9.10.** *Let $f : X \to S$ be a quasi-compact morphism of schemes and let $g : Y \to S$ be a morphism of schemes. Then the projection $p_2 : X \times_S Y \to Y$ in the pullback*

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & S \end{tikzcd}$$

*is quasi-compact as well.*

**Example 3.9.11.** The identity map $\mathrm{id}_X : X \to X$ is always quasi-compact (even if $X$ is not). This is because if $\{U_i \mid i \in I\}$ is an open cover of $X$, $\mathrm{id}_X^{-1}(U_i) = U_i$ is open affine and hence quasi-compact by Proposition 3.1.18.

**Example 3.9.12.** Let $X$ be a scheme over a field $K$, i.e., a scheme $f : X \to \operatorname{Spec} K$. Then $f$ is quasi-compact if and only if $X$ is quasi-compact. This follows immediately because $f^{-1}(\operatorname{Spec} K) = X$. More generally $X$ is any scheme then $X$ is quasi-compact if $X$ is an $S$-scheme for any quasi-compact scheme $S$.

**Example 3.9.13.** Any isomorphism $f : X \to Y$ is quasi-compact for largely the same reasons that $f$ is.

**Proposition 3.9.14** ([33, Exercises II.3.3.a, II.3.3.b])**.** *A morphism $f : X \to Y$ of schemes is finite type if and only if it is both locally of finite type and quasi-compact. In particular for any affine open subscheme $V \cong \operatorname{Spec} B$ of $Y$, the preimage $f^{-1}(V)$ can be covered by finitely many open affine subschemes $U_i \cong \operatorname{Spec} A_i$ where each $A_i$ is a finitely generated $B$-algebra.*

*Proof.* We first show the if and only if statment. For this note that if $f$ is finite type then it is definitionally locally of finite type; that it is quasi-compact follows from the fact that each preimage of the affine cover can be covered by finitely many open affine subschemes. Alternatively if $f$ is quasi-compact and locally of finite type then there is an affine open cover $\{V_i \mid i \in I, V_i \cong \operatorname{Spec} B_i\}$ of $Y$ for which each pullback scheme $f^{-1}(V_i)$ is both quasi-compact and covered by affine opens $U_{ij} \cong \operatorname{Spec} A_{ij}$ for which each $A_{ij}$ is a finitely generated $B_i$ algebra. It then follows from Propositions 3.9.3 and 3.9.9 that it suffices to take finitely many of the the affines $U_{ij}$ to cover $f^{-1}(V_i)$, which gives the result. □

In particular, this allows us to deduce the following finite generation result on finite type morphisms.

**Proposition 3.9.15** ([33, Exercise II.3.3.c])**.** *If $f : X \to Y$ is a morphism of finite type then for every affine open $V \to Y$ and for every affine open $U \to f^{-1}(V)$ with $V \cong \operatorname{Spec} B$ and $U \cong \operatorname{Spec} A$, $A$ is a finitely generated $B$ algebra.*

The importance of Proposition 3.9.15 is really in its conclusion and for how we'll use it in applications. Because it says that finite type morphisms $f : X \to S$ give the manifold structure of $X$ as gluing together finitely generated algebras over open patches of $Y$, we will be partiularly interested in studying finite type maps over affine schemes $S = \operatorname{Spec} A$ for Noetherian rings $A$.[51] In the case we have such a finite type map $f : X \to \operatorname{Spec} A$ for a Noetherian ring $A$ we get the following observations:

- For any open affine cover $U_i$ of $X$ with $U_i \cong \operatorname{Spec} A_i$, each ring $A_i$ is a finitely generated $A$-algebra.
- Because $A$ is Noetherian, finitely generated is the same thing as finitely presented. Thus each algebra $A_i$ satisfies $A_i \cong A[x_1, \cdots, n_i]/(f_1, \cdots, f_{m_i})$ for some $n_i, m_i \in \mathbb{N}$.

[51] While life does not have to be Noetherian in general (for instance, the rings of holomorphic functions on a domain $U \subseteq \mathbb{C}$ are not generically Noetherian, nor are the rings we meet in the the study of perfectoid geometry and other aspects of $p$-adic geometry/ultrametric geometry), the theory of varieties is a theory of schemes over a Noetherian ring. Similarly, many objects studied in arithmetic geometry (such as Dedekind schemes, finite field arithmetic, and the theory of local fields) do take place in a Noetherian setting, so for most users of algebraic geoemtry the Noeotherian setting is the only setting that matters. It's analogous to the joke/observation: you can tell the difference between a number theorist and a geometer by asking if the word "Galois" means "finite normal and separable field extension" or "normal and separable aglebraic field extension."

What we conclude about this is that finite type schemes over Noetherian bases $S$ are manifolds built out of solutions to finitely many polynomials in finitely many variables! It is for this reason we will need to know about finite type schemes when we get to varieties. For now we'll record a few formal results that we'll use later. To set up proving many of these properties, we'll follow a very helpful strategy for proving things about classes of morphisms as outlined in various places in [27]. This will not only give us slick proofs of various pullbacks and compositions existing, but also gives us a strategy for checking when certain classes of morphisms have certain properties as well as a reduction of properties to check to answer questions like "when is the pullback $f \times_S g$ a morphism that has this desired property?"

For the time being we'll assume that $\mathcal{P}$ is a property that morphisms of schemes and $S$-schemes can have. We also assume that we can form a class of maps $(\mathbf{Sch}_{/S})_1^{\mathcal{P}} = \{f \in (\mathbf{Sch}_{/S})_1 \mid f \text{ has property } \mathcal{P}\}$. We will not make any further assumptions about this class of maps, but we will indiscriminately allow ourselves to quantify over this class, choose elements of this class, and assume that we get into no logical problems about this class existing and issues that may or may not arise with the interactions of the formal logic and this class being potentially "large."[52]

**Lemma 3.9.16** ([27, Proposition 3.5.7])**.** *Let $\mathcal{P}$ be a property of morphisms of schemes and consider the following two propositions:*

1. *If $f : X \to Z$ and $g : Y \to W$ are morphisms of $S$-schemes which have property $\mathcal{P}$ then $f \times_S g$ and $g \times_S f$ also have property $\mathcal{P}$.*
2. *If $f : X \to Y$ is a morphism of $S$-schemes with property $\mathcal{P}$ and $t : T \to S$ is any morphism of schemes then every base change $f \times_S \mathrm{id}_T : X \times_S T \to Y \times_S T$ and $\mathrm{id}_T \times_S f : T \times_S X \to T \times_S Y$ has property $\mathcal{P}$.*

*If for every scheme $X$ the identity morphism $\mathrm{id}_X$ has property $\mathcal{P}$ then* (1) $\implies$ (2)*. If having composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ with property $\mathcal{P}$ implies that $g \circ f$ has property $\mathcal{P}$ then* (2) $\implies$ (1)*. In particular if every identity $\mathrm{id}_X$ has property $\mathcal{P}$ and morphisms with property $\mathcal{P}$ are closed with respect to composition then Proposition* (1) *is equivalent to Proposition* (2)*.*

*Proof.* To prove (1) $\implies$ (2) assume that every identity morphism has property $\mathcal{P}$ and that if $f : X \to Z$ and $g : Y \to W$ are morphisms of $S$-schemes with property $\mathcal{P}$, so are $f \times_S g$ and $g \times_S f$. Assume that $f : X \to Y$ has property $\mathcal{P}$ and let $t : T \to S$ be a map of schemes. Then the maps $\mathrm{id}_T \times_S f$ and $f \times_S \mathrm{id}_T$ have property $\mathcal{P}$, as desired.

To prove (2) $\implies$ (1) assume that if $f : X \to Y$ is a morphism of $S$-schemes with property $\mathcal{P}$ every base change of $f$ along $t : T \to S$, $\mathrm{id}_T \times_S f$ and $f \times_S \mathrm{id}_T$ have property $\mathcal{P}$ as well. Assume also that morphisms with property $\mathcal{P}$ are closed with respect to composition. Now let $f : X \to Z$ and $g : Y \to W$ be morphisms with property $\mathcal{P}$. Because we can factor $f \times_S g$ as

$$X \times_S Y \xrightarrow{\mathrm{id}_X \times_S g} X \times_S W \xrightarrow{f \times_S \mathrm{id}_W} Z \times_S W$$

and both the morphisms above have property $\mathcal{P}$, it follows that $f \times_S g$ has property $\mathcal{P}$ as well.

Finally, the equivalence (1) $\iff$ (2) follows immediately if $\mathrm{id}_X$ has property $\mathcal{P}$ for every $S$-scheme $X$ and if morphisms with property $\mathcal{P}$ are closed with respect to composition, as in this case both propositions (1) $\implies$ (2) and (2) $\implies$ (1) are true. ☐

To further elaborate on this strategy we give the following set up that we'll use for finite type morphisms later:

**Proposition 3.9.17.** *Let $\mathcal{P}$ be a property of morphisms of schemes and consider the list of propositions:*

[52] If you like, just assume we've thrown a massive Grothendieck universe at the problem. I feel it's important to bring up issues like this, but then be honest about how we treat the foundations and move forward without spending too much time on the underlying set theory, as we can always enrich our universe of sets. What matters, at least to me, is the interaction of the objects and the way the theory develops.

1. *Every closed immersion has property $\mathcal{P}$.*

2. *The composition of any two morphisms with property $\mathcal{P}$ has property $\mathcal{P}$.*

3. *If $f : X \to Z$ and $g : Y \to W$ are morphisms of $S$-schemes with property $\mathcal{P}$ then so is $f \times_S g$.*

4. *If $t : T \to S$ is any morphism of schemes and $f : X \to Y$ is a morphism of $S$-schemes with property $\mathcal{P}$ then each morphism $f \times_S \mathrm{id}_T : X \times_S T \to Y \times_S T$ has property $\mathcal{P}$ as well.*

5. *If the composition $g \circ f$ of morphisms $f : X \to Y, g : Y \to Z$ has property $\mathcal{P}$ and $g$ is separated then $f$ has property $\mathcal{P}$ as well.*

6. *If a morphism $f : X \to Y$ has property $\mathcal{P}$ then so does $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$.*

*If* (1) *and* (2) *are true then* (3) *is equivalent to* (4)*. Moreover, if* (1), (2), *and* (3) *are all true then* (5) *and* (6) *are true as well.*

*Proof.* The first claim follows from Lemma 3.9.16 and the fact that identity morphisms $\mathrm{id}_X$ are closed immersions. For the second claim we assume properties (1), (2), and (3) all hold. To establish (5) assume that $g \circ f$ is a composite of morphisms $f : X \to Y$ and $g : Y \to Z$ for which $g$ is separated and $g \circ f$ has property $\mathcal{P}$. We now consider the commuting diagram:

$$\begin{tikzcd}
X \arrow[dr, "{\langle f, \mathrm{id}_X \rangle}"'] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, equal] & & \\
& X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "g \circ f"'] & Z
\end{tikzcd}$$

Because this shows that $p_2 = (g \circ f) \times_Z \mathrm{id}_Y$ and both $(g \circ f)$ and $\mathrm{id}_Y$ have the property $\mathcal{P}$, so does $p_2$ by (3). Now note that by Corollary 3.7.16 because $g$ is separated we have that $\langle f, \mathrm{id}_X \rangle$ is a closed immersion; by (1) it follows that $\langle f, \mathrm{id}_X \rangle$ has property $\mathcal{P}$ as well. However, this the implies that

$$f = p_2 \circ \langle \mathrm{id}_X, f \rangle$$

is the composite of two morphisms with property $\mathcal{P}$, so by (2) the map $f$ has property $\mathcal{P}$ as well.

To establish (6) consider the commuting diagram

$$\begin{tikzcd}
X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d, "\varepsilon_X"'] & Y_{\mathrm{red}} \arrow[d, "\varepsilon_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

of $S$-schemes. Recall that the morphisms $\varepsilon_X$ and $\varepsilon_Y$ are closed immersions and hence have property $\mathcal{P}$ by (1). Moreover, both morphisms $\varepsilon_X$ and $\varepsilon_Y$ are also separated by Proposition 3.7.14. Now since the diagram commutes and $f$ has property $\mathcal{P}$ by assumption, the composite $f \circ \varepsilon_X = \varepsilon_Y \circ f_{\mathrm{red}}$ has property $\mathcal{P}$ by (2). Finally by (5) it follows that $f_{\mathrm{red}}$ has property $\mathcal{P}$ as well. □

Let us see the power of this set up in action by using it as a short cut technique to proving the following proposition about finite type morphisms.

**Proposition 3.9.18.** *In what follows let $S$ be an arbitrary base scheme.*

1. *Every closed immersion is finite type.*

2. *The composition of any two finite type morphisms is finite type.*

3. *If $f : X \to Z$ and $g : Y \to W$ are finite type morphisms of $S$-schemes then $f \times_S g : X \times_S Y \to Z \times_S W$ is a finite type morphism of $S$-schemes.*

4. *If $f : X \to Y$ is a finite type morphism of $S$-schemes and $t : T \to S$ is any morphism of schemes then the morphism $f \times_S T : X \times_S \mathrm{id}_T \to Y \times_S T$ is finite type.*

5. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a composable pair of morphisms of $S$-schemes for which $g \circ f$ is finite type and $g$ is separated then $f$ is finite type.*

6. *If $f : X \to Y$ is finite type then so is $f_{\mathrm{red}}$.*

*Proof.* We use Proposition 3.9.17 to show that this holds. In particular, we need only verify claims (1), (2), and (4).

We begin by establishing (1). Let $f : X \to Y$ be a closed immersion. To establish that $f$ is finite type it suffices by Proposition 3.9.15 to assume that $Y$ is affine.[53] In this case we can write $Y = \operatorname{Spec} A$ and note that $f : X \to \operatorname{Spec} A$ is a closed immersion of an affine scheme. It then follows from Proposition 3.4.19 that there is an ideal $\mathfrak{a}$ of $A$ for which there is an isomorphism of schemes $X \cong \operatorname{Spec}(A/\mathfrak{a})$. Using the affine cover $\operatorname{Spec} A$ of $\operatorname{Spec} A$ we find the pullback scheme $f^{-1}(\operatorname{Spec} A)$ is $\operatorname{Spec} A/\mathfrak{a}$ and hence we just need to check that $A/\mathfrak{a}$ is finite type over $A$. However, this is immediate as the map $A \to A/\mathfrak{a}$ is a surjection and hence so is the map $A[x] \xrightarrow{x \mapsto 0} A \to A/\mathfrak{a}$.

We now establish (2). Let $f : X \to Y$ and $g : Y \to Z$ be finite type $S$-morphisms and let $\{V_i \to Z \mid i \in I\}$ be an affine cover of $Z$ and write $V_i \cong \operatorname{Spec} A_i$. Fix an $i \in I$. Consider the pullback

$$(g \circ f)^{-1}(V_i) = f^{-1}\left(g^{-1}(V_i)\right).$$

First we use that $g$ is finite type to give an open affine cover $U_{ij}$ of $g^{-1}(V_i)$ for which each affine scheme $U_{ij} \cong \operatorname{Spec} B_{ij}$ has $B_{ij}$ as a finite type $A_i$ algebra. Moreover, by the quasi-compactness of $g$ (cf. Proposition 3.9.14) we write $g^{-1}(V_i) = \cup_{j=1}^{n} U_{ij}$ (after potentially reindexing and redefining the $j$'s to be natural numbers instead of arbitrary indexes). Note that this shows

$$(g \circ f)^{-1}(V_i) = f^{-1}\left(g^{-1}(V_i)\right) = f^{-1}\left(\bigcup_{j=1}^{n} U_{ij}\right) = \bigcup_{j=1}^{n} f^{-1}(U_{ij}).$$

Now fix $j$. Using that $f$ is finite type in the same way as above we get that we can write an open cover

$$f^{-1}(U_{ij}) = \bigcup_{k=1}^{m_j} U_{ijk}$$

for affine subschemes $U_{ijk}$ of $f^{-1}(U_{ij})$ where if $U_{ijk} \cong \operatorname{Spec} C_{ijk}$ then each $C_{ijk}$ is a finite type $B_{ij}$ algebra. Putting these togehter lets us write

$$(g \circ f)^{-1}(V_i) = \bigcup_{j=1}^{n} f^{-1}(U_{ij}).f^{-1}(U_{ij}) = \bigcup_{j=1}^{n} \bigcup_{k=1}^{m_j} U_{ijk}$$

which is a finite union of affine subschemes. To complete the argument we just need to argue that $C_{ijk}$ is a finite type $A_i$ algebra. However, this is routine; using the surjection $A_i[x_1, \cdots, x_{n_j}] \to B_{ij}$ together with the surjection $B_{ij}[x_1, \cdots, x_{n_{jk}}] \to C_{ijk}$ we get a surjection

$$A_i[x_1, \cdots, x_{n_j}, y_1, \cdots, y_{n_{jk}}] \to B_{ij}[y_1, \cdots, y_{n_{jk}}] \to C_{ijk},$$

[53] The fancy language for this is that "finite type maps are affine-local on the target."

as desired.

We close by establishing (4).

Finish the proof Geoff!

□

Let us now see how to use finite type schemes in practice. We'll begin by giving some preliminary definitions and constructions for use later. Of particular interest to us will be Noetherian topological spaces and schemes, which are an important class of schemes and spaces. In fact, we'll see that the world of varities is a Noehterian world, so having a handle on the finiteness that this gives us will be helpful for later.

**Definition 3.9.19.** A topological space $X$ is Noetherian if for every ascending chain of open subsets

$$U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots$$

there exists an $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ $U_n = U_{n+k}$.

**Remark 3.9.20.** An equivalent formulation of a Noetherian space $X$ is that $X$ satisfies the descending chain condition on closed subspaces, i.e., for all chains of closed subspets

$$V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$$

there is an $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ $V_n = V_{n+k}$.

**Example 3.9.21.** If $A$ is a Noetherian ring then the space $|\operatorname{Spec} A|$ is Noetherian (sadly this is not an if and only if — the next example has a non-Noetherian ring with a Noetherian underlying space).

**Example 3.9.22.** For an example of a non-Noetherian ring with a Noetherian topological spectrum we let $K$ be a field and $A$ be the ring

$$A := \frac{K[x_n : n \in \mathbb{N}]}{(x_n^2 : n \in \mathbb{N})}.$$

Then $A$ is certainly not Noetherian, as the chain

$$(x_1) \subseteq (x_1, x_2) \subseteq (x_1, x_2, x_3) \subseteq \cdots$$

does not stabilize. However note that the only prime ideal in $A$ is the ideal $\mathfrak{m} = (x_n : n \in \mathbb{N})$[54] so we have that $|\operatorname{Spec} A| = \{\mathfrak{m}\}$ (which is finite and hence Noetherian).

**Example 3.9.23.** We'll now show an example where a non-Noetherian ring does have a non-Noetherian spectrum. Consider the ring $\mathbb{Z}[x_n : n \in \mathbb{N}]$ and the non-stabilizing ascending chain of ideals:

$$(x_1) \subseteq (x_1, x_2) \subseteq (x_1, x_2, x_3) \subseteq \cdots$$

This leads to a non-stabilizing chain of open sets:

$$D(x_1) \subseteq D(x_1, x_2) \subseteq D(x_1, x_2, x_3) \subseteq \cdots$$

It would stabilize if and only if there is an $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $\sqrt{(x_1, \cdots, x_n)} = \sqrt{(x_1, \cdots, x_{n+k})}$ which cannot happen for any $n \in \mathbb{N}$ if $k \geq 1$. In particular, note that $|\operatorname{Spec} A|$ is quasi-compact but not Noetherian.

[54] This can be shown by noting that since every $x_n$ is nilpotent, every prime ideal needs to contain $\mathfrak{m}$ while $A/\mathfrak{m} \cong K$ so $\mathfrak{m}$ is maximal.

**Example 3.9.24.** The space $[0,1]$ under its subspace topology of $\mathbb{R}$ is not Noetherian. One way to check this is to note that the chain of opens

$$\left(0, \frac{1}{2}\right) \subseteq \left(0, \frac{2}{3}\right) \subseteq \left(0, \frac{3}{4}\right) \subseteq \cdots,$$

i.e., our opens $U_n$ are given by

$$U_n := \left(0, 1 - \frac{1}{n+2}\right).$$

Then this chain of opens does not stabilize for any $n \in \mathbb{N}$. Note that this space is also compact but not Noetherian.

These examples lead us to a characterization of Noetherian topological spaces which we do not prove.

**Proposition 3.9.25.** *A topological space $X$ is Noetherian if and only if every open subspace $U \subseteq X$ is quasi-compact.*

As we've seen in Example 3.9.22, we can have non-Noetherian rings $A$ which have Noetherian spectra. Thus if we want to describe (locally) Noetherian schemes as being those schemes $X$ which have covers by Noetherian spaces we will not have the correct local finiteness conditions on $X$ that being Noetherian gives. Instead, we'll have to instead insist upon the scheme arising as the gluing of spectra of Noetherian schemes.

**Definition 3.9.26.** A scheme $X$ is locally Noetherian if there is an open affine cover

$$X = \bigcup_{i \in I} U_i$$

for which each affine scheme $U_i \cong \operatorname{Spec} A_i$ is the spectrum of a Noetherian ring $A_i$.

**Definition 3.9.27.** A scheme $X$ is Noetherian if it is locally Noetherian and quasi-compact.

Our first result on locally Noethrian schemes says that every open affine subscheme of a locally Noetherian scheme is the spectrum of a Noetherian ring. This is the Noetherian analogue of Propositions 3.9.3 and 3.9.4, but we'll present a complete proof in this case (please read the footnote if you'd like an explanation for my inconsistent choices here).[55] Before doing so, however, we need one ring theoretic lemma.

**Lemma 3.9.28.** *Let $A$ be a commutative ring with identity and let $f_1, \cdots, f_n \in A$ such that $(f_1, \cdots, f_n) = A$ and each ring $A[f_i^{-1}]$ is Noetherian. Then $A$ is Noetherian.*

*Proof.* We first establish the identity, for any ideal $\mathfrak{a} \trianglelefteq A$,

$$\mathfrak{a} = \bigcap_{i=1}^{n} \lambda_{f_i}^{-1}\left(A[f_i^{-1}]\lambda_{f_i}(\mathfrak{a})\right) =: \mathfrak{b}.$$

For this note that it is routine[56] to show that $\mathfrak{a} \subseteq \mathfrak{b}$, so we only need to show that $\mathfrak{b} \supseteq \mathfrak{a}$. Fix an element $b \in \mathfrak{b}$ and note that we can write, for each $i$,

$$\lambda_{f_i}(b) = \frac{a_i}{f_i^{m_i}}$$

[55] This is largely because there is a complete proof in [33], which we will follow and mimic. While I think it is an important geometric rite of passage, as I mentioned earlier, to prove these sorts of propositions by hand, when one is well-known in the literature (and what is better known than being in the manly book of Noetherian schemes) I feel less bad about presenting the proof and taking away a rite of passage from a future geometer. That being said, it can be helpful to have one such proof to follow when do the proof yourself.

[56] The trick is to calculate $\lambda_{f_i}(a)$ for each $f_i$ and note that the fraction $a/1$ is in each preimage ideal.

where $m_i \geq 1$ and $a_i \in \mathfrak{a}$. Let $m = \max\{m_i \mid 1 \leq i \leq n\}$. We then can find $s_i \in \mathbb{N}$ for which the equation

$$f_i^{s_i}(f_i^m b - a_i) = 0$$

holds in $A$ for each $i$. We then define $s := \max\{s_i \mid 1 \leq i \leq n\}$ so that we have the equations

$$f_i^s(f_i^m b - a_i) = 0$$

in $A$ for all $i$. Then using that $D(f_i) = D(f_i^t)$ for all $t \in \mathbb{N}$ with $t \geq 1$ we have $A = (f_1^{m+s}, \cdots, f_n^{m+s})$. Write

$$1 = \sum_{i=1}^{n} \alpha_i f_i^{m+s}.$$

Then

$$b = \left(\sum_{i=1}^{n} \alpha_i f_i^{m+s}\right) b = \sum_{i=1}^{n} \alpha_i f_i^{m+s} b = \sum_{i=1}^{n} \alpha_i f_i^s a_i \in \mathfrak{a},$$

so $\mathfrak{a} = \mathfrak{b}$.

We now show that $A$ is Noetherian. For this consider the ideal $\mathfrak{a}$ and note that in each ring $A[f_i^{-1}]$ the ideal

$$\mathfrak{a}_{f_i} := A[f_i^{-1}]\lambda_{f_i}(\mathfrak{a})$$

is finitely generated in $A[f_i^{-1}]$. We then can lift each of these ideals to a finitely generated ideal

$$\lambda_{f_i}^{-1}(\mathfrak{a}_i)$$

in $A$. It then follows that the finite intersection of these ideals is finitely generated as well, so we conclude that

$$\mathfrak{a} = \bigcap_{i=1}^{n} \lambda_{f_i}^{-1}(\mathfrak{a}_i)$$

is finitely generated as well. Therefore $A$ is Noetherian. $\square$

**Proposition 3.9.29** ([33, Proposition II.3.2])**.** *Let $X$ be a locally Noetherian scheme and let $U$ be an open affine subscheme of $X$ with $U \cong \operatorname{Spec} A$. Then $A$ is Noetherian.*

*Proof.* $\implies$ : This direction of the proof is striaghtforward, as if every affine scheme of $X$ is the spectrum of a Noetherian ring then $X$ is immediately locally Noetherian.

$\impliedby$ : We first show how to reduce the proof to the case that $X$ is an affine scheme. First find an open affine subcover

$$X = \bigcup_{i \in I} V_i$$

where $V_i \cong \operatorname{Spec} B_i$ for a Noetherian ring $B_i$. Then because each localizaiton $B_i[f^{-1}]$ is a Noetherian ring and the set

$$\mathcal{D}_{B_i} = \{D(f) \mid f \in B_i\}$$

is a basis of open affines for $\operatorname{Spec} B_i$ (cf. Proposition 3.1.14) we get an open basis of affines for $X$. Explicitly, writing $V_{i,f}$ as the open affine subscheme of $V_i$ with $V_{i,f} \cong D(f) \cong \operatorname{Spec} B_i[f^{-1}]$ for $f \in B_i$, we have

$$\mathcal{D}_X := \bigcup_{i \in I} \bigcup_{f \in B_i} V_{i,f}.$$

This implies that for any open affine subscheme $U$ of $X$, we can find an open covering of $U$ by Noetherian affine schemes. As such, we can assume that $X$ is an affine scheme and proceed.

Assume now that $X$ is a locally Noetherian affine scheme and write $X = \operatorname{Spec} A$ for some ring $A$. Consider an open cover

$$X = \bigcup_{i \in I} U_i$$

where $U_i \cong \operatorname{Spec} B_i$ and each ring $B_i$ is Noetherian. Because the set

$$\mathcal{D}_A = \{D(f) \mid f \in A\}$$

is an open basis to $|\operatorname{Spec} A|$ we can find an $f \in A$ for which $D(f) \subseteq U_i$. Then we have a commuting diagram

$$\begin{tikzcd}
 & \operatorname{Spec} B \arrow[ddr, bend left] & \\
 & U \arrow[u, "\cong"'] \arrow[dr] & \\
D(f) \arrow[uur, bend left] \arrow[ur] \arrow[rr] & & \operatorname{Spec} A
\end{tikzcd}$$

of open immersions of schemes. Write $\overline{f}$ for the image of $f$ in $B$ under the map induced by the diagram above. From the commutativity and the fact that the bottom edge is a localization, it follows that $A[f^{-1}] \cong B[\overline{f}^{-1}]$. Because $B$ is Noetherian, $B[\overline{f}^{-1}]$, and hence $A[f^{-1}]$, are also is Noetherian. Thus we find a cover of $\operatorname{Spec} A$ be open affines $D(f) \cong \operatorname{Spec} A[f^{-1}]$ where each ring $A[f^{-1}]$ is Noetherian. Moreover, because $\operatorname{Spec} A$ is quasi-compact we can write

$$\operatorname{Spec} A = \bigcup_{i=1}^{n} D(f_i) \cong \bigcup_{i=1}^{n} \operatorname{Spec} A[f_i^{-1}].$$

From this we see that the $f_i$ generate the unit ideal in $A$ and each localization $A[f_i^{-1}]$ is Noetherian. Applying Lemma 3.9.28 we get that $A$ is Noetherian which completes the proof. ☐

**Corollary 3.9.30.** *If $X$ is a Noetherian scheme and $U$ is an open affine subscheme of $X$ then $U \cong \operatorname{Spec} A$ for a Noetherian ring $A$.*

We'll now see the power that being locally Noetherian gives. In particular, locally Noetherian schemes and maps betweent them are very strong, as they force a lot of finiteness structure that roughly comes from the fact that each affine subspace looks like solution spaces to finitely many polynomial equations over some Noehterian coefficient rings.

**Proposition 3.9.31.** *Let $f : X \to Y$ be a finite type morphism. If $Y$ is Noetherian (respectively locally Noetherian) then $X$ is Noetherian (respectively locally Noetherian).*

*Proof.* We recall that by Proposition 3.9.14 since $f$ is locally finite type and quasi-compact. Thus if $Y$ is Noetherian (and hence quasi-compact and locally Noetherian), it follows immediately that $X$ is quasi-compact as well; note that this is also the only place we use the full Noetherian hypothesis in the statement of the proposition. Combined with the proof we present below, this will give the Noetherian statement of the proposition.

We now only need to verify that $X$ is locally Noetherian if $Y$ is locally Noetherian, so assume that $Y$ is locally Noetherian. By taking open affine covers of $Y$ and using Proposition 3.9.29 we can assume that $f : X \to Y$ is a finite type map for $Y$ an affine Noetherian scheme. Write $Y$ as a finite union

$$Y = \bigcup_{i=1}^{n} V_i \cong \bigcup_{i=1}^{n} \operatorname{Spec} A_i$$

where each $A_i$ is a Noetherian ring. Then each pullback scheme $f^{-1}(V_i)$ can be covered by open affine subschemes $U_{ij} \cong \operatorname{Spec} B_{ij}$ where $B_{ij}$ is a finitely generated $A_i$-algebra because $f$ is locally finite type. However, then $B_{ij}$ is a Noetherian $A_i$-algebra so each pullback scheme $f^{-1}(V_i)$ is locally Noetherian. Finally, because

$$X = f^{-1}(Y) = \bigcup_{i=1}^{n} f^{-1}(V_i) = \bigcup_{i=1}^{n} \bigcup_{j \in J_i} U_{ij}$$

where each $U_{ij}$ is an affine Noetherian scheme. Thus $X$ is locally Noetherian, as desired. □

**Corollary 3.9.32.** *If $f : X \to S$ is a finite type map and $t : T \to S$ is a map of schemes with $T$ Noetherian (respecitively locally Noetherian) then $X \times_S T$ is Noetherian (respectively locally Noetherian).*

*Proof.* Consider the pullback diagram:

$$\begin{tikzcd} X \times_S T \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ T \arrow[r, "t"'] & S \end{tikzcd}$$

By Proposition 3.9.18 the map $p_2$ is finite type over $T$. Now, because $T$ is Noetherian and $p_2 : X \times_S T \to T$ is finite type, apply Proposition 3.9.31 to conclude. □

One truly helpful corollary that we can also deduce involves triangles

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\nu_X"'] & & Y \arrow[dl, "\nu_Y"] \\ & S & \end{tikzcd}$$

where $S$ is a locally Noetherian base and the schemes $X$ is a finite type $S$-scheme, we'll be able to deduce that $f$ must finite type as well. We do not prove this corollary here, but instead refer the reader to [27, Corollary 6.3.9]. Note that this is formally similar to Proposition 3.7.20

**Corollary 3.9.33.** *Let $\nu_X : X \to S$ be a finite type morphism where $S$ is a locally Noetherian scheme. Then any morphism $f : X \to Y$ of $S$-schemes is finite type.*

We now close our discussion of finite type schemes by introducing the category of finite type schemes over a locally Noetherian base scheme and proving a finite type analogue of Theorem 3.8.18. This will be of particular help to us when we introduce and study varieties later, as well as giving you tools for establishing various results if you're feeling adventurous and want to generalize what varieties away from reduced schemes.

**Definition 3.9.34.** Let $S$ be a locally Noetherian base scheme. We then define the category of finite type $S$-schemes, $\mathbf{Sch}_{/S}^{\text{f.t.}}$, as follows:

- Objects: $S$-schemes $X$ for which the structure morphism $\nu_X : X \to S$ is finite type.
- Morphisms: Morphisms of $S$-schemes.
- Composition and Identities: As in $\mathbf{Sch}_{/S}$.

**Proposition 3.9.35.** *If $X$ is a finite type scheme over a locally Noetherian scheme $X$ then every locally closed subscheme $V$ of $X$ is an object of $\mathbf{Sch}_{/S}^{\text{f.t.}}$. In particular, the reduction $X_{\text{red}}$ of $X$ is an object of $\mathbf{Sch}_{/S}^{\text{f.t.}}$.*

*Proof.* This follows immediately from the fact that open and closed imersions are finite type and that compositions of finite type maps are also finite type by Part (2) of Proposition 3.9.18. □

**Proposition 3.9.36.** *If $f \in \mathbf{Sch}_{/S}^{\text{f.t.}}(X, Y)$ and if $S$ is locally Noetherian then $f : X \to Y$ is finite type.*

*Proof.* Consider the commuting triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\nu_X"'] & & Y \arrow[dl, "\nu_Y"] \\ & S & \end{tikzcd}$$

induced by the fact that $f$ is a morphism of $S$-schemes. Because $\nu_X = \nu_Y \circ f$ is finite type, it follows from Corollary 3.9.33 that $f$ is finite type. $\square$

In what follows we'll need the category $\mathbf{RedSch}_{/S}^{\text{f.t.}}$ of reduced finite type $S$-schemes. This is as straightforward to define as $\mathbf{RedSch}_{/S}$, but we do so explicitly to dispel ambiguity.

**Definition 3.9.37.** The category $\mathbf{RedSch}_{/S}^{\text{f.t.}}$ of reduced finite type $S$-schemes is defined as follows:

- Objects: $S$-schemes $X \in \mathbf{Sch}_{/S}^{\text{f.t.}}$ which are also reduced.
- Morphisms: Morphisms of $S$-schemes in $\mathbf{Sch}_{/S}^{\text{f.t.}}$ for which the domain and codomain are reduced schemes.
- Composition and Identities: As in $\mathbf{Sch}_{/S}$.

As before, $\mathbf{RedSch}_{/S}^{\text{f.t.}}$ is a full subcategory of $\mathbf{Sch}_{/S}^{\text{f.t.}}$ and there is an inclusion functor $\text{incl} : \mathbf{RedSch}_{/S}^{\text{f.t.}} \to \mathbf{Sch}_{/S}^{\text{f.t.}}$ which is the identity on objects and morphisms. It also follows

**Lemma 3.9.38.** *The restriction of the reduction functor* red *to the category of finite type $S$-schemes is a functor*

$$\text{red} : \mathbf{Sch}_{/S}^{\text{f.t.}} \to \mathbf{RedSch}_{/S}^{\text{f.t.}}.$$

*Proof.* This is a routine check that if $X$ is a finite type $S$-scheme then $X_{\text{red}}$ is a finite type $S$-scheme. However, this is immediate by Part (2) of Proposition 3.9.18 and Proposition 3.9.35 as the structure map of $X_{\text{red}}$ satisfies

$$\nu_{X_{\text{red}}} = \nu_X \circ \varepsilon_X.$$

$\square$

**Proposition 3.9.39.** *Let $S$ be a locally Noetherian scheme. The category $\mathbf{RedSch}_{/S}^{\text{f.t.}}$ is a coreflective subcategory of $\mathbf{Sch}_{/S}^{\text{f.t.}}$. That is, there is an adjunction:*

$$\begin{tikzcd} \mathbf{Sch}_{/S}^{\text{f.t.}} \arrow[rr, bend right, "\text{red}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{RedSch}_{/S}^{\text{f.t.}} \arrow[ll, bend right, "\text{incl}"'] \end{tikzcd}$$

*Sketch.* This follows mutatis mutandis to the proof of Theorem 3.8.18 save for the fact that if every $S$-scheme is assumed to be finite type then every map and scheme in sight is also finite type over $S$. $\square$

**Proposition 3.9.40.** *If $S$ is a locally Noetherian scheme then the category $\mathbf{Sch}_{/S}^{\text{f.t.}}$ is finitely complete.*

*Proof.* Since the terminal object $\text{id}_S : S \to S$ is trivially finte type over $S$, we must show that for any cospan of $X \xrightarrow{f} Z \xleftarrow{g} Y$ of finite type $S$-schemes the pullback $X \times_Z Y$ is a finite type $S$-scheme as well. However, this is immediate as the structure map $X \times_Z Y$ is equivalently defined as

$$\nu_{X \times_Z Y} = \nu_X \circ p_1 = \nu_Z \circ f \circ p_1 = \nu_Z \circ g \circ p_2 = \nu_Y \circ p_2,$$

where the morphisms $p_1$ and $p_2$ are the first and second projections of the pullback, the maps $p_1$ and $p_2$ are finite type by Part (3) of Proposition 3.9.18, and the composition of finite type maps is finite type by Part (2) of Proposition 3.9.18. $\square$

You may be wondering (and it is natural to do so) if while the proof above shows that $\mathbf{Sch}^{\text{f.t.}}_{/S}$ is finitely complete only, is it in fact actually infinitely complete? The answer to this question is negative, which we show below. Essentially infinite limits destroy the "finite" part of finite type.

**Example 3.9.41.** Here is an example of an infinite limit of finite type schemes over a locally Noetherian base scheme $S$ which is not finite type. Let $S = \operatorname{Spec} K$ for a field $K$ and note that since $K$ is a field $S$ is Noetherian. Consider the scheme

$$\mathbb{A}^{\infty}_S = \prod_{n \in \mathbb{N}} \mathbb{A}^1_S \cong \operatorname{Spec} K[x_n : n \in \mathbb{N}].$$

This affine scheme is the infinite product of the $\mathbb{A}^1_S$ but is not finite type over $\operatorname{Spec} K$ because there is no $m \in \mathbb{N}$ with a surjection $K[x_0, \cdots, x_m] \to K[x_n : n \in \mathbb{N}]$. Note that this same argument works for any nonempty locally Noetherian scheme $S$, save that we instead work affine locally in the case that $S$ is non-affine.

**Corollary 3.9.42.** *The category $\mathbf{RedSch}^{\text{f.t.}}_{/S}$ has all finite limits and pullbacks in $\mathbf{RedSch}^{\text{f.t.}}_{/S}$ are determined by the formula*

$$X. \overset{\mathbf{RedSch}^{\text{f.t.}}_{/S}}{\times}{}_Z\, Y \cong \left( X \overset{\mathbf{Sch}^{\text{f.t.}}_{/S}}{\times}{}_Z\, Y \right)_{\text{red}}$$

*Proof.* This follows from the fact that the reduction functor $\text{red} : \mathbf{Sch}^{\text{f.t.}}_{/S} \to \mathbf{RedSch}^{\text{f.t.}}_{/S}$ creates all limits. $\square$

## 3.10 Varieties

We now can finally define and introduce varieties. Our main goals in this section are twofold:

1. Define varieties and give two flavours of examples: algebraic examples and arithmetic examples (as well as examples that mix the two).
2. Get to know the category $\mathbf{Var}_{/K}$. In particular, we want to show this category is finitely complete but emphasize (yet again) its pullback is not the pullback in $\mathbf{Sch}_{/K}$.

We will not go too far into the general theory of varieties, but instead we will focus on collating and combining the facts we do know based on our exploration of reduced, speararated, and finite type schemes. In particular, we'll focus more on seeing *why* varieties are as wide-reaching and ubiquitous as they are in modern mathematics than the exact properties being a variety gives you. Let us begin by seeing 13 examples (and non-examples) of varieties for good luck[57] before getting into our short discussion on the category of varieties.

**Definition 3.10.1.** Let $K$ be a field. A variety over $K$ is a reduced, separated scheme of finite type over $\operatorname{Spec} K$. A morphism of varieties is a morphism of underlying schemes. We write $\mathbf{Var}_{/K}$ for the category of varieties over $K$.

**Example 3.10.2.** For any $n \in \mathbb{N}$ affine $n$-space $\mathbb{A}^n_K := \operatorname{Spec} K[x_1, \cdots, x_n]$ is a $K$-variety.

**Example 3.10.3.** For any $n \in \mathbb{N}$ projective $n$-space $\mathbb{P}^n_K$ is a $K$-variety.

**Example 3.10.4.** The scheme

$$\mathbb{G}_{m,K} := \operatorname{Spec} \left( \frac{K[x,y]}{(xy-1)} \right)$$

is a $K$-variety.

[57] I'm not superstitious, nor am I mildly stitious.

**Example 3.10.5.** If $L/K$ is any finite field extension then $\operatorname{Spec} L$ is a variety over $K$.

**Example 3.10.6.** More generally, if $\{L_0, \cdots, L_n\}$ is a collection of finite field extensions of $K$, the scheme

$$X = \coprod_{i=0}^{n} \operatorname{Spec} L_i$$

is a $K$-variety.

**Example 3.10.7.** If $L/K$ is an infinite field extension then $\operatorname{Spec} L$ is not a $K$-variety (in this case $L$ is not finite type over $K$).

**Example 3.10.8.** For any $n \geq 2$ the scheme $\operatorname{Spec} K[x]/(x^n)$ is not a $K$-variety.

**Example 3.10.9.** For any finite field extension $L/K$ the scheme

$$X = \operatorname{Spec}\left(\frac{L[x,y]}{(xy-1)}\right)$$

is a $K$-variety.

**Example 3.10.10.** For any $n \geq 1$ the scheme

$$\mathrm{GL}_{n,K} := \operatorname{Spec}\left(\frac{K[x_{ij}, y : 1 \leq i,j \leq n]}{(\det(x_{ij})y - 1)}\right)$$

is a $K$-variety. Similarly, the scheme

$$\mathrm{SL}_{n,K} := \operatorname{Spec}\left(\frac{K[x_{ij} : 1 \leq i,j \leq n]}{(\det(x_{ij}) - 1)}\right)$$

is a $K$-variety as well. We can even make more complicated Lie groups into varieties by noting that the algebraic conditions on the determinants are polynomial conditions in the entries of the matrices. For instance, recall that $\mathrm{SO}_n(K)$ is the group of $n \times n$ orthogonal matrices with entries in $K$, i.e., the group of $n \times n$ matrices $(a_{ij})$ for which $\det(a_{ij}) = 1$ and

$$(a_{ij})(a_{ij})^t = (a_{ij})(a_{ji}) = I.$$

Note, however, that

$$(a_{ij})(a_{ji}) = \left(\sum_{j=1}^{n} a_{ij}a_{jk}\right)$$

so we can represent $\mathrm{SO}_n(K)$ as the $K$-points[58] of the $K$-variety

$$\mathrm{SO}_{n,K} = \operatorname{Spec}\left(\frac{K[x_{ij} : 1 \leq i,j \leq n]}{\mathfrak{a}}\right)$$

where $\mathfrak{a}$ is the ideal

$$\mathfrak{a} = \left(\det(x_{ij}) - 1, \sum_{j=1}^{n} x_{ij}x_{ji} - 1, \sum_{j=1}^{n} x_{ij}x_{jk} \;\middle|\; 1 \leq i,k \leq n; i \neq k\right).$$

We'll see later that this is a much more difficult way of defining the scheme $\mathrm{SO}_{n,K}$; however, I wanted to give an explicit presentation of it before giving the more abstract one to show how to go about schemifying

[58]For any scheme $X$ and any scheme $Y$, the $Y$-points of $X$ are the hom-set $X(Y) = \mathbf{Sch}(Y, X)$. We won't discuss this too much in these notes, but we'll give a short digression on the functor of points when we discuss group objects.

your Lie theory. It's also worth noting that $\mathrm{SO}_{n,K}$ is a closed subvariety of $\mathrm{SL}_{n,K}$. Each relation defining $\mathrm{SL}_{n,K}$ (namely the condition $\det(x_{ij}) - 1 = 0$) is present in the ideal $\mathfrak{a}$, so there is a surjection of rings

$$\frac{K[x_{ij} : 1 \leq i, j \leq n]}{(\det(x_{ij}) - 1)} \to \frac{K[x_{ij} : 1 \leq i, j \leq n]}{\mathfrak{a}}$$

and hence a closed immersion of $K$-varieties $\mathrm{SO}_{n,K} \to \mathrm{SL}_{n,K}$.

**Example 3.10.11.** If $f(x, y) \in K[x, y]$ is an irreducible polynomial then the scheme

$$C = \operatorname{Spec}\left(\frac{K[x, y]}{(f)}\right)$$

is a $K$-variety. Similarly, for any finite field extension $L/K$ the scheme

$$C_L = \operatorname{Spec}\left(\frac{L[x, y]}{(f)}\right)$$

is a $K$-variety.

**Example 3.10.12.** More generally, if $f_1, \cdots f_m \in K[x_1, \cdots, x_n]$ is a collection of irreducible polynomials then

$$X = \operatorname{Spec}\left(\frac{K[x_1, \cdots, x_n]}{(f_1, \cdots, f_m)}\right)$$

is a $K$-variety.

**Example 3.10.13.** Let $X$ be a $K$-scheme for which there is an isomorphism

$$\operatorname{Spec} K^{\mathrm{sep}} \times_K X \cong \operatorname{Spec}\left(\frac{K^{\mathrm{sep}}[x, y]}{(xy - 1)}\right),$$

where $K^{\mathrm{sep}}$ is a separable closure of $K$. Then $X$ is a $K$-variety (such a scheme is a quasi-split torus over $K$ — such objects are important in the representation theory of $p$-adic groups).

**Example 3.10.14.** I'm going to give a bad example here because I'm not going to explain this completely, as such an explanation requires the Proj (cf. [21], [33], or [73] for details) construction and the theory of graded/homogeneous ideals to make complete sense of. The basic idea, however, is that curves and surfaces in $\mathbb{P}^n_K$ are cut out by homogeneous polynomials and the like. For instance, we can describe a projective elliptic curve as a degree 3 regular curve in $\mathbb{P}^2_K$, i.e., the solution space in $\mathbb{P}^2_K$ whose local coordinates are described by the homogeneous polynomial equation

$$y^2 z = x^3 + axz^2 + bz^3$$

for some $a, b \in K$ (assuming the characteristic of $K$ is not 2 and not 3). This just says that affine-locally the curve looks like a solution space of the form

$$Y^2 = X^3 + aX^2 + b$$

after a suitable change of variables involving whether or not $z, y$, or $x$ is locally invertible (i.e., is $z = 1/y$ or is $z = 1/x$ in your patch).

**Example 3.10.15.** Let $p$ be an integer prime and consider the function field $K = \mathbb{F}_p(t)$. Then the scheme

$$X = \operatorname{Spec}\left(\frac{\mathbb{F}_p(t)[x, y]}{(y^2 - x^p + t)}\right),$$

which is a curve cut out by the equation $y^2 = x^p - t$, is a $K$-variety. For arithmetically minded people: this is the only explicit non-smooth variety we've presented (which will make more sense later on when we discuss smoothness). The reason why this variety is not smooth is because the field $K = \mathbb{F}_p(t)$ is not perfect so the extension of $\mathbb{F}_p(t)$ induced by the equation $x^p - t$ is not separable (and hence is ramified, i.e., singular).

We now move to close off this (overly long) chapter by discussing the category of $K$-varieties. In particular, we'll show that the category $\mathbf{Var}_{/K}$ has all finite limits and can be exhibited as a coreflective subcategory of the category of separated finite type $S$-schemes $\mathbf{SepSch}^{\text{f.t.}}_{/S}$.[59] To begin this process, however, we'll explicitly define the category $\mathbf{SepSch}^{\text{f.t.}}_{/S}$.

**Definition 3.10.16.** The category $\mathbf{SepSch}^{\text{f.t.}}_{/S}$ of separated finite type $S$-schemes is defined as follows:

- Objects: $S$-schemes $X$ whose structure morphisms $\nu_X : X \to S$ are separated and finite type.
- Morphisms: For any two $X, Y \in (\mathbf{SepSch}^{\text{f.t.}}_{/S})_0$, $\mathbf{SepSch}^{\text{f.t.}}_{/S}(X,Y) := \mathbf{Sch}_{/S}(X,Y)$.
- Composition and Identities: As in $\mathbf{Sch}_{/S}$.

As usual, if $S = \operatorname{Spec} A$ we'll often write $\mathbf{Sch}^{\text{f.t.}}_{/A}$ for the category $\mathbf{Sch}^{\text{f.t.}}_{/\operatorname{Spec} A}$.

**Remark 3.10.17** (Achtung!)**.** Sometimes the definition of variety in the literature is distinct from what we have given in this set of notes. While I have gone with the definition of variety in [73], as it is the one I have worked with and seen most frequently in practice, other authors may use different definitions than what we use here. For instance, in [33] a variety is an integral separated scheme of finite type over *an algebraically closed* field $K$ (when working over algebraically closed fields there are good reasons why we can drop just asking for reduced schemes and instead work with integral schemes)[60] while in the Stacks Project they take varieties simply to be integral separated schemes of finite type over a field. In [47] varieties are simply defined to be objects in the category $\mathbf{Sch}^{\text{f.t.}}_{/\operatorname{Spec} K}$, but in doing so we lose the Hausdorffness that we ask our varieties to have. It is also possible to take a $K$-variety to be an object in the category $\mathbf{SepSch}^{\text{f.t.}}_{/\operatorname{Spec} K}$ so that the bugs of having a pullback of varieties being distinct from the pullback of schemes get fixed by just not worrying about it (and acknowledging that our variety theory is now very different from classical variety theory as cut out by Hilbert's Nullstellensatz). In any case, I want to warn you that when you see/meet a variety, you should just check the context of how it is used and in what setting it is used before blindly assuming something (although, in my experience, if you assume reduced separated scheme of finite type you probably won't be too far off the mark or at least any example you care about will probably have these properties — all Lie groups do, for instance).

**Lemma 3.10.18.** *There is an inclusion functor* $\operatorname{incl} : \mathbf{Var}_{/K} \to \mathbf{SepSch}^{\text{f.t.}}_{/K}$ *and the restriction of the reduction functor to* $\mathbf{Sch}^{\text{f.t.}}_{/K}$ *is a functor* $\operatorname{red} : \mathbf{Sch}^{\text{f.t.}}_{/K} \to \mathbf{Var}_{/K}$.

*Proof.* This follows from the proofs of Corollary 3.8.24 and Lemma 3.9.38. ☐

**Theorem 3.10.19.** *Let $K$ be a field. Then the category of $K$-varieties,* $\mathbf{Var}_{/K}$ *is a coreflective subcategory of* $\mathbf{SepSch}^{\text{f.t.}}_{/K}$*, i.e., there is an adjunction:*

$$\begin{tikzcd} \mathbf{SepSch}^{\text{f.t.}}_{/K} \arrow[rr, bend right, "\operatorname{red}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{Var}_{/K} \arrow[ll, bend right, "\operatorname{incl}"'] \end{tikzcd}$$

*Proof.* This follows mutatis mutandis from the proofs of Theorem 3.8.18, Corollary 3.8.24, and Proposition 3.9.39 and noting that in this case we are in all situations described above simultaneously. ☐

[59] To my limited knowledge, this perspective is not taken on varieties in the literature and is at best folkloric and at worst never explicitly stated. I wanted to be explicit and clear in these notes, however, so we'll go through this discussion in detail. It will feel like "more of the same," however, as we'll be showing the subcategory of reduced schemes in $\mathbf{SepSch}^{\text{f.t.}}_{/S}$ once again is coreflective.

[60] In my eyes, the best reason is you no longer have to worry about products of field extensions not remaining integral (so your category of varieties is *only* worried about the algebraic data and does not care about arithmetic data). For instance, the scheme $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ corresponding to the product $\operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ is not integral even though the scheme $\operatorname{Spec} \mathbb{C}$ is integral.

**Corollary 3.10.20.** *The category* $\mathbf{Var}_{/K}$ *is finitely complete and the pullback of varities is determined by the isomorphism*

$$X \overset{\mathbf{Var}_{/K}}{\times}{}_{Z} Y \cong \left( X \overset{\mathbf{SepSch}^{\mathrm{f.t.}}_{/K}}{\times}{}_{Z} Y \right)_{\mathrm{red}}$$

*Proof.* Once again this follows from the fact that $\mathbf{SepSch}^{\mathrm{f.t.}}_{/K}$ is finitely complete (combine Propositions 3.7.24 and 3.9.40 and note that both being finite type and separated are preserved by pullback) and because the reduction functor $\mathrm{red} : \mathbf{SepSch}^{\mathrm{f.t.}}_{/K} \to \mathbf{Var}_{/K}$ creates all limits. □

We now close with a frequently useful but immediate fact about varieties over a field $K$.

**Proposition 3.10.21.** *Let* $X$ *be a* $K$*-variety. Then* $X$ *is a quasi-compact scheme.*

*Proof.* Because $S = \operatorname{Spec} K$ is Noetherian and $X$ is a $K$-variety, the structure map $\nu_X : X \to S$ is a separated finite type map over a scheme $S$ with $|S| \cong \{*\}$. Because $\nu_X$ is finite type, it is locally of finite type and quasi-compact. Thus $\nu_X^{-1}(S) = X$ is quasi-compact as well. □

## 3.11 Quasi-Separated Morphisms

This short section is mainly aimed towards experts and for use later on when we discuss quasi-coherent sheaves. I've included it for completeness, but readers who are only interested in varieties or in a surface-level understanding of the full weight of scheme theory are advised to skip this subsection on a first reading.

Quasi-separated morphisms are a weaker form of morphism than separated morphisms which, while less pleasant than the Hausdorff-like properties that separated morphisms give us, at least allow us to do a "compact approximation" of our covers by separated patches. We'll see that in analogy to Proposition 3.7.18, the intersection of affines of a quasi-separated scheme are the union of finitely many affine open subschemes (as opposed to being bang-on affine).

**Definition 3.11.1.** A morphism $f : X \to Y$ of schemes is quasi-separated if and only if the diagonal morphism

$$\Delta_X|_Y : X \to X \times_Y X$$

is quasi-compact.

**Proposition 3.11.2.** *Quasi-separated morphisms are stable under base change, i.e., if* $f : X \to S$ *is quasi-separated and if* $g : Y \to S$ *is a morphism of schemes then* $p_2 : X \times_S Y \to Y$ *is quasi-separated.*

*Proof.* Write the pullback square as

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

and proceed as in the proof of Part (5) of Proposition 3.7.14 to get the pullback square:

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "\Delta_Z|_S"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\Delta_X|_S"] \\ P \times_Y P \arrow[r, "p_1 \times p_1"'] & X \times_S X \end{tikzcd}$$

The result now follows from Proposition 3.9.10. □

**Proposition 3.11.3.** *Let* $f : X \to S$ *be a quasi-separated morphism of schemes for an affine scheme* $S$ *and let* $U$ *and* $V$ *be affine opens of* $X$*. Then* $U \cap V$ *may be covered by a finite number of affine opens.*

*Proof.* Recall that $U \times_X V \cong U \cap V$. Now proceed as in the proof of Proposition 3.7.18 the pullback diagram:

$$\begin{tikzcd} U \times_X V \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \times_S V \arrow[d] \\ X \arrow[r, "\Delta_X|_S"'] & X \times_S X \end{tikzcd}$$

Because $\Delta_X|_S$ is quasi-compact by virtue of $f$ being quasi-separated, it follows from Proposition 3.9.10 that the map $U \times_X V \to U \times_S V$ is quasi-compact as well. However, since $U \times_S V$ is an affine scheme (as it is a pullback of affine schemes) it is an open affine cover of itself (and in fact, by Proposition 3.1.18, quasi-compact as well). Thus, by Proposition 3.9.9 it follows that $p^{-1}(U \times_S V) = U \times_X V \cong U \cap V$ is quasi-compact as well. It is now immediate that $U \cap V$ can be covered by finitely many affine open subschemes. □

Finally, we close our brief discussion on quasi-separated morphisms by showing that separated morphisms are quasi-separated.[61]

**Lemma 3.11.4.** *Let $i : V \to X$ be a closed immersion. Then $i$ is quasi-compact.*

*Proof.* Since $i$ is a closed immersion, $|V|$ may be assumed to be a closed subspace of $|X|$ (as $|i|$ is a homeomorphism on to a closed subspace of $|X|$). From here the result follows from the fact that closed subspaces of quasi-compact spaces remain quasi-compact. □

**Proposition 3.11.5.** *Let $f : X \to S$ be a separated morphism. Then $f$ is quasi-separated.*

*Proof.* Because $f$ is separated the diagonal $\Delta_X|_S : X \to X \times_S X$ is a closed immersion. The result now follows from Lemma 3.11.4. □

[61] Which is a good sanity check to do, honestly.

# Appendix A

# Sites and Sheaves on Sites

One of the most important aspects about sheaves, at least from a geometric perspective, is that they allow us to learn local information about global structure. Sheaves allow us to capture many local properties of geometric objects that would otherwise elude us, as they allow us to get at the local structure of an object and those that interact with it in a way that classical techniques do not allow. Furthermore, by generalizing these sheaf-theoretic tools to a categorical setting, we can use geometric reasoning to study logic (cf. using the $\neg\neg$-topology to do set-theoretic forcing and determine when logics are classical; see [36] or [54] for elementary details), as well as to study spaces and their interactions in ways that set-theoretic methods do not see (cf. the introduction of the étale topology in algebraic geometry to do étale cohomology, or $\ell$-adic cohomology, and hence to solve the Weil conjectures). Sheaves also allow one to get towards a generalized homotopy theory that not only allows one to capture strange notions of covers, but also allows one to cast homotopy theory into totally disconnected spaces in a way that is not utterly uninteresting (see Chapter 8 of [36] for details). We will begin this article in the completely and utterly standard way by motivating sheaves through showing how they behave with respect to covers of a topological space and how they can be used to capture étale spaces over $X$, i.e., spaces $p : E \to X$ such that $p$ is a local homeomorphism. From here we will generalize this massively and then show how to take these covers into Grothendieck topologies and then define sheaves in a general fashion. Before diving right in, however, we will make a definition of convenience so that we can talk about a presheaf throughout this entire article. We will also cite a couple facts about the category of presheaves on a fixed category.

**Proposition A.0.1.** *If $\mathscr{C}$ is a category then $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a locally small topos. In particular, $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is complete, cocomplete, and Cartesian closed.*

An important aspect of this fact is that limits and colimits in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ are calculated pointwise (locally). In particular, if $\{P_i \mid i \in I\}$ is a family of presheaves and $\{\alpha_{ij} : P_i \to P_j \mid i, j \in I\}$ is a family of morphisms (by definition, natural transformations) between presheaves, then the limit

$$\varprojlim_{i \in I} P_i$$

is calculated by defining, for all $U \in \operatorname{Ob}\mathscr{C}$,

$$\left(\varprojlim_{i \in I} P_i\right)(U) := \varprojlim_{i \in I}(P_i(U)).$$

Calculating colimits is, of course, done in the same manner, and the right adjoint $[P, -]$ of the product functor $(-) \times P$ on $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is induced by the equation, for all $U \in \operatorname{Ob}\mathscr{C}$,

$$[P, F](U) := [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]\big(\mathscr{C}(-, U) \times P, F\big).$$

We also recall the following definition for the sake of completeness.

**Definition A.0.2.** A category $\mathcal{E}$ is an elementary topos (which we will instead abbreviate as a topos) if and only if $\mathcal{E}$ is finitely complete, Cartesian closed, and admits a subobject classifier $\Omega$ (cf. Definition A.2.20).

## A.1 Sites, Sieves, and Sheaves

One of the important aspects of sheaves on a space we saw in the subsection above is that the covers above are geometrically determined by having the same restriction to mutual intersections, i.e., if $U$ and $V$ are open subsets of some ambient topological space then

$$\rho_{U,U\cap V} = \rho_{V,U\cap V},$$

and furthermore that this restriction is uniquely determined on their embeddings $V \to X, U \to X$. While it may be difficult to see the appropriate categorical generalization to determine this in the first place, note that if $X$ is a topological space and $U$ and $V$ are open subsets of $X$, then we can describe $U \cap V$ as the pullback

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

where each arrow is simply the standard open immersion of topological spaces. In this way, to naïvely categorically generalize topologies and covers to categories we will work with categories that have all pullbacks; we will see later when we get to study sieves that this can be bypassed, but for the moment we will insist upon it.

The axioms of a cover we will insist upon will come from three fairly reasonable geometric insights:

1. If $\varphi : V \to U$ is an isomorphism in $\mathscr{C}$, then it had better be a cover;
2. If $\{\varphi_i : U_i \to U \mid i \in I\}$ is a cover of $U$ and, for each $i \in I$, $\{\psi_{ij} : V_{ij} \to U_i \mid j \in J_i\}$ is a cover of $U_i$, then the set of composite maps
$$\{\varphi_i \circ \psi_{ij} : V_{ij} \to U \mid i \in I, j \in J_i\}$$
had better be a cover of $U$;
3. If $\{\varphi_i : U_i \to U \mid i \in I\}$ is a cover of $U$ and if $\rho : V \to U$ is any map, then when we consider all intersections
$$\begin{tikzcd} V \times_U U_i \arrow[r, "\pi_{2,i}"] \arrow[d, "\pi_{1,i}"'] & U_i \arrow[d, "\varphi_i"] \\ V \arrow[r, "\rho"'] & U \end{tikzcd}$$
simultaneously, we had better have that $\{\pi_{1,i} : V \times_U U_i \to V \mid i \in I\}$ is a cover of $V$.

**Definition A.1.1** ([36], [54])**.** A *Grothendieck pretopology $\tau$ (or a basis $\tau$ to a Grothendieck topology)* on a category $\mathscr{C}$ with fibre products is a collection of sets $\tau(U) \subseteq \mathcal{P}^2(\operatorname{Codom} U)$ (called basic covers of $U$) such that:

1. For all isomorphisms $\varphi$ in $\mathscr{C}$ with $\operatorname{Codom}(\varphi) = U$, the set $\{\varphi : V \to U\} \in \tau(U)$;
2. If $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ and if $\{\psi_{ij} : V_{ij} \to U_i \mid j \in J_i\} \in \tau(U_i)$ for all $i \in I$, then
$$\{\varphi_i \circ \psi_{ij} : V_{ij} \to U \mid i \in I, j \in J_i\} \in \tau(U);$$
3. If $\rho \in \mathscr{C}(V,U)$ and if $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ then $\{\pi_{1,i} : V \times_U U_i \to V \mid i \in I\} \in \tau(V)$, where $\pi_{1,i}$ is the projection in the pullback:
$$\begin{tikzcd} V \times_U U_i \arrow[r, "\pi_{2,i}"] \arrow[d, "\pi_{1,i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_i \arrow[d, "\varphi_i"] \\ V \arrow[r, "\rho"'] & U \end{tikzcd}$$

**Example A.1.2.** Let $\mathscr{C} := \mathbf{Open}(X)$ be the open lattice of a topological space $X$. Then define a pretopology $\tau$ on $\mathbf{Open}(X)$ by saying that $\{U_i \to U \mid i \in I\} \in \tau(U)$ if and only if

$$\bigcup_{i \in I} U_i = U.$$

We can easily check conditions (1) and (2): Since the only isomorphisms in a poset category are the identity, (1) follows from the fact that

$$U = \bigcup_{i \in \{i\}} U,$$

while if $\{U_i \to U \mid i \in I\} \in \tau(U)$ and for all $i \in I, \{U_{ij} \to U_i \mid j \in J_i\} \in \tau(U_i)$ then

$$\bigcup_{i \in I} \bigcup_{j \in J_i} U_{ij} = \bigcup_{i \in I} \left( \bigcup_{j \in J_i} U_{ij} \right) = \bigcup_{i \in I} U_i = U$$

shows that $\{U_{ij} \to U \mid i \in I, j \in J_i\} \in \tau(U)$. Finally, (3) holds from the following: If $\{U_i \to U \mid i \in I\} \in \tau(U)$ and if $V \to U$ is an arrow in $\mathbf{Open}(X)$, then $U_i \times_U V = U \cap V$. Moreover, the arrow $U_i \times_U V \to V$ is a subset inclusion of $U \cap V \to V$, so since the $U_i$ cover $U$ by De Morgan's Laws we have that

$$\bigcup_{i \in I} (U_i \times_U V) = \bigcup_{i \in I} (U_i \cap V) = \left( \bigcup_{i \in I} U_i \right) \cap V = U \cap V.$$

However, since $V \to U$ is an arrow in $\mathbf{Open}(X)$, $V \subseteq U$ and so $V \cap U = V$ and hence $\{U_i \times_U V \to V \mid i \in I\} \in \tau(V)$.

**Example A.1.3.** Let $\mathscr{C} = \mathbf{AffSch}$ be the category of affine schemes and define a pretopology *fét* on $\mathscr{C}$ by saying that a collection of morphisms of schemes $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau_{\text{fét}}(U)$ if and only if the map

$$[\varphi_i]_{i \in I} : \coprod_{i \in I} U_i \to U$$

is surjective (so the $\varphi_i$ are jointly surjective) and such that each $\varphi_i$ is finite étale. This is the *finite étale* pretopology on $\mathbf{AffSch}$.

**Example A.1.4.** Let $\mathscr{C} = \mathbf{Sch}$ and define the flat pretopology *fppf* (for "fidèlement plat de présentation finie," meaning faithfully flat and of finite presentation) to be given by saying that if $U \in \operatorname{Ob} \mathbf{Sch}$ is affine, then $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau_{\text{fppf}}(U)$ if and only if the $\varphi_i$ are jointly surjective, each $X_i$ is affine, and each $\varphi_i$ is flat and finitely presented. For arbitrary schemes $X$, we say that $\{\varphi_i : X_i \to X \mid i \in I\} \in \tau_{\text{fppf}}(U)$ if and only if the cover happens to be an fppf cover after base changing to an open affine subscheme of $X$.

Now that we have described pretopologies, and even seen a few examples, it would be nice to see how to define sheaves on pretopologies. We can do this largely in the same way that we define sheaves in the case of a topological space, but we will see that pretopologies have an unfortunate imprecision: Sometimes it is the case that distinct pretopologies give rise to the same categories of sheaves. This can lead to some frustrating consequences, but after we show some examples of distinct pretopologies giving rise to the same sheaves, we will show how to avoid this imprecision: Through Grothendieck topologies!

**Definition A.1.5.** We say that a presheaf $F$ on $\mathscr{C}$ is a *sheaf with respect to the pretopology* $\tau$ if given any covering family $\{U_i \xrightarrow{\varphi_i} U \mid i \in I\} \in \tau(U)$, the diagram

$$\begin{tikzcd} FU \arrow[r, "\langle F\varphi_i \rangle_{i \in I}"] & \prod_{i \in I} FU_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j \in I} F(U_i \times_U U_j) \end{tikzcd}$$

is an equalizer in $\mathbf{Set}$.

Note that the morphisms $p$ and $q$ come from the following pairing maps: If $i$ and $j$ are any fixed indeces of $I$, then there are maps $(\pi_1)_{ij} : U_i \times_U U_j \to U_i$ and $(\pi_2)_{ij} : U_i \times_U U_j \to U_j$; iterating over all such pairs allows us to consider the diagrams

$$\begin{tikzcd} FU_i \arrow[r, "F(\pi_1)_{ij}"] & F(U_i \times_U U_j) \\ \prod_{i \in I} FU_i \arrow[u, "\pi_i"] \arrow[r, "\langle F(\pi_1)_{ij} \rangle_{i \in I}"'] & \prod_{i,j \in I} F(U_i \times_U U_j) \arrow[u, "\pi_{ij}"'] \end{tikzcd}$$

and

$$\begin{tikzcd} FU_j \arrow[r, "F(\pi_2)_{ij}"] & F(U_i \times_U U_j) \\ \prod_{i \in I} FU_i \arrow[u, "\pi_2"] \arrow[r, "\langle F(\pi_2)_{ij} \rangle_{j \in I}"'] & \prod_{i,j \in I} F(U_I \times_U U_j) \arrow[u, "\pi_{ij}"'] \end{tikzcd}$$

The two of these together induce $p$ and $q$ as

$$p := \langle F(\pi_1)_{ij} \circ \pi_i \rangle_{i,j \in I}$$

and

$$q := \langle F(\pi_2)_{ij} \circ \pi_j \rangle_{i,j \in I},$$

respectively.

**Definition A.1.6.** A morphism of sheaves $F$ and $G$ on a pretopology $\tau$ is simply a natural transformation $\varphi : F \to G$ in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$. The category full subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ of all $\tau$-sheaves is denoted by $\mathbf{Shv}(\mathscr{C}, \tau)$.

**Remark A.1.7** (Achtung!)**.** It can happen that two pretopologies $\tau, \tau'$ on $\mathscr{C}$ have $\mathbf{Shv}(\mathscr{C}, \tau) = \mathbf{Shv}(\mathscr{C}, \tau')$ even if $\tau \neq \tau'$. Grothendieck pretopologies do not uniquely determine categories of sheaves, so we can't every say that $\tau$ is *the* pretopology generating a sheaf category $\mathbf{Shv}(\mathscr{C}, \tau')$ without throwing adjectives[1] at the pretopology itself.

**Example A.1.8.** Let $\tau$ be a Grothendieck pretopology on $\mathscr{C}$ and define a new pretopology $\tau'$ by saying, for $U \in \operatorname{Ob} \mathscr{C}$, that $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau'(U)$ if and only if there exists an index set $J_i$ for each $i \in I$ there is a set of morphisms $\{\psi_{ij} : U_{ij} \to U_i \mid j \in J_i\}$ such that $\{\varphi_i \circ \psi_{ij} : U_{ij} \to U \mid i \in I, j \in J_i\} \in \tau(U)$. One can then show that a $\tau$-sheaf is a $\tau'$-sheaf, and vice-versa.

**Example A.1.9.** If $\mathscr{C} = \mathbf{AffSch}$, consider the fppf pretopology given above, i.e., $\tau$ is defined by saying that $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ if and only if the $\varphi_i$ are jointly surjective and each $\varphi_i$ is flat and finitely presented. Then we define a pretopology $\tau'$ by saying that $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau'(U)$ if and only if $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ and each $\varphi_i$ is quasi-finite. Then $\tau$ and $\tau'$ generate the same sheaf categories by [31, Corollary 17.16.2].

As the above example shows, it can be quite frustrating to try and work with sheaves of pretopologies because it can be the case that some very different looking covers can generate the same sheaves. However, if one works with Grothendieck topologies, they remove this frustration altogether! However, to understand Grothendieck topologies we first need to learn about sieves.

**Definition A.1.10.** A *sieve $S$ on an object* $U \in \operatorname{Ob} \mathscr{C}$ is a subfunctor of $\mathbf{y}(U) := \mathscr{C}(-, U)$.

[1] Such as "maximal."

**Remark A.1.11.** We will make a common abuse of notation and when working with sieves; while I do not defend the notation itself, this abuse is common in the literature and does frequently make the manipulation easier, so it is worth seeing at least so that you are not totally lost when reading other books on sites, sheaves, and toposes. A sieve $S$ on $U \in \operatorname{Ob}\mathscr{C}$ can be described as a set of morphisms, all with codomain $U$, such that if $f : V \to U \in S$ and $g : W \to V$, then $f \circ g \in S$, i.e., the following deduction holds:

$$\frac{f \in S \qquad g \in \operatorname{Mor}\mathscr{C}\,.\,\operatorname{Dom}(f) = \operatorname{Codom}(g)}{f \circ g \in S}$$

To see why, consider that since $S$ is a subfunctor of $\mathscr{C}(-, U)$, for all $V \in \operatorname{Ob}\mathscr{C}$, $S(V) \subseteq \mathscr{C}(V, U)$ and for all $\varphi : W \to V$, there is a commuting diagram

$$\begin{tikzcd} S(W) \arrow[r, "i_W"] & \mathscr{C}(W,U) \\ S(V) \arrow[u, "\varphi^*"] \arrow[r, "i_V"'] & \mathscr{C}(V,U) \arrow[u, "\varphi^*"'] \end{tikzcd}$$

where the natural transformation $i$ is induced by the subset inclusion $S(X) \subseteq \mathscr{C}(X, U)$ for all $X \in \operatorname{Ob}\mathscr{C}$, and $\varphi^*$ acts by pre-composition by $\varphi$. Defining

$$S := \bigcup_{V \in \operatorname{Ob}\mathscr{C}} S(V)$$

then allows us to see from the naturality square above that if $f \in S$, then for all $g \in \operatorname{Mor}\mathscr{C}$ with $\operatorname{Codom}(g) = \operatorname{Dom}(f)$, $f \circ g \in S$.

On the other hand, it is possible to build a subfunctor of $\mathscr{C}(-, U)$ from a pre-composition closed subset of $\operatorname{Codom} U$ by taking $S(V) := \{\varphi \in S \mid \operatorname{Dom}(\varphi) = V\}$, which shows that sieves and pre-composition closed subsets of $\operatorname{Codom} U$ are one and the same.

**Example A.1.12.** Let $X$ be a topological space and let $\mathscr{C} = \mathbf{Open}(X)$ be the open lattice of $X$. If $C = \{U_i \to X \mid i \in I\}$ is an open cover of $X$ then there is an induced sieve $(C)$ on $\mathscr{C}$ which is determined by, for an open $V \subseteq X$,

$$(C) = \begin{cases} \varnothing & \text{if} \nexists\, i \in I.\, V \subseteq U_i; \\ \{*\} & \text{if} \exists\, i \in I.\, V \subseteq U_i. \end{cases}$$

In this way sieves work just as the word implies: they provide us with "holes" to test a cover, and something fits through the sieve (has non-empty maps) if and only if it fits through the holes the cover sets up. A visualization of this is given in Figure A.1.

**Definition A.1.13.** If $S$ is a sieve on $U \in \operatorname{Ob}\mathscr{C}$ and if $\rho \in \mathscr{C}(V, U)$, then the *pullback sieve* $\rho^*(S)$ is the set

$$\rho^*(S) := \{\varphi \in \operatorname{Mor}\mathscr{C} \mid \operatorname{Codom}(\varphi) = V, \rho \circ \varphi \in S\}.$$

**Lemma A.1.14.** *If $S$ is a sieve on $U$ and if $\rho \in \mathscr{C}(V, U)$ then $\rho^*(S)$ is a sieve on $V$.*

*Proof.* This is immediate from the fact that $S$ is pre-composition closed. □

**Definition A.1.15.** A *Grothendieck topology* $J$ on a category $\mathscr{C}$ is a collection, for all $U \in \operatorname{Ob}\mathscr{C}$, of *$J$-covering sieves*[2] on $U$ called $J(U)$ such that:

1. The maximal sieve
$$\operatorname{Codom} U = \mathbf{y}(U) \in J(U);$$

[2]By covering sieve we simply mean a declared or chosen sieve.

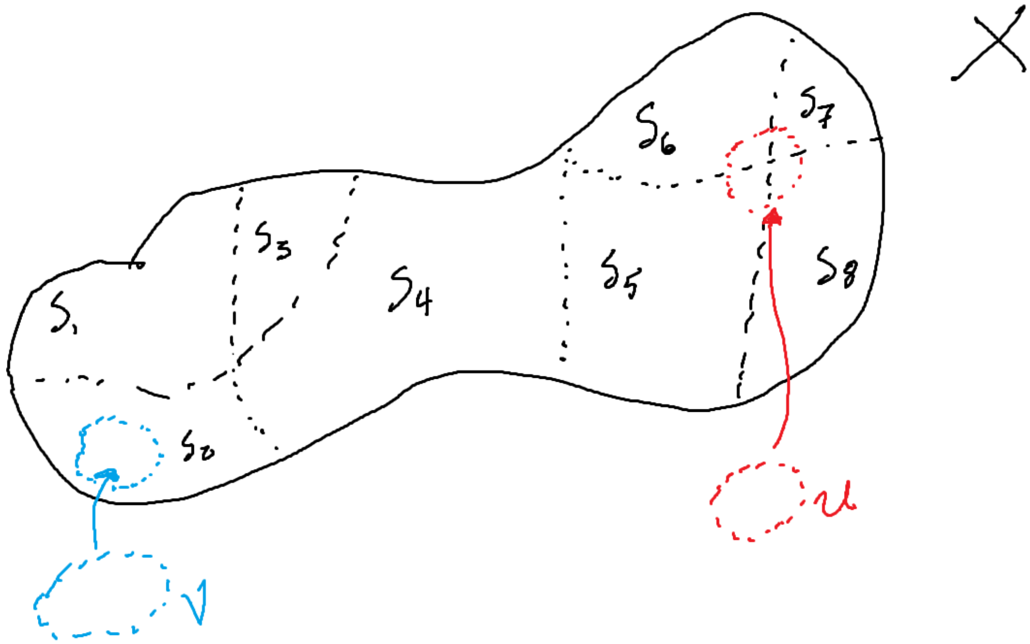


Figure A.1: In this picture we have a topological space $X$ covered by open sets $S_1 - S_8$ with two other open sets labeled $U$ (in red) and $V$ (in blue). The set $U$ does not fit through the sieve $S$ generated by $U$ (so $S(U) = \varnothing$) because $U \nsubseteq S_i$ for any $1 \leq i \leq 8$ while the set $V$ does fit through $S$ (so $S(V) = \{*\}$) because $V \subseteq S_2$.

2. If $S \in J(U)$ and if $\rho : V \to U$, the $\rho^*(S) \in J(V)$;

3. If $S \in J(U)$ and if $R$ is any sieve on $U$ such that for all $\varphi : \operatorname{Dom}\varphi \to U \in S$, $\varphi^*(R) \in J(\operatorname{Dom}\varphi)$, then $R \in J(U)$.

**Remark A.1.16.** The definition of a Grothendieck and its covering sieves can be rephrased in terms of arrows. In this way we think of a sieve $S$ on $U$ covering a morphism $\rho : V \to U$ if $\rho \in S$. In this way we can rephrase a Grothendieck topology as a collection of covering sieves $S \in J(U)$, for all $U \in \operatorname{Ob}\mathscr{C}$, such that:

1. If $S$ is a sieve on $U$ and if $f : V \to U \in S$, then $S$ covers $f$;

2. If $S$ covers $f : V \to Y$, it covers any composite $f \circ g : W \to U$, where $g : W \to V$.

3. If $S$ covers $f : V \to U$ and $R$ is a sieve on $U$ which covers every $\rho \in S$, then $R$ covers $f$.

**Remark A.1.17.** One useful aspect of a Grothendieck topology is that we no longer have to assume that the underlying category $\mathscr{C}$ has pullbacks. This means that we can naïvely find more Grothendieck topologies than we can find pretopologies. An alternative approach to fix the pretopologies-require-pullbacks problem is to embed $\mathscr{C} \to [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ through the Yoneda Embedding $\mathbf{y}$ and then place a pretopology on $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ instead.

**Definition A.1.18.** A *site* is a choice of category $\mathscr{C}$ and Grothendieck topology $J$ on $\mathscr{C}$, and is denoted $(\mathscr{C}, J)$.

We say that if $(\mathscr{C}, J)$ is a site and if $R, S \in J(U)$ for some $U \in \operatorname{Ob}\mathscr{C}$ then a sieve refines $R$ and $S$ if $T \in J(U)$ and $T \subseteq R, T \subseteq S$. The next proposition we will describe shows that any two covering sieves on an object $U \in \operatorname{Ob}\mathscr{C}$ have a common refinement which is also a covering sieve of $U$.

**Proposition A.1.19.** *If $(\mathscr{C}, J)$ is a site and $R, S \in J(U)$, then $R \cap S$ is a sieve and $R \cap S \in J(U)$.*

*Proof.* The fact that $R \cap S$ is a sieve is immediate, while the fact that $R \cap S \in J(U)$ follows from Condition (3) of a Grothendieck topology. □

**Example A.1.20.** If $\mathscr{C}$ is a category, the trivial topology on $\mathscr{C}$ has the property that $S \in J(U)$ if and only if $S = \operatorname{Codom} U$. Evidently, $J$ is the smallest topology on $\mathscr{C}$.

One unfortunate aspect of the axioms of a Grothendieck topology, as opposed to a Grothendieck pretopology, is that at a surface level they seem to have nothing, or at least very little, to do with each other. However, we will now show how to build a Grothendieck topology from a pretopology, as well as how to find a pretopology that generates a given topology (when the underlying category has pullbacks, anyway).

**Definition A.1.21.** Let $\mathscr{C}$ be a category with fibre products and let $\tau$ be a pretopology on $\mathscr{C}$. Now define a collection of covering sieves $J$ on $\mathscr{C}$ by saying that, for some object $U \in \operatorname{Ob}\mathscr{C}$, a sieve $S \in J(U)$ if and only if there exists a cover $K \in \tau(U)$ such that $K \subseteq S$. The collection $J$ is then said to be the *Grothendieck topology generated by $\tau$.*

**Theorem A.1.22.** *The collection $J$ defined in Definition A.1.21 is a Grothendieck topology on $\mathscr{C}$.*

*Proof.* (1): Begin by observing that since $\{\mathrm{id}_U : U \to U\}$ is an isomorphism, $\{\mathrm{id}_U\} \in \tau(U)$, so $\tau(U) \neq \varnothing$. Moreover, if $\operatorname{Codom} U$ is the maximal sieve on $U$, then $\{\mathrm{id}_U\} \subseteq \operatorname{Codom} U$, so $\operatorname{Codom} U \in J(A)$.

(2): Let $S \in J(U)$ and let $\rho \in \mathscr{C}(V, U)$. Find a cover $C := \{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ such that $K \subseteq S$. Then define $C'$ to be the cover

$$C' := \{\pi_{1,i} : V \times_U U_i \to V \mid i \in I\},$$

where the pullback morphisms come from the diagram:

$$\begin{tikzcd} V \times_U U_i \arrow[r, "\pi_{2,i}"] \arrow[d, "\pi_{1,i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_i \arrow[d, "\varphi_i"] \\ V \arrow[r, "\rho"'] & U \end{tikzcd}$$

Moreover, note that $C' \in \tau(V)$ and that from the commutativity of the pullback diagram we have that

$$\rho \circ \pi_{1,i} = \varphi_i \circ \pi_{2,i} \in S.$$

Thus each $\pi_{1,i} \in \rho^*(S)$ so $C' \subseteq \rho^*(S)$, proving that $\rho^*(S) \in J(V)$.

(3): Let $U \in \operatorname{Ob}\mathscr{C}$ and assume that $R$ is a sieve on $U$ such that there exists an $S \in J(U)$ with the property that for all $\rho \in S, \rho^*(R) \in J(\operatorname{Dom}\rho)$. So, since $S \in J(U)$, there exists a cover

$$K := \{\varphi_i : U_i \to U \mid i \in I\} \in \tau U$$

with $K \subseteq S$. Then, for all $i \in I$, we have that $\varphi_i^*(R) \in J(U_i)$; for each fixed $i \in I$, fix a cover

$$K_i := \{\varphi_{ij} : U_{ij} \to U_i \mid j \in J_i\} \in \tau U_i$$

such that $K_i \subseteq \varphi_i^*(R)$. Observe on one hand that it follows by definition that for each $\varphi_{ij} \in K_i, \varphi_i \circ \varphi_{ij} \in R$, while because each $K_i \in \tau U_i$ and because $K \in \tau U$, the cover

$$C := \{\varphi_i \circ \varphi_{ij} : U_{ij} \to U \mid i \in I, j \in J_i\} \in \tau U.$$

Thus, because each composite $\varphi_i \circ \varphi_{ij} \in R$, $C \subseteq R$ and so $R \in J(U)$. □

Now, in order to show how to give a maximal pretopology that generates a Grothendieck topology, assume that $\tau$ is a pretopology on a category $\mathscr{C}$ with fibre products, and let $C \in \tau(U)$. Now define

$$(C) := \{\varphi \circ \psi \mid \varphi \in C, \operatorname{Dom} \varphi = \operatorname{Codom} \psi\};$$

it is easy to show that $(C)$ is a sieve, and that $(C)$ is intimately related to $C$.

**Definition A.1.23.** If $\tau$ is a pretopology on $\mathscr{C}$ and $C \in \tau(U)$ for some $U \in \operatorname{Ob} \mathscr{C}$, then we say that $(C)$ is *the sieve on $U$ generated by $C$.*

**Proposition A.1.24.** *Let $\mathscr{C}$ be a category with fibre products and let $(\mathscr{C}, J)$ be a site. Then there exists a unique maximal pretopology $\tau$ which generates $J$.*

*Sketch.* Define the pretopology $\tau$ by saying that a cover $C := \{\varphi_i : U_i \to U \mid i \in I\}$ satisfies the rule

$$C \in \tau(U) \iff (C) \in J(U).$$

Showing that $\tau$ is a pretopology is straightforward using the three axioms of the topology $J$. Furthermore, the maximality of $\tau$ with respect to inclusion is also straightforward. $\square$

**Example A.1.25.** If $\mathscr{C} = \mathbf{AffSch}$, then the finite étale, étale, and fppf topologies are the topologies on $\mathscr{C}$ generated by the corresponding pretopologies given in Examples A.1.3 and A.1.4.

We will now move on from discussing pretopologies and topologies to discuss the sheaves they induce. Just like how a sheaf on a space respects the fact that the restrictions along the inclusions $U \cap V \to U$ and $U \cap V \to V$ should be in a very real sense "the same," we would like it to be the case for sheaves on a topology to be the same on the "inclusions" $\pi_1 : U_i \times_U U_j \to U_i$ and $\pi_2 : U_i \times_U U_j \to U_j$ whenever the maps $\varphi_I : U_i \to U$ and $\varphi_j : U_j \to U$ are covered by a sieve $S \in J(U)$. However, this description unfortunately has the requirement that $\mathscr{C}$ admit fibre products, and we should instead be able to do this in any category. As such, we are going to work with covering sieves $S \in J(U)$ and work with gluing conditions that use the fact that $f \circ g \in S$ for all $f \in S$ and $g \in \operatorname{Mor} \mathscr{C}$ with $\operatorname{Dom} f = \operatorname{Codom} g$ to generalize how to deal with intersections in categories without fibre products.

In order to examine how this general gluing is best gone about, assume that $(\mathscr{C}, J)$ is a site and let $S \in J(U)$ be a covering sieve of $U$. Then, for any presheaf $P \in [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ we can consider the object

$$\prod_{f \in S} P(\operatorname{Dom} f),$$

which should play the role of considering the product of a family $PU_i$ for a cover $\{U_i \to U \mid i \in I\}$. We now, perhaps unfortunately, need to replace the fibre product induced intersections in order to proceed with defining general sheaves on $J$. However, this is not as daunting as one may expect; since $S$ is a sieve, for all $g \in \operatorname{Mor} \mathscr{C}$, whenever $f \in S$ if $\operatorname{Dom} f = \operatorname{Codom} g$ then $f \circ g \in S$. Thus we should make sure that if a morphism $f \in S$ also is equal to a composite $f' \circ g$ for $f' \in S$ and some $g \in \operatorname{Mor} \mathscr{C}$, the action of $P$ had better be the same on its "gluings." That is, we consider the doubly indexed product

$$\prod_{\substack{f \in S, g \in \operatorname{Mor} \mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P(\operatorname{Dom} g)$$

in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, together with two maps from the product of the $P(\operatorname{Dom} f)$, for $f \in S$, that encapsulate the fact that $P$ had better not destroy inclusions. To how to go about this we will construct two parallel morphisms

$$\begin{tikzcd} \prod\limits_{f \in S} P(\operatorname{Dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{\substack{f \in S, g \in \operatorname{Mor} \mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P(\operatorname{Dom} g). \end{tikzcd}$$

where $p$ and $q$ are induced in two different ways that had best be the same in order for $P$ to respect gluings of covers.

In order to see how to define the maps $p$ and $q$, assume that an element

$$(x_f)_{f\in S} \in \prod_{f\in S} P(\operatorname{Dom} f)$$

is given. On one hand, we can define a map simply based on consider the action on the $P(\operatorname{Dom} f)$ induced by the sieve axiom; that is, define $p$ by the equation

$$p(x_f)_{f\in S} := (x_{f\circ g})_{f\in S, g\in \operatorname{Mor}\mathscr{C}, \operatorname{Dom} f = \operatorname{Codom} g}.$$

The map $q$, on the other hand, is defined by the action of $P(g)$ on each of the $x_f$; that is, whenever $g \in \operatorname{Mor}\mathscr{C}$ with $\operatorname{Dom} f = \operatorname{Codom} g$, there is a map $P(g) : P(\operatorname{Dom} f) \to P(\operatorname{Dom} g)$; taking the pairing map indexed over all such pairs then gives a morphism which is defined by the equation

$$q(x_f)_{f\in S} := \big(P(g)(x_f)\big)_{f\in S, g\in \operatorname{Mor}\mathscr{C}, \operatorname{Dom} f = \operatorname{Codom} g}.$$

In this way, if $S$ is a covering sieve on $U$, we can produce a commuting diagram

$$\begin{tikzcd} PU \arrow[r, "e"] & \prod\limits_{f\in S} P(\operatorname{Dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{\substack{f\in S, g\in\operatorname{Mor}\mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P\operatorname{Dom} g \end{tikzcd}$$

where $e$ is simply the pairing map of the presheaf morphism attached to each morphism in $S$, i.e.,

$$e = \langle P(f)\rangle_{f\in S}.$$

In order for $P$ to be a sheaf, it had better be the case that the actions of $p$ and $q$ be the same on $PU$. However, this is the same thing as saying that the diagram above is an equalizer! This motivates our definition of a sheaf on a general Grothendieck topology, and shows how it is constructed.

**Definition A.1.26.** Let $(\mathscr{C}, J)$ be a site. We then say that a presheaf $P \in [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a *sheaf on the $J$-topology, or a $J$-sheaf,* if for every $U \in \operatorname{Ob}\mathscr{C}$ and for every $S \in J(U)$, the induced diagram

$$\begin{tikzcd} PU \arrow[r, "e"] & \prod\limits_{f\in S} P(\operatorname{Dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{\substack{f\in S, g\in\operatorname{Mor}\mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P\operatorname{Dom} g \end{tikzcd}$$

is an equalizer in $\mathbf{Set}$.

**Remark A.1.27.** This remark serves to connect the definition of sheaf given classically (cf. [36] and [54]) to the equalizer definition given above, as well as so that we can use some of that classical theory to prove various things about sheaves in general. In the classical language, a sheaf was defined as a presheaf $P$ such that whenever $S$ was a sieve on $U \in \operatorname{Ob}\mathscr{C}$, given a natural transformation $\alpha : S \to P$, $P$ was a sheaf if there was a unique natural transformation $\beta : \mathscr{C}(-, U) \to P$ making the diagram

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d] & P \\ \mathscr{C}(-,U) \arrow[ur, dashed, "\exists!\beta"'] & \end{tikzcd}$$

commute.

The reasons that our two definitions of sheaf are equivalent are as follows: If $S$ is a sieve on $U \in \operatorname{Ob}\mathscr{C}$ and $P$ is a presheaf on $\mathscr{C}$, a natural transformatio $\alpha : S \to P$ is a natural transformation assigning, for each $f \in S$ an element $x_f \in P(\operatorname{Dom} f)$ such that if $g \in \mathscr{C}(W, \operatorname{Dom} f)$ then $P(g)(x_f) = x_{f\circ g}$. That is to say, a natural transformation $\alpha$ allows us to infer

$$\frac{V \xrightarrow{f} U \in S(V) \subseteq \mathscr{C}(V,U)}{x_f \in P(V), x_f = \alpha(f)}$$

such that:

$$\frac{W \xrightarrow{g} V \in \operatorname{Mor}\mathscr{C} \qquad V \xrightarrow{f} U \in S}{P(g)(x_f) = x_{f\circ g}}$$

In this way a unique extension of $\alpha : S \to P$ to a natural transformation $\mathscr{C}(-,U) \to P$ is the same as a unique element $x \in P(U)$ such that if $f \in S$ then

$$P(f)(x) = x_f.$$

Given such a condition, it then follows that if such an $x$ is given, $e(x) = (x_f)_{f\in S}$ *and*

$$\begin{aligned}(p\circ e)(x) &= p(e(x)) = p(x_f)_{f\in S} = (x_{f\circ g})_{f\in S, g\in \operatorname{Mor}\mathscr{C}, \operatorname{Dom} f = \operatorname{Codom} g} \\ &= (P(g)(x_f))_{f\in S, g\in \operatorname{Mor}\mathscr{C}, \operatorname{Dom} f = \operatorname{Codom} g} = q(x_f)_{f\in S} \\ &= (q\circ e)(x),\end{aligned}$$

while the universal property of the equalizer follows from the uniqueness of the extension. Similarly, if we have sheaf we have defined, the unique (because equalizers in **Set** are monic, of course) lift of the $(x_f)_{f\in S} = e(x)$, for some $x \in P(U)$, defines the extension of the natural transformation $\alpha : S \to P$ induced by the sheaf axiom and the morphism $p$: $(x_f)_{f\in S} := (\alpha(f))_{f\in S}$.

It is unfortunate, but one difficulty that comes with working with the full weight of sheaves on a site $(\mathscr{C}, J)$ is that it can be quite awkward and very complicated to check that for all covering sieves $S$ on $U \in \operatorname{Ob}\mathscr{C}$, the diagram

$$\begin{tikzcd} PU \arrow[r, "e"] & \prod_{f\in S} P(\operatorname{Dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{\substack{f\in S, g\in\operatorname{Mor}\mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P\operatorname{Dom} g \end{tikzcd}$$

is an equalizer. This is especially the case when you know very little about general covering sieves, and instead only know things about certain covers in a given pretopology $\tau$. However, in the case where you do have a pretopology $\tau$ generating $J$, your task is greatly simplified: you only need to check on the covers of $\tau(U)$ instead. This is the content of our next theorem, which has a simple and straightforward proof that only requires one to unravel the definitions.

**Theorem A.1.28.** *Let $\mathscr{C}$ be a category with fibre products and let $(\mathscr{C}, J)$ be a site generated by the pretopology $\tau$. Then $F$ is a $J$-sheaf if and only if for all covers $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$, the diagram*

$$\begin{tikzcd} FU \arrow[r, "e"] & \prod_{i\in I} FU_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j\in I} F(U_i \times_U U_j) \end{tikzcd}$$

*is an equalizer.*

**Remark A.1.29.** In the proof of the theorem above, we will frequently be referring to the pullbacks $U_i \times_U U_j$ arising from morphisms $\varphi_i : U_i \to U$ and $\varphi_j : U_j \to U$ in a cover $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$. Thus, when we write the morphisms $\pi_{1,ij} : U_i \times_U U_j \to U_i$ and $\pi_{2,ij} : U_i \times_U U_j \to U_j$ we mean the first and second projections coming from the diagram

$$\begin{tikzcd} U \times_U U_j \arrow[r, "\pi_{2,ij}"] \arrow[d, "\pi_{1,ij}"'] & U_j \arrow[d, "\varphi_j"] \\ U_i \arrow[r, "\varphi_i"'] & U \end{tikzcd}$$

in $\mathscr{C}$.

*Proof.* $\Longrightarrow$: Assume that $F$ is a $J$-sheaf and let $C := \{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ be a cover of $U$. Now find a collection of elements $\{x_i \in FU_i \mid i \in I\}$ such that for all $i, j \in I$ the equation

$$F(\pi_{1,ij})(x_i) = F(\pi_{2,ij})(x_j)$$

holds. Now consider the sieve

$$(C) = \{\varphi_i \circ \psi \mid i \in I, \psi \in \operatorname{Mor}\mathscr{C}\} = \{\rho : V \to U \mid \exists\, i \in I, \exists\, \psi \in \operatorname{Mor}\mathscr{C}\,.\, \rho = \varphi_i \circ \psi\},$$

which is a $J$-sieve because $\tau$ generates $J$. We will now show that the desired diagram is an equalizer via consideration of this sieve.

Define a morphism $\alpha : (C) \to F$ in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ by, for all $\rho \in (C)$,

$$\alpha(\rho) := F(\psi)(x_i)$$

whenever $\rho = \varphi_i \circ \rho$. To see that this is well-defined, assume that $\varphi_i \circ \psi = \rho = \varphi_j \circ \psi'$ for some $i, j \in I$ and some $\psi, \psi' \in \operatorname{Mor}\mathscr{C}$. Then, from the universal property of the pullback, there exists a unique morphism $\theta : \operatorname{Dom}\rho \to U_i \times_U U_j$ making the diagram

$$\begin{tikzcd}
\operatorname{Dom}\rho \arrow[dr, dashed, "\exists!\theta"] \arrow[drr, bend left, "\psi'"] \arrow[ddr, bend right, "\psi"'] & & \\
& U_i \times_U U_j \arrow[r, "\pi_{2,ij}"'] \arrow[d, "\pi_{1,ij}"] & U_j \arrow[d, "\varphi_j"] \\
& U_i \arrow[r, "\varphi_i"'] & U
\end{tikzcd}$$

commute in $\mathscr{C}$. However, it then follows that $\psi = \pi_{1,ij} \circ \theta$ and $\psi' = \pi_{2,ij} \circ \theta$. This then allows us to calculate that

$$F(\psi)(x_i) = F(\pi_{1,ij} \circ \theta)(x_i) = F(\theta)\big(F(\pi_{1,ij})(x_i)\big) = F(\theta)\big(F(\pi_{2,ij})(x_j)\big) = F(\pi_{2,ij} \circ \theta)(x_j) = F(\psi')(x_j),$$

from whence it follows that $\alpha$ is well-defined. Thus, since $\alpha$ is a morphism and $F$ is a sheaf, because the diagram

$$FU \xrightarrow{e} \prod_{f \in (C)} F(\operatorname{Dom} f) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{\substack{f \in (C), g \in \operatorname{Mor}\mathscr{C} \\ \operatorname{Codom} g = \operatorname{Dom} f}} F(\operatorname{Dom} g)$$

is an equalizer, there exists a unique $x \in FU$ such that

$$F(\rho)(x) = \alpha(\rho)$$

for all $\rho \in (C)$. Moreover, since $\varphi_i \in (C)$ for all $i \in I$,

$$F(\varphi_i)(x) = x_i.$$

To see this is the unique such element of $FU$, assume that $y \in FU$ such that $F(\varphi_i)(y) = x_i$ for all $i \in I$. Then for any $\rho \in (C)$ we have that $\rho = \varphi_i \circ \psi$ for some $i \in I$ and for some $\psi$, so it follows that

$$F(\rho)(y) = F(\varphi_i \circ \psi)(y) = F(\psi)(x_i) = \alpha(\rho) = F(\rho)(x).$$

Thus, using that $e$ is monic (because it is an equalizer), we derive that $x = y$ and so the diagram

$$FU \xrightarrow{e} \prod_{i \in I} FU_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j \in I} F(U_i \times_U U_j)$$

is an equalizer.

$\Longleftarrow$ : Fix an arbitrary $S \in J(U)$ and find a cover $C := \{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ such that $C \subseteq S$. Find a morphism $\alpha : S \to F$ and write $\alpha(\rho) = y_\rho$ for all $\rho \in S$. Then, by the naturality of $\alpha$, we have that

$$F(\pi_{1,ij})(\alpha(\varphi_i)) = F(\pi_{1,ij})(y_{\varphi_i}) = F(\pi_{2,ij})(y_{\varphi_j}) = F(\pi_{2,ij})(\alpha(\varphi_j))$$

for all $i, j \in I$. Thus there exists a unique $x \in FU$ such that for all $i \in I$, the equation

$$F(\varphi_i)(x) = y_{\varphi_i}$$

holds.

We now need only to show that for all $\rho \in S$, $F(\rho)(x) = \alpha(\rho)$. So, fix a $\rho \in S$ and consider the pullbacks

$$\begin{tikzcd}
\operatorname{Dom}\rho \times_U U_i \arrow[r, "\pi_{2,i\rho}"] \arrow[d, "\pi_{1,i\rho}"'] & U_i \arrow[d, "\varphi_i"] \\
\operatorname{Dom}\rho \arrow[r, "\rho"'] & U
\end{tikzcd}$$

Because $C$ is a cover, by the pullback axiom for pretopologies, the set

$$C' := \{\pi_{1,i\rho} : \operatorname{Dom}\rho \times_U U_i \to \operatorname{Dom}\rho \mid i \in I\} \in \tau(\operatorname{Dom}\rho).$$

Thus, for all $\varphi_i \in C$ we have that

$$\begin{aligned}
F(\rho \circ \pi_{1,i\rho})(x) = F(\varphi_i \circ \pi_{2,i\rho})(x) = F(\pi_{2,i\rho})\big(F(\varphi_i)(x)\big) = F(\pi_{2,ij})(\alpha(\varphi_i)) \\
= \alpha(\varphi_i \circ \pi_{2,ij}) = \alpha(\rho \circ \pi_{1,i\rho}) = F(\pi_{1,i\rho})(\alpha(\rho)).
\end{aligned}$$

Now fix $i, j \in I$ and consider the pullback $P_{ij}$

$$\begin{tikzcd}
P_{ij} \arrow[r, "\widetilde{\pi}_{2,ij}"] \arrow[d, "\widetilde{\pi}_{1,ij}"'] & \operatorname{Dom}\rho \times_U U_j \arrow[d, "\pi_{1,j\rho}"] \\
\operatorname{Dom}\rho \times_U U_i \arrow[r, "\pi_{1,i\rho}"'] & \operatorname{Dom}\rho
\end{tikzcd}$$

in $\mathscr{C}$. This diagram allows us to compute that

$$F(\widetilde{\pi}_{1,ij})\big(F(\rho \circ \pi_{1,i\rho})(y)\big) = F(\widetilde{\pi}_{2,ij})\big(F(\rho \circ \pi_{1,j\rho})(y)\big).$$

which in turn, because $C' \in \tau(\operatorname{Dom}\rho)$, allows us to conclude that the diagram

$$\begin{tikzcd}
F(\operatorname{Dom}\rho) \arrow[r, "e_\rho"] & \prod_{i \in I} F(\operatorname{Dom}\rho \times_U U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j \in I} F(P_{ij})
\end{tikzcd}$$

is an equalizer. However, this implies that there is a unique factorization making the diagram

$$\begin{tikzcd}
F(\operatorname{Dom}\rho) \arrow[r, "e_\rho"] & \prod_{i \in I} F(\operatorname{Dom}\rho \times_U U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j \in I} F(P_{ij}) \\
FU \arrow[u, dashed, "\exists!"] \arrow[ur] & &
\end{tikzcd}$$

and so it must follow that $F(\rho)(y) = \alpha(\rho)$, which allows us to conclude that the diagram

$$\begin{tikzcd}
FU \arrow[r, "e"] & \prod_{f \in S} P(\operatorname{Dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{\substack{f \in S, g \in \operatorname{Mor}\mathscr{C} \\ \operatorname{Codom} g = \operatorname{Dom} f}} F(\operatorname{Dom} g)
\end{tikzcd}$$

is an equalizer as well. This proves that $F$ is a $J$-sheaf and concludes the proof of the theorem. $\square$

**Remark A.1.30.** Note that a consequence of Theorem A.1.28 above is the fact that different topologies can generate the same sheaves. This shows, in fact, that if $\tau$ and $\tau'$ are pretopologies that generate the same topology $J$, then they generate the same sheaves, *and* how you would go about checking it!

## A.2 The Category of Sheaves and Sheafification

Now that we have made the acquaintance of sheaves on a site $(\mathscr{C}, J)$, we would like to study some of the basic properties of the category of $J$-sheaves. This will be especially important as we begin to study stacks, so becoming familiar with sheaf categories now will be to our advantage later.

We will begin our examination of sheaves by first defining the category of $J$-sheaves and then showing that the category of sheaves has all **Set**-indexed limits. From there we will discuss the associated sheaf functor and use this to show that it allows us to conclude that the category of sheaves admits all colimits. Afterwards, we will discuss the Day Reflection Theorem (cf. Theorem A.2.17 below) and use this, together with the Associated Sheaf Functor (also known as the sheafification functor), to prove that the category of $J$-sheaves is Cartesian Closed.

**Definition A.2.1.** Let $(\mathscr{C}, J)$ be a site. A morphism between $J$-sheaves $F$ and $G$ is then a natural transformation $\alpha : F \to G$ in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, and the category of all $J$-sheaves, $\mathbf{Shv}(\mathscr{C}, J)$, is the full subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ comprised of $J$-sheaves.

**Proposition A.2.2.** *All* **Set***-indexed limits exist in* $\mathbf{Shv}(\mathscr{C}, J)$.

*Proof.* Begin by letting $I$ be a category for which $\mathrm{Ob}\, I$ is a set and consider a family $\{P_i \mid i \in I\}$ of $J$-sheaves. Now let $P$ be the limit

$$P := \varprojlim_{i \in I} P_i$$

of some diagram $D : I \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$; note that the limit $P$ exists because $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos and hence complete. Now consider that for all $i \in I$, for all $U \in \mathrm{Ob}\, \mathscr{C}$, and for all $S \in J(U)$, we have that the diagram

$$\begin{tikzcd} P(U) \arrow[r, "e"] & \prod_{f \in S} P_i(\mathrm{Dom}\, f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{\substack{f \in S, g \in \mathrm{Mor}\, \mathscr{C} \\ \mathrm{Dom}\, f = \mathrm{Codom}\, g}} P_i(\mathrm{Dom}\, g). \end{tikzcd}$$

is an equalizer. Taking the limit allows us to produce a diagram

$$\begin{tikzcd} \prod_{f \in S} P(\mathrm{Dom}\, f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{\substack{f \in S, g \in \mathrm{Mor}\, \mathscr{C} \\ \mathrm{Dom}\, f = \mathrm{Codom}\, g}} P(\mathrm{Dom}\, g). \end{tikzcd}$$

which commutes because it commutes for each $P_i(U)$; now, because $P$ is a limit, because an equalizer is a limit, and because limits commute with each other, it then follows that the above diagram is an equalizer diagram and so $P$ is a $J$-sheaf. $\square$

From here we will move on to discuss the Associated Sheaf/Sheafification Functor. To do this, we will go use Grothendieck's Double Plus construction, which, while being technical, has the benefit of making it immediate as to why the functor is flat. We need one definition before we will proceed.

**Definition A.2.3.** Let $(\mathscr{C}, J)$ be a site. We then say that a presheaf $P$ on $\mathscr{C}$ is a *separated presheaf* if for each object $U \in \mathscr{C}$ and for each sieve $S \in J(U)$, given a natural transformation $\alpha : S \to P$, there is *at most one* natural transformation $\beta : \mathscr{C}(-, U) \to P$ making the diagram

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d] & P \\ \mathscr{C}(-, U) \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$

commute.

We will proceed now by defining a $(-)^+$ functor on $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ which will construct a separated presheaf out of a presheaf; while this does not get us to sheaves directly, this plus functor will get us halfway there. Additionally, it is here that we will be able to prove the flatness of the sheafification functor as a simple corollary of the flatness of the plus functor.

**Definition A.2.4.** If $(\mathscr{C}, J)$ is a site and $P$ is a presheaf on $\mathscr{C}$, define a presheaf $P^+$ via the equation

$$P^+(U) := \varinjlim_{S \in J(U)} [\mathscr{C}^{\text{op}}, \mathbf{Set}](S, P),$$

where $J(U)$ is regarded as a poset with respect to inclusion of sieves. If $\varphi \in \mathscr{C}(V, U)$, define $P^+(\varphi)$ by taking the colimit over the induced function $\varphi^* : J(U) \to J(V)$ which sends a sieve $S \in J(U)$ to $\varphi^*(S) \ni J(V)$.

**Remark A.2.5.** The colimit defining $P^+$ is a filtered colimit: If $x = \{\alpha(f) \mid f \in S\}$ and $y = \{\beta(g) \mid g \in R\}$, for sieves $R, S \in \alpha$ and for natural transformations $\alpha : S \to P$ and $\beta : R \to P$, then $x \simeq y$ in $P^+(U)$ if and only if there exists a refinement $T \subseteq R \cap S$ such that $T \in J(U)$ and for all $\varphi \in T$, $\alpha(\varphi) = \beta(\varphi)$; that is to say, there is a sieve $T$ on $U$ such that $\alpha$ and $\beta$ are "eventually equal" on $T$. Moreover, if $\varphi$ is any natural transformation of presheaves $P \to Q$, then the post-composition morphism

$$[\mathscr{C}^{\text{op}}, \mathbf{Set}](\mathrm{id}_S, \varphi) : [\mathscr{C}^{\text{op}}, \mathbf{Set}](S, P) \to [\mathscr{C}^{\text{op}}, \mathbf{Set}](S, Q),$$

for all $S \in J(U)$ and for all $U \in \operatorname{Ob} \mathscr{C}$, then gives a morphism $\varphi^+ : P^+ \to Q^+$. A straightforward but tedious check shows that this assignment is compatible with the colimit operation, i.e., for all objects $U \in \operatorname{Ob} \mathscr{C}$, there is a corresponding morphism

$$\varinjlim_{S \in J(U)} \mathscr{C}(\mathrm{id}_S, \varphi) = \varphi_U^+ : P^+(U) \to Q^+(U),$$

which is natural in $U$. Taking these observations together shows that the $(-)^+$ assignment determines a functor, yielding the content of the proposition below.

**Proposition A.2.6.** *The assignment $(-)^+ : [\mathscr{C}^{\text{op}}, \mathbf{Set}] \to [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ is a functor.*

In order to proceed with studying the plus functor, we would like to show that there is a natural transformation $\eta : \mathrm{id}_{[\mathscr{C}^{\text{op}}, \mathbf{Set}]} \to (-)^+$ which plays well with $J$-sheaves. To show how to define $\eta$, let $P \in \operatorname{Ob} [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ be a presheaf and let $U \in \operatorname{Ob} \mathscr{C}$. Then define $\eta_U : PU \to P^+U$ by

$$\eta_U(x) = \{P(f)(x) \mid f \in \operatorname{Codom} U\}.$$

Note that this represents an equivalence class in $P^+(U)$, and if $\varphi \in \mathscr{C}(V, U)$ for all $x \in PU$,

$$P^+(\varphi)(\eta_U(x)) = P^+(\varphi)(\{P(f)(x) \mid f \in \operatorname{Codom} U\}).$$

However, since $P^+(\varphi)$ is induced by sending sieves $S$ in $J(U)$ to $\varphi^*(S) \in J(V)$, we find that $P(\varphi)^+$ acts on the sieve $\operatorname{Codom} U$ by

$$\varphi^*(\operatorname{Codom} U) = \{g \in \operatorname{Mor} \mathscr{C} \mid \varphi \circ g \in \operatorname{Codom} U\} = \{g \in \operatorname{Mor} \mathscr{C} \mid \operatorname{Codom} g = V\} = \operatorname{Codom} V$$

so we calculate that

$$\begin{aligned} P^+(\varphi)(\eta_U(x)) &= P^+(\varphi)(\{P(f)(x) \mid f \in \operatorname{Codom} U\} = \{P(\varphi \circ g)(x) \mid g \in \operatorname{Codom} V\} \\ &= \{P(g)\big(P(\varphi)(x)\big) \mid g \in \operatorname{Codom} V\} = \eta_V(P(\varphi)(x)) \end{aligned}$$

which shows that the diagram

$$\begin{tikzcd} PU \arrow[r, "\eta_U"] \arrow[d, "P(\varphi)"'] & P^+U \arrow[d, "P^+\varphi"] \\ PV \arrow[r, "\eta_V"'] & P^+V \end{tikzcd}$$

commutes. This proof yields the lemma below that $\eta$ is a natural transformation, which in turn helps us get our study of the Double Plus Functor running.

**Lemma A.2.7.** *The family of functions $\eta_U : PU \to P^+U$, for all $U \in \operatorname{Ob}\mathscr{C}$, defined by*

$$\eta_U(x) := \{P(f)(x) \mid f \in \operatorname{Codom} U\}$$

*determines a natural transformation $\eta : \mathrm{id}_{[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]} \to (-)^+$.*

We will now use a structural lemma about sheaves and how they behave with respect to the natural transformation $\eta$. It turns out that we can determine whether or not a presheaf is both separated *or* a $J$-sheaf by studying whether or not the map $\eta : P \to P^+$ is monic or an isomorphism.

**Lemma A.2.8.** *Let $(\mathscr{C}, J)$ be a site and let $P$ be a presheaf on $\mathscr{C}$. Then:*

1. *$P$ is separated if and only if $\eta : P \to P^+$ is monic;*

2. *$P$ is a $J$-sheaf if and only if $\eta : P \to P^+$ is an isomorphism.*

*Proof.* (1): Begin by observing that if $x, y \in P(U)$, for some $U \in \operatorname{Ob}\mathscr{C}$, then $\eta(x) = \eta(y)$ implies that there is a covering sieve $S \in J(U)$ such that for all $f \in S$, $P(f)(x) = P(f)(y)$. We can then conclude that $x = y$ if and only if there is at most one natural transformation $\beta : \mathscr{C}(-, U) \to P$ extending the diagram below along the dotted arm:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & P \\ \mathscr{C}(-,U) \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$

Finally, the claim for (2) follows by using the surjecivity of $\eta$ to give the existence of a lift, while the fact that it is monic implies that there is at most one such lift. ☐

**Lemma A.2.9.** *The functor $(-)^+$ is flat.*

*Proof.* Since $(-)^+$ is determined pointwise by a filtered colimit in **Set** and filtered colimits in set commute with finite limits, it follows immediately that $(-)^+$ is flat because limits in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ are computed pointwise in **Set**. ☐

**Corollary A.2.10.** *The functor $(-)^{++} := (-)^+ \circ (-)^+$ is flat.*

**Lemma A.2.11.** *If $F$ is a $J$-sheaf and if $P \in \operatorname{Ob}[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, then if $\varphi \in [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](P, F)$ there exists a unique $\varphi^\sharp \in [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](P^+, F)$ making the diagram*

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[dr, "\varphi"'] & P^+ \arrow[d, dashed, "\exists ! \varphi^\sharp"] \\ & F \end{tikzcd}$$

*commute.*

*Proof.* We will begin by exploring how $P^+$ acts on its elements in order to determine necessary conditions on the map $\varphi^\sharp$. Begin by noting that if $x \in P^+U$, then we can find a collection of elements $\{x_f \mid f \in S\}$ induced by elements of a natural transformation from some covering sieve $S \in J(U)$ to $P$ which represent $x$. i.e.,

$$x = [\{x_f \mid f \in S\}].$$

Now let $\rho : V \to U$ be a morphism in $S$ and recall that

$$\eta_V(x_\rho) = \{P(k)(x_\rho) \mid k \in \operatorname{Codom} V\}$$

and that $P^+(\rho)$ acts on $x$ by

$$P^+(\rho)(x) = P^+(\rho)(\{x_f \mid f \in S\}) = \{x_{\rho\circ g} \mid g \in \rho^*(S)\}.$$

Moreover, by the naturality of $\eta$ we have that

$$\eta_V(x_\rho) = P^+(\rho)(\{x_f \mid f \in S\}).$$

Thus, if $\varphi^\sharp$ were to exist, it would be uniquely determined by the fact that $\varphi^\sharp$ must map the equivalence class of $\{x_f \mid f \in S\}$ to the unique $y \in FU$ satisfying the equations

$$F(\rho)(y) = F(\rho)\Big(\varphi^\sharp\big(\{x_f \mid f \in S\}\big)\Big) = \varphi^\sharp\left(\Big[P^+(\rho)(\{x_f \mid f \in S\})\Big]\right) = \varphi^\sharp(\eta_V(x_\rho)) = \varphi(x_\rho)$$

for all $\rho \in S$. This implies that $y$ is a unique lift of the transformation $S \to F$ induced by the factorization

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & P \arrow[d, "\varphi"] \\ & F \end{tikzcd}$$

of natural transformations; however, this unique lift exists because $F$ is a $J$-sheaf and because the family $\{\varphi(x_\rho) \mid \rho \in S\}$ is induced from the factorization above. $\square$

Unfortunately, this lemma does not prove that $(-)^+$ is a left adjoint because it *only* provides a factorization for $J$-sheaves, not for all presheaves. However, if we knew that $(-)^+$ was a sheaf, we'd be done because this says that the plus functor would be left adjoint to the inclusion $\iota : \mathbf{Shv}(\mathscr{C}, J) \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$. However, all is not lost! We will show below that $P^+$ is separated, and then show that if $P$ is separated $P^{++}$ is an isomorphism. This, together with the fact that $\eta|_{\mathbf{Shv}(\mathscr{C},J)} \cong \mathrm{id}_{\mathbf{Shv}(\mathscr{C},J)}$, will allow us to conclude that the Double Plus Functor is left adjoint to the inclusion of $\mathbf{Shv}(\mathscr{C}, J)$ into the presheaf topos $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.

**Lemma A.2.12.** *If $P$ is a presheaf on a site $(\mathscr{C}, J)$, then $P^+$ is separated.*

*Proof.* We begin by observing that in order to show that $P^+$ is separated, it suffices to show that if $x, y \in P^+U$ for some $U \in \operatorname{Ob}\mathscr{C}$ such that there exists a sieve $Q \in J(U)$ with the property that $P^+(h)(x) = P^+(h)(y)$ for all $h \in Q$, then $x = y$. As such, find sieves $S, R \in J(U)$ and natural transformations, say $\alpha : S \to P^+$, $\beta : R \to P^+$, which represent $x$ and $y$ in $P^+U$, respectively. Then we have the equalities, if we set $x_f := \alpha_{\operatorname{Dom} f}(f)$ and $y_g := \beta_{\operatorname{Dom} g}(g)$ for all $f \in S, g \in R$, then

$$x = [\{x_f \mid f \in S\}]$$

and

$$y = [\{y_g \mid g \in R\}]\,.$$

Let $\rho \in Q$ be given and write $\rho : V \to U$. Then since we have that $P^+(\rho)(x) = P^+(\rho)(y)$, there exists a cover $T_\rho \subseteq \rho^*(S) \cap \rho^*(R)$ with $T_\rho \subseteq J(V)$ such that $x_{\rho\circ f'} = y_{\rho\circ f'}$ for all $f' \in T_\rho$. However, by the transitivity axiom, the family

$$T := \{\rho \circ f \mid \rho \in Q, f \in T_\rho\}$$

is a $J$-covering sieve of $U$ with $T \subseteq R \cap S$. Moreover, for all $\varphi \in T$ we have that $P^+(\varphi)(x)0 = P^+(\varphi)(y)$ so it follows that $x = y$ in $P^+U$. This shows that $P^+$ is separated. $\square$

**Lemma A.2.13.** *If $P$ is a separated presheaf on a site $(\mathscr{C}, J)$, then $P^+$ is a $J$-sheaf.*

*Proof.* Let $\alpha : S \to P^+$ be a natural transformation for $S$ a $J$-covering sieve on $U$. For each $f \in S$, define

$$x_f := \alpha_{\operatorname{Dom} f}(f) \in P^+ \operatorname{Dom} f$$

and note that since $\alpha$ is a natural transformation, the family $(x_f)_{f\in S}$ satisfies the equation

$$\begin{aligned} q(x_f)_{f\in S} &= \big(P^+(g)(x_f)\big)_{f\in S, g\in \operatorname{Mor}\mathscr{C}, \operatorname{Codom} g = \operatorname{Dom} f} = \big((P^+(g)(\alpha_{\operatorname{Dom} f}(f))\big)_{f\in S, g\in \operatorname{Mor}\mathscr{C}, \operatorname{Codom} g = \operatorname{Dom} f} \\ &= \big(\alpha_{\operatorname{Dom} g}(f\circ g)\big)_{f\in S, g\in \operatorname{Mor}\mathscr{C}, \operatorname{Codom} g = \operatorname{Dom} f} = \big(x_{f\circ g}\big)_{f\in S, \operatorname{Mor}\mathscr{C}, \operatorname{Codom} g = \operatorname{Dom} f} \\ &= p(x_f)_{f\in S} \end{aligned}$$

where $p$ and $q$ are defined as in the sheaf diagram

$$\prod_{f\in S} P^+(\operatorname{Dom} f) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{\substack{f\in S, g\in \operatorname{Mor}\mathscr{C} \\ \operatorname{Codom} g = \operatorname{Dom} f}} P^+ \operatorname{Dom} g$$

We now need to show how to construct a single element $y \in P^+U$ that has $P(f)(y) = x_f$ for all $f \in S$. Before doing this, however, we need to consider how to represent the naturality condition on the $(x_f)_{f\in S}$ in the sets $P^+ \operatorname{Dom} g$. To do this, note that because $x_f \in P^+ \operatorname{Dom} f$ there exists a $J$-covering sieve $S_f \in J(\operatorname{Dom} f)$ such that $x_f$ is represented by the set

$$x_f = [\{(x_f)_g \mid g \in S_f\}] = [\{(\alpha_{\operatorname{Dom} f}(f))_g \mid g \in S_f\}]$$

where we write $(x_f)_g := (\alpha_{\operatorname{Dom} f}(f))_g$. Fix now some $\rho : \operatorname{Dom}\rho \to \operatorname{Dom} f$ in $\mathscr{C}$. Then because

$$P^+(\rho)(\alpha(f)) = \alpha(f\circ\rho)$$

it follows that in $P^+(\operatorname{Dom}\rho)$ there is a similarity between the sets

$$\{(\alpha(f))_{\rho\circ g} \mid g \in \rho^*(S_f)\} \simeq \{(\alpha(f\circ\rho))_g \mid g \in S_{f\circ\rho}\},$$

i.e., the sets represent the same equivalence class in the colimit. So there exists a sieve $T_{f,\rho} \subseteq \rho^*(S_f) \cap S_{f\circ\rho}$ such that for all $k \in T_{f,\rho}$ we have that

$$(\alpha(f))_{\rho\circ k} = (\alpha(f\circ\rho))_k.$$

We will use the above representations to construct an element $y$ in $P^+U$ equalizing $p$ and $q$. Define the sieve $R$ by the equation

$$R := \{f \circ g \mid f \in S, g \in S_f\}$$

and note that since $S \in J(U)$ and, for all $f \in S$, $S_f \in J(\operatorname{Dom} f)$, it follows from the transitivity axiom for topologies that $R \in J(U)$ as well. Now define $Y \in P^+U$ via

$$y_{f\circ g} := (\alpha(f))_g.$$

To see that $y$ is a well-defined element of $P^+U$, we must show that it does not depend on the choice of factorization for $f \circ g$. To this end, assume that there exists $h \in S$ and $k \in S_h$ such that $f \circ g = h \circ k$. Then we derive that if $\rho \in T_{f,g} \cap T_{h,k}$, we have that

$$P^+(\rho)\big((\alpha(f))_g\big) = ((\alpha(f))_{g\circ\rho} = (\alpha(f\circ g))_\rho = (\alpha(h\circ k))_\rho = (\alpha(h))_{k\circ\rho} = P^+(\rho)\big((\alpha(h))_k\big).$$

Because $T_{f,g} \cap T_{h,k}$ is a $J$-covering sieve and $P$ is a separated sheaf, it follows then $(\alpha(f))_g = (\alpha(h))_k$. Defining the natural transformation $\beta : R \to P$ via the equation $\beta(\rho) := y_\rho$ for all $\rho \in R$ then yields that the set

$$y := [\{y_\rho \mid \rho \in R] \in P^+U.$$

Then we compute that for all $f \in S$,

$$P^+(f)(y) = \alpha(f)$$

by construction, so

$$e(y) = \langle P^+(f)\rangle_{f\in S}(y) = (P^+(f)(y))_{f\in S} = (\alpha(f))_{f\in S}$$

and by design $p \circ e = q \circ e$. Moreover, since $P$ is separated, it follows that $y$ is the unique such element that maps through $e$ to $\alpha$; it then follows that

$$\begin{tikzcd} P^+U \arrow[r, "e"] & \prod\limits_{f\in S} P(\operatorname{Dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{\substack{f\in S, g\in \operatorname{Mor}\mathscr{C} \\ \operatorname{Codom} g = \operatorname{Dom} f}} P^+(\operatorname{Dom} g) \end{tikzcd}$$

is an equalizer diagram, which proves that $P^+$ is a $J$-sheaf. $\square$

**Corollary A.2.14.** *The functor $(-)^{++} : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ takes values in $\mathbf{Shv}(\mathscr{C}, J)$, i.e., $(-)^{++}$ factors as:*

$$\begin{tikzcd} {[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]} \arrow[rr, "(-)^{++}"] \arrow[dr, "(-)^{++}"'] & & {[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]} \\ & \mathbf{Shv}(\mathscr{C}, J) \arrow[ur, "\iota"'] & \end{tikzcd}$$

**Corollary A.2.15.** *The functor $(-)^{++} : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Shv}(\mathscr{C}, J)$ is flat and left adjoint to the inclusion $\iota : \mathbf{Shv}(\mathscr{C}, J) \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.*

*Proof.* The flatness of $(-)^{++}$ follows from the fact that $(-)^{+}$ is flat, while the fact that $(-)^{++}$ is left adjoint to $\iota$ follows from the fact that there is a universal property for $\eta' : \mathrm{id}_{[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]} \to \iota \circ (-)^{++}$ given by the equation

$$\eta'_P := \eta_{P^+} \circ \eta_P$$

and then using the induced universal property for $\eta'$. $\square$

**Corollary A.2.16.** *The counit of adjuction $\varepsilon : (-)^{++} \circ \iota \to \mathrm{id}_{\mathbf{Shv}(\mathscr{C}, J)}$ is an isomorphism.*

*Proof.* This is immediate from the fact that $\iota$ is a fully faithful right adjoint. $\square$

Finally, to complete this section and our introductory study of $\mathbf{Shv}(\mathscr{C}, J)$, we need to discuss the Day Reflection Theorem. It is an intuitive theorem on symmetric monoidal categories, but gives a nice way of showing that $\mathbf{Shv}(\mathscr{C}, J)$ is Cartesian Closed by relying only on the Cartesian Monoidal structure of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ (which is guaranteed by the fact that $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos).

**Theorem A.2.17** (Day Reflection Theorem; cf. [17])**.** *Let $R : \mathscr{C} \to \mathscr{D}$ be a fully faithful right adjoint with left adjoint $L : \mathscr{D} \to \mathscr{C}$. Then if $(\mathscr{D}, \otimes, I)$ is a closed symmetric monoidal structure on $\mathscr{D}$ and let*

$$(\eta, \varepsilon) : (-) \otimes Y \dashv [Y, -] : \mathscr{D} \to \mathscr{D}$$

*describe the tensor/internal hom adjunction in $\mathscr{D}$. Then, for all $U \in \operatorname{Ob}\mathscr{C}$ and all $V \in \operatorname{Ob}\mathscr{D}$, if any of the following natural transformations are invertible, then they all are:*

1. $\eta_{[V,RU]} : [V, RU] \to (R \circ L)[V, RU]$;
2. $[\eta_V, \mathrm{id}_{RU}] : [(R \circ L)V, RU] \to [V, RU]$;
3. $L(\eta_V \otimes \mathrm{id}_{V'}) : L(V \times V') \to L((R \circ L)V \otimes V')$;
4. $L(\eta_V \otimes \eta_{V'}) : L(V \otimes V') \to L((R \circ L)V \otimes (R \circ L)V')$.

*In particular, if $\mathscr{D}$ is Cartesian Closed and if $L$ preserves products, then $\mathscr{C}$ is an exponential ideal of $\mathscr{D}$.*

**Corollary A.2.18.** *If $\mathscr{C}$ is a reflexive subcategory of $\mathscr{D}$ with reflector $L : \mathscr{D} \to \mathscr{C}$, if $\mathscr{D}$ is Cartesian Closed, and if $L$ preserves products, then $\mathscr{C}$ is Cartesian Closed.*

**Corollary A.2.19.** *The category $\mathbf{Shv}(\mathscr{C}, J)$ is Cartesian Closed.*

*Proof.* Since $\mathbf{Shv}(\mathscr{C}, J)$ is a reflective subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ with flat reflector $(-)^{++}$, and since $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a Cartesian closed, $\mathbf{Shv}(\mathscr{C}, J)$ is Cartesian closed. □

Finally, we conclude this appendix by showing that for any site $(\mathscr{C}, J)$ the category of sheaves $\mathbf{Shv}(\mathscr{C}, J)$ is a topos. That is, we must show that $\mathbf{Shv}(\mathscr{C}, J)$ is finitely complete, Cartesian closed, and has a subobject classifier. Because we already know that $\mathbf{Shv}(\mathscr{C}, J)$ is complete (cf. Proposition A.2.2) and Cartesian closed (cf. Corollary A.2.19), we must show that $\mathbf{Shv}(\mathscr{C}, J)$ has a subobject classifier. For this we will first define subjobject classifiers before proving their existence.

**Definition A.2.20.** Let $\mathscr{C}$ be a category with a terminal object $\top$. A subobject classifier, if it exists, is an object $\Omega$ equipped with a morphism $\mathsf{true} : \top \to \Omega$ such that if $m : A \to B$ is any monomorphism in $\mathscr{C}$ there is a unique map $\chi_m : B \to \Omega$ called the classifying map of $m$ for which the diagram

$$\begin{tikzcd} A \arrow[r, dashed, "\exists!"] \arrow[d, "m"'] & \top \arrow[d, "\mathsf{true}"] \\ B \arrow[r, "\exists! \chi_m"'] & \Omega \end{tikzcd}$$

is a pullback in $\mathscr{C}$.

**Remark A.2.21.** It is helpful to think of the object $\Omega$ as an object of generalized truth values in $\mathscr{C}$ and the morphism $\mathsf{true} : \top \to \Omega$ as determining when a proposition (or subobject) is true in $\mathscr{C}$. The classifying map $\chi_m$ of a monic $m$ can in this way be thought of as a $\mathscr{C}$-valued indicator function which states that the piece of $B$ that the morphism $m$ cuts out can be classified internally to the language of the category $\mathscr{C}$.

**Example A.2.22.** In $\mathscr{C} = \mathbf{Set}$ the subobject classifier is $\Omega = \{0, 1\}$, the truth map $\mathsf{true} : \{*\} \to \Omega$ is given by $\mathsf{true}(*) := 1$, and the classifying map $\chi_m : B \to \Omega$ of a monic $m : A \to B$ is

$$\chi_m(b) := \begin{cases} 1 & \text{if} \, \exists \, a \in A. \, b = m(a); \\ 0 & \text{else.} \end{cases}$$

We omit the proof of the next lemma and proposition, as they are straightforward but tedious checks.

**Lemma A.2.23.** *Let $\mathscr{C}$ be a category. Then $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ has a subobject classifier $\Omega$ where $\Omega : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is defined by*

$$\Omega(U) := \mathrm{Sub}(\mathbf{y}(U)),$$

*i.e., $\Omega$ sends an object $U$ to its set of sieves on $U$. The morphism $\mathsf{true} : \top \to \Omega$ is defined by $\mathsf{true}_U(*) = \mathbf{y}(U)$ for all $U \in \mathscr{C}_0$.*

**Proposition A.2.24.** *If $(\mathscr{C}, J)$ is a site then $\mathbf{Shv}(\mathscr{C}, J)$ has a subobject classifier $\Omega$ given on objects by*

$$\Omega(U) := J(U)$$

*and truth map $\mathsf{true} : \top \to \Omega$ given by*

$$\mathsf{true}_U(*) = \mathbf{y}(U)$$

*for all $U \in \mathscr{C}_0$.*

**Theorem A.2.25.** *For any site $(\mathscr{C}, J)$ the category $\mathbf{Shv}(\mathscr{C}, J)$ is a topos.*

*Proof.* That $\mathbf{Shv}(\mathscr{C}, J)$ is complete follows from Proposition A.2.2, that it is Cartesian closed follows from Corollary A.2.19, and that $\mathbf{Shv}(\mathscr{C}, J)$ has a subobject classifier follows from Proposition A.2.24. $\square$

**Definition A.2.26.** A category $\mathcal{E}$ is a Grothendieck topos if there exists a site $(\mathscr{C}, J)$ and an equivalence of categories $\mathcal{E} \simeq \mathbf{Shv}(\mathscr{C}, J)$.

# Appendix B

# Localizations of Categories

In this chapter we will introduce and discuss the theory of localizations of categories. This should be seen as simultaneously a generalization of the conditions that allow us to localize and invert classes of elements in a noncommutative unital ring (cf. [43], Chapters 9 and 10, for details), as well as asking if we can formally go backwards along the directions of certain morphisms in a category. We will likely not go through the Quillen model theory around such theories[1], but whenever possible I will point to appropriate references in this direction. This is because we can see this "can I go back?" question as very much a *homotopical* one, as it effectively can be seen as a way of localizing maps along certain deformations.

At the moment[2], this chapter serves two purposes: First, we want to be as self-contained as possible, and having the theory of (categorical) localization is very useful for cohomology and things of that nature; Second, this is a place to consider various localization generalities that we will largely ignore later on[3], at least when no real difficulties arise.

## B.1 Localizations of Categories: The Basics

We begin the serious mathematical study with a review of localization of a ring with identity. When we localize a ring $R$ at a set $S$, we ask to form a ring $S^{-1}R$ which is "minimally enriched" (in the sense that it is universally defined) in such a way that every element in $S$ has been turned into a multiplicative unit in $S^{-1}R$, together with a map $R \to S^{-1}R$ that we think of as plopping the ring into a constructed ring of fractions. In particular, this asks that if $T$ is a unital ring and if $\varphi : R \to T$ is a ring homomorphisms and if $\lambda_S : R \to S^{-1}R$ is the canonical map[4], then there must be a unique $\psi : S^{-1}R \to T$ making the diagram

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "\lambda_S"'] & & T \\ & S^{-1}R \arrow[ur, dashed, "\exists!\psi"'] & \end{tikzcd}$$

commute in **Ring**. However, there are myriad difficulties (which we will see later) involving the description of a localization of a category. For instance, if we view a unital ring $R$ as a category $\mathscr{R}$ with one object, then a localization of $R$ at a set $S$ of elements corresponds to a localization of $\mathscr{R}$ at a set $S$ of morphisms. In this case we do *not* generally have a description of elements of $S^{-1}R$ (and hence the morphisms of $S^{-1}\mathscr{R}$) of the

[1] Although we should. I will try to go back in time and fix this if at all possible

[2] I should see if later on this is still the moment. In other words, are we living in the moment or are we living in the past?

[3] I will try to avoid set-theoretic issues as much as possible, but will try to point out where we should be careful about such things and where I am *not* being careful about such things. This is a mathematical "hold my beer" moment.

[4] For whatever that happens to mean. It's not like the word "canonical" is overused in math or anything. In fact, it's a normal and completely regular word to use!

form $s^{-1}r$; instead generic elements of $S^{-1}R$ take the form

$$\sum_{i=1}^{n}\prod_{j_i=1}^{m_i} s_{j_i}^{-1}r_{j_i},$$

i.e., elements are sums of words and cannot be generically simplified into a two-letter format. This means categorically that we should expect our morphisms of $S^{-1}\mathscr{R}$ to look like zig-zags

$$\begin{tikzcd}
* \arrow[dr, "s_1^{-1}"'] & & * \arrow[dr, dash] & & * \arrow[dr, "s_n^{-1}"'] & & * \\
& * \arrow[ur, "r_1"'] & & \cdots \arrow[ur] & & * \arrow[ur, "r_n"'] &
\end{tikzcd}$$

where $*$ is the object of $\mathscr{R}$.

However, despite this difficulty, it is worth having localizations of categories. It *can* be the case that morphisms have a simple description, and we will work to prove when this is the case, both in the case in which we localize morphisms at an equivalence relation and in the case in which we localize at a class of homotopies.

Let us now move to define what we mean by a localization of categories. This definition will be used throughout the rest of this article.

**Definition B.1.1.** Let $\mathscr{C}$ be a category and let $W$ be a class[5] of morphisms. The *localization of $\mathscr{C}$ at $W$* is a category $W^{-1}\mathscr{C}$ with the following properties:

1. The objects of $W^{-1}\mathscr{C}$ are the objects of $\mathscr{C}$, and there is a functor $\lambda_W : \mathscr{C} \to W^{-1}\mathscr{C}$ which is the identity on objects;

2. For all $s \in W$, $\lambda_W(s)$ is an isomorphism of $W^{-1}\mathscr{C}$;

3. For any category $\mathscr{D}$ together with a functor $F : \mathscr{C} \to \mathscr{D}$ for which $F(s)$ is an isomorphism, there exists a unique functor $\widetilde{F} : W^{-1}\mathscr{C} \to \mathscr{D}$ making the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "F"] \arrow[dr, "\lambda_W"'] & & \mathscr{D} \\
& W^{-1}\mathscr{C} \arrow[ur, dashed, "\exists!\widetilde{F}"'] &
\end{tikzcd}$$

commute in **Cat**.

We will now move to prove that localizations exist[6] for any class of morphisms $W$ and for any category $\mathscr{C}$. After this we will move to discuss quotients of categories[7] and then towards what is called a calculus of (left/right) fractions on a category. Both these classes of localizations admit particularly nice formal descriptions, so it is these on which we will focus; in fact, we will use both these constructions to show that the derived category can be written as a quotient followed by a fractional localization, as opposed to a one-step localization, and show that this gives it a much cleaner description.

[5] Here is a place where I am ignoring set-theoretic issues. However, one should think of $W$ as a class of weak equivalences in a model category.

[6] At least when we consider *sets* of morphisms. The set-theory minded reader should look away here, because I will take the convention that we have enriched universes as much as possible and not really pay much attention to foundational issues. It is very likely that I will say this works for small categories and just say later on that this works for a category of chain complexes, or even that we can define equivalence relations on proper classes without any worry.

[7] Note that a categorical quotient can mean different things than "quotient category," especially to algebraic geometers.

The following lemma is well-known, as it appears as an example in Section IV.2 of [53], and is an important ingredient our proof of the existence of localizations. It provides the left adjoint to the forgetful functor $U : \mathbf{Cat} \to \mathbf{Graph}$, $P : \mathbf{Graph} \to \mathbf{Cat}$;[8] the category $P(G)$ is called, dually, the *path category* and the *free category on* $G$ depending on one's tastes. The path category $P(G)$ of a graph $G$ has the following description:

- Objects: Vertices $v$ of the graph $G$;
- Morphisms: Paths (ordered lists) $[e_1, \cdots, e_n]$ of composable edges $e_i$ of $G$;
- Composition: Concatenation of paths, i.e., $[e_1, \cdots, e_n] \circ [f_1, \cdots, f_m] = [f_1, \cdots, f_m, e_1, \cdots, e_n]$[9];
- Identities: The empty path $[]_v$ starting and ending at the vertex $v$ is the identity at $v$.

**Lemma B.1.2** (Section IV.2 of [53])**.** *The forgetful functor $U : \mathbf{Cat} \to \mathbf{Graph}$ has a left adjoint $P : \mathbf{Graph} \to \mathbf{Cat}$ whose unit of adjunction $\eta_G : G \to U(P(G))$ is given by sending a vertex $v \in V(G)$ to itself and sends an edge $e \in E(G)$ to the singleton path $[e]$.*

As an immediate consequence to this lemma, we find that for any category $\mathscr{C}$ there is a homomorphism of graphs
$$U(\mathscr{C}) \xrightarrow{\eta_{U(\mathscr{C})}} U(P(U(\mathscr{C})))$$
which is, unfortunately, not the image of a functor $\mathscr{C} \to P(U(\mathscr{C}))$. However, we can make this into a functor by introducing the language of quotient categories, and then further make candidate categories for localizations $W^{-1}\mathscr{C}$ at classes $W$ by instead of simply considering all possible paths in $\mathscr{C}$, formally inverting the morphisms in $W$ and then allowing our paths to move along the formal inverses of $W$.

The lemma we move on to prove provides the technical grounding for us to define what it means to have a quotient category. In particular, we show here that with this lemma we can define a quotient category by modding out morphisms along an equivalence relation, exactly as one would hope.

**Lemma B.1.3.** *Assume that $\mathscr{C}$ is a category such that for all $X, Y \in \mathscr{C}_0$ there exists an equivalence relation $\simeq_{X,Y}$ on $\mathscr{C}(X,Y)$ with the property that whenever $f \simeq_{X,Y} g$, for all $h \in \mathscr{C}(W,X)$ and for all $k \in \mathscr{C}(Y,Z)$, we have that*
$$f \circ h \simeq_{W,Y} g \circ h$$
*and*
$$k \circ f \simeq_{X,z} k \circ g.$$
*Then there exists a category $\mathscr{D}$ with $\mathscr{D}_0 = \mathscr{C}_0$ and $\mathscr{D}(X,Y) = \mathscr{C}(X,Y)_{/\simeq_{X,Y}}$. Moreover, there is always a full and essentially surjective functor $q : \mathscr{C} \to \mathscr{D}$.*

*Proof.* Define the category $\mathscr{D}$ by the following assignment:

- Objects: $X \in \mathscr{D}_0$ if and only if $X \in \mathscr{C}_0$;
- Morphisms: For all $X, Y \in \mathscr{D}_0$, we define $\mathscr{D}(X,Y) := \mathscr{C}(X,Y)_{/\simeq_{X,Y}}$;
- Composition: For any $[f] \in \mathscr{D}(X,Y)$ and any $[g] \in \mathscr{D}(Y,Z)$, define
$$[g] \circ [f] := [g \circ f];$$

[8] Thanks to Kristaps Balodis for making me realize this oversight and abuse of universe on my part. The category **Graph** considered here is as "locally small" as the category **Cat** above, i.e., if $\mathscr{C}$ is a locally small category and $\mathscr{C}$ is an object in **Cat**, then we allow graphs $G$ to be "locally small" in the following sense: if $G = (V, E)$ is a pair of vertices and edges, we allow $V$ and $E$ to both be proper classes with the condition that for any two vertices $v, v' \in V$, the class $E(v, v') = \{e \in E \mid s(e) = v, t(e) = v'\}$ is actually a set.

[9] The silliness of this definition comes from the fact that we are writing our morphisms in application order, while paths are listed in diagrammatic order. A smarter man would have rectified this notationally, but I have instead decided to keep the notation in conflict because sometimes conflict in life is necessary. Sometimes this conflict is also self-imposed.

- Identities: For any $X \in \mathscr{D}_0$, $\mathrm{id}_X = [\mathrm{id}_X]$.

We now need only verify that this does indeed give a category. It follows immediately from the definition that if composition is well-defined, it is immediately associative while our proposed identities are indeed identities. As such, it suffices to prove that composition is well-defined.

Suppose that $f, g \in \mathscr{C}(X,Y)$ with $f \simeq_{X,Y} g$ and that $h, k \in \mathscr{C}(Y,Z)$ with $h \simeq_{Y,Z} k$. It then follows from the naturality of the equivalence relations with respect to composition that

$$k \circ f \simeq_{X,Z} h \circ f \simeq_{X,Z} h \circ g$$

and

$$h \circ f \simeq_{X,Z} k \circ f \simeq_{X,Z} k \circ g.$$

Thus we derive that

$$[h] \circ [f] = [h \circ f] = [k \circ g] = [k] \circ [g],$$

proving that composition is well-defined. This shows that the category $\mathscr{D}$ exists. Finally, the full functor $q : \mathscr{C} \to \mathscr{D}$ is defined on objects by

$$q(X) = X$$

for all objects $X \in \mathscr{C}_0$ and by

$$q(f) = [f]$$

for all morphisms $f \in \mathscr{C}_1$. This is evidently functorial by the definition of $\mathscr{D}$, and it is full because the map

$$q_* : \mathscr{C}(X,Y) \to \mathscr{D}(qX, qY)$$

is equivalent to the quotient map

$$\widehat{q} : \mathscr{C}(X,Y) \to \mathscr{C}(X,Y)_{/\simeq_{X,Y}},$$

which is surjective by construction. Finally, essential surjectivity is clear because every object of $\mathscr{D}$ is equal to $q(X)$. $\square$

**Definition B.1.4.** A *quotient category (of a category $\mathscr{C}$)* is a category isomorphic to the category $\mathscr{D}$ constructed in Lemma B.1.3, provided such a category exists.

Before we see examples of quotient categories, we will first see their universal property, as it will be useful in our development of localization below.

**Lemma B.1.5.** *If $\mathscr{C}$ is a category admitting a quotient category $q : \mathscr{C} \to \overline{\mathscr{C}}$ and if $F : \mathscr{C} \to \mathscr{D}$ is a functor such that $F\varphi = F\psi$, then there exists a unique functor $\widetilde{F} : \overline{\mathscr{C}} \to \mathscr{D}$ making the diagram*

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathscr{D} \\ & \overline{\mathscr{C}} \arrow[ur, dashed, "\exists!\widetilde{F}"'] & \end{tikzcd}$$

*commute.*

*Sketch.* This lemma is immediate, so we only sketch its proof. Because $\mathscr{C}_0 = \overline{\mathscr{C}}_0$, $\widetilde{F}X = FX$ for all objects $X$ of $\mathscr{C}$. Moreover, because the functor $F$ identifies morphisms which are equivalent to each other through the relations $\simeq$, defining $\widetilde{F}[\varphi] = F\varphi$ is well-defined, determines a functor by construction, and gives the commuting diagram. Finally, the uniqueness of $\widetilde{F}$ follows from the fact that the assignment of $\widetilde{F}$ on morphisms is necessarily unique by the universal property of quotient sets. $\square$

**Example B.1.6.** If $\mathscr{C}$ is any category, then $\mathscr{C}$ is a quotient of itself.

**Example B.1.7.** Let $\mathscr{T}$ be the category of triply connected topological spaces. Then the naïve homotopy category, $h\,\mathscr{T}$, defined by:

- Objects: Triply connected spaces;
- Morphisms: Homotopy classes of morphisms $f : X \to Y$;
- Composition: As in $\mathscr{T}$, but up to homotopy;
- Identities: Homotopy classes of the identity function;

is a quotient category of $\mathscr{T}$.

We now move to prove that every category $\mathscr{C}$, together with an arbitrary class of morphisms $W \subseteq \mathscr{C}_1$, admits a localization at $W$. We will do this in two steps: The first will be to construct a path category with arrows formally going against the flow of $W$, and the second will be to define a suitable quotient of this path category which will play the role of the localization. We begin by making a definition of this formally inverted path category:

Let $\mathscr{C}$ be a category and let $W \subseteq \mathscr{C}_1$. Then define the graph $U_W(G)$ to be given as follows:

- Vertices: $X \in \mathscr{C}_0$;
- Edges: $f \in \mathscr{C}_1$ and for all $s \in S$, an edge $s^{-1}$ where the source of $s^{-1}$ is the target of $s$ and vice-versa.

We then define the category $P(\mathscr{C}, W)$ to by

$$P(\mathscr{C}, W) := P(U_W(\mathscr{C})).$$

This is the category we will quotient to build the localization $W^{-1}\mathscr{C}$; the reason we have to quotient this category is because the map $\mathscr{C} \to W^{-1}\mathscr{C}$ given by $X \mapsto X$ and $f \mapsto [f]$ is not a functor; it does not preserve identities! Thus we must quotient $P(\mathscr{C}, W)$ by a suitable equivalence relation in order to get the correct category.

**Lemma B.1.8.** *Define a series of relations on $P(\mathscr{C}, W)_1$ by defining:*

- $[]_X \simeq [\mathrm{id}_X]$ *for all* $X \in \mathscr{C}_0$*;*
- $[e_1, \cdots, e_n] \simeq [f_1, \cdots f_m]$ *for all* $e_i, f_j \in \mathscr{C}_1$ *if and only if* $e_n \circ \cdots \circ e_1 = f_m \circ \cdots \circ f_1$*;*
- *for all* $w \in W$*,* $[w, w^{-1}] \simeq [\mathrm{id}_{\mathrm{Dom}(w)}]$ *and* $[w^{-1}, w] \simeq [\mathrm{id}_{\mathrm{Codom}(w)}]$*.*

*The $\simeq$ defines an equivalence relation on each hom-set $P(\mathscr{C}, W)(X, Y)$. Furthermore, if $[f] \simeq [g]$ in $P(\mathscr{C}, W)(X, Y)$ and if $[h] \in P(\mathscr{C}, W)(Y, Z)$ then $[h] \circ [f] \simeq [h] \circ [g]$.*

*Proof.* We first verify that each $\simeq$ is reflexive. Let $[e_1, \cdots, e_m] \in P(\mathscr{C}, W)(X, Y)$ be a morphism in $P(\mathscr{C}, W)$. Then since $e_m \circ \cdots \circ e_1 = e_m \circ \cdots \circ e_1$ it follows that $[e_1, \cdots, e_m] \simeq [e_1, \cdots, e_m]$. Similarly, $[]_X \simeq []_X$ for all $X \in \mathscr{C}_0$ because $\mathrm{id}_X = \mathrm{id}_X$ and $[]_X \simeq [\mathrm{id}_X]$. The verification for formal inverses $[w^{-1}] \simeq [w^{-1}]$ follows from the fact that $[w] \simeq [w]$.

Let us now verify the symmetry of $\simeq$; note that it suffices to prove this for the case in which a path $[f_1, \cdots, f_m] \simeq [e_1, \cdots, e_n]$ is given along edges of the form $f_i, e_j \in \mathscr{C}_1$. To see the symmetry note that if $[f_1, \cdots, f_m] \simeq [e_1, \cdots, e_n]$ then $f_m \circ \cdots \circ f_1 = e_n \circ \cdots \circ e_1$, and by the symmetry of equality of morphisms, we find that $e_n \circ \cdots \circ e_1 = f_m \circ \cdots \circ f_1$ and hence $[e_1, \cdots, e_n] \simeq [f_1, \cdots, f_m]$.

We now verify the transitivity of $\simeq$; as before, it suffices to verify this for the case in which all edges in the paths we consider come from $\mathscr{C}_1$. Assume that $[e_1, \cdots, e_k] \simeq [f_1, \cdots f_m]$ and that $[f_1, \cdots, f_m] \simeq [g_1, \cdots, g_n]$. It then follows that $g_k \circ \cdots \circ g_1 = f_m \circ \cdots \circ f_1$ and $f_m \circ \cdots \circ f_1 = g_n \circ \cdots \circ g_1$. Using the transitivity of equality of morphisms we derive that $e_k \circ \cdots \circ e_1 = g_n \circ \cdots \circ g_1$ and hence that $[e_1, \cdots, e_k] \simeq [g_1, \cdots, g_n]$. This in turn shows that $\simeq$ is an equivalence relation.

Finally, we show that these equivalence relations preserve composition of morphisms. Note that if we have paths $[e_1, \cdots, e_m] \simeq [f_1, \cdots, f_n]$ and if $[g_1, \cdots, g_k]$ is a path which may be pre-composed by the two prior paths then we find that

$$[g_1, \cdots, g_k] \circ [e_1, \cdots, e_m] \simeq [e_1, \cdots, e_m, g_1, \cdots, g_k] \simeq [g_k \circ \cdots \circ g_1 \circ e_m \circ \cdots \circ e_1];$$

similarly,

$$[g_1, \cdots, g_k] \circ [f_1, \cdots, f_n] \simeq [g_k \circ \cdots \circ g_1 \circ f_n \circ \cdots \circ f_1].$$

Because we assumed $[e_1, \cdots, e_m] \simeq [f_1, \cdots, f_n]$, it follows that

$$f_n \circ \cdots \circ f_1 = e_m \circ \cdots \circ e_1.$$

Thus we get that

$$g_k \circ \cdots \circ g_1 \circ e_m \circ \cdots \circ e_1 = g_k \circ \cdots \circ g_1 \circ f_n \circ \cdots \circ f_1$$

and so

$$[g_k \circ \cdots \circ g_1 \circ e_m \circ \cdots \circ e_1] \simeq [g_k \circ \cdots \circ g_1 \circ f_n \circ \cdots \circ f_1].$$

Using transitivity gives that

$$[g_1, \cdots, g_k] \circ [e_1, \cdots, e_m] \simeq [g_1, \cdots, g_k] \circ [f_1, \cdots, f_n]$$

and hence completes the proof of the lemma. □

The lemma above shows that the quotient category of $P(\mathscr{C}, W)$ at $\simeq$ exists by Lemma B.1.3. As such we define the category

$$\mathscr{A}_W := P(\mathscr{C}, W)_{/\simeq}.$$

It now remains to show that this indeed satisfies the universal property required of the localization of $\mathscr{C}$ at $W$.

**Lemma B.1.9.** *The category $\mathscr{A}_W$ is a localization of $\mathscr{C}$ at the class $W$ of morphisms.*

*Proof.* We first show that there is a functor $\lambda : \mathscr{C} \to \mathscr{A}_W$ which is the identity on objects and sends morphisms $w \in W$ to isomorphisms in $\mathscr{A}_W$. Define $\lambda : \mathscr{C} \to \mathscr{A}_W$ by setting $\lambda(X) := X$ for all $X \in \mathscr{C}_0$ and by defining $\lambda(f) := [f]$ for all $f \in \mathscr{C}_1$.

Let us show that $\lambda$ is well-defined. Since

$$(\mathscr{A}_W)_0 = (P(\mathscr{C}, W)_{/\simeq})_0 = P(\mathscr{C}, W)_0 = \mathscr{C}_0,$$

$\lambda$ is the identity function on objects and hence well-defined; thus we need only show the assignment on morphisms is well-defined. However, since $\mathscr{A}_W = P(\mathscr{C}, W)_{/\simeq}$ and two paths $[e_1, \cdots, e_n] \simeq [f_1, \cdots, f_m]$ if and only if $e_n \circ \cdots \circ e_1 = f_m \circ \cdots \circ f_1$ and since $\mathrm{id}_X = []_X = [\mathrm{id}_X]$ in $\mathscr{A}_W$, $\lambda$ is well-defined on morphisms as if $f = g$ then $\lambda(f) = [f]$, $\lambda(g) = [g]$, $[f] \simeq [g]$, and hence $\lambda(f) = \lambda(g)$. The verification that $\lambda$ is a functor follows similarly; if $g \circ f$ is defined in $\mathscr{C}$ then because

$$[g] \circ [f] \simeq [g \circ f]$$

we get that $\lambda(g) \circ \lambda(f) = \lambda(g \circ f)$, while the identification $[\mathrm{id}_X] \simeq []_X$ gives $\lambda(\mathrm{id}_X) = \mathrm{id}_{\lambda X}$. Thus $\lambda$ is a functor which is the identity function on objects.

We now verify the universal property asked of $\mathscr{A}_W$. First let $F : \mathscr{C} \to \mathscr{D}$ be a functor such that $F(w)$ is an isomorphism in $\mathscr{D}$ for all $w \in W$. Define the assignment

$$\widetilde{F} : \mathscr{A}_W \to \mathscr{D}$$

by $\widetilde{F}X = FX$ for all $X \in (\mathscr{A}_W)_0 = \mathscr{C}_0$ and defining $\widetilde{F}$ on morphisms as follows: Given a singleton path $[f] \in (\mathscr{A}_W)_1$, set

$$\widetilde{F}[f] := \Big\{ F(f)^{-1} \quad \text{if} \, \exists \, g \in W. [f] = [g]^{-1};$$

and extend $\widetilde{F}$ multiplicatively through multiplication, i.e., set

$$\widetilde{F}([e_m] \circ \cdots \circ [e_1]) := \widetilde{F}[e_m] \circ \cdots \circ \widetilde{F}[e_1]$$

for all edges $[e_i]$; note that since every path can be written as a composition of edges, it suffices to describe $\widetilde{F}$ on morphisms in the above way. Note that because $F$ is a functor and we have defined $\widetilde{F}$ to preserve composition of morphisms, once we have shown $\widetilde{F}$ to be well-defined, it is immediate that it is a functor. However, the well-definition of $\widetilde{F}$ on objects follows from the fact that $F$ is well-defined on objects, and using that $F$ sends all morphisms in $w$ to isomorphisms gives that $\widetilde{F}$ is well-defined on morphisms. Thus $\widetilde{F}$ is a functor. Furthermore, a routine check shows that the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "\lambda"'] & & \mathscr{D} \\ & \mathscr{A}_W \arrow[ur, "\widetilde{F}"'] & \end{tikzcd}$$

commutes in **Cat**. To see that $\widetilde{F}$ is unique we simply observe that the object assignment may be induced from the universal property of a quotient set, and so must be unique (cf. Lemma B.1.5). Thus it follows that the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "\lambda"'] & & \mathscr{D} \\ & \mathscr{A}_W \arrow[ur, dashed, "\exists! \widetilde{F}"'] & \end{tikzcd}$$

commutes. This proves the lemma. $\square$

**Theorem B.1.10.** *Let $\mathscr{C}$ be a category and let $W \subseteq \mathscr{C}_1$ be any class of morphisms. Then $W^{-1}\mathscr{C}$ exists.*

*Proof.* Define $W^{-1}\mathscr{C} := \mathscr{A}_W$, where $\mathscr{A}_W$ is the category constructed in Lemma B.1.9. Then the functor $\lambda : \mathscr{C} \to W^{-1}\mathscr{C}$ and the category $W^{-1}\mathscr{C}$ have the desired properties of a localization by Lemma B.1.9, allowing us to conclude the theorem. $\square$

While this shows that the localization category exists, it, unfortunately, does not tell us anything about the nature of the localization other than the fact that $W^{-1}\mathscr{C}$ is the universal $W$-inverting category. For instance, it can be the case that $W^{-1}\mathscr{C}$ is now a large category (and in fact one needs to be very careful to ensure that this is not the case), or it can be the case that the category $W^{-1}\mathscr{C}$ becomes degenerate in some sense. We will see many of these examples later on, but for the moment we show a class of such degenerate cases.

**Proposition B.1.11.** *Let $\mathscr{C}$ be a category and let $e \in \mathscr{C}(X,X)$ be an absorbing element, i.e., for all endomorphisms $f \in \mathscr{C}(X,X)$, $f \circ e = e = e \circ f$. Then if $W$ contains $e$, $\mathscr{C}(X,X)_{/\simeq} = \{[e]\}$.*

*Proof.* Let $f \in \mathscr{C}(X,X)$ be arbitrary. Then we have that

$$[f] = \mathrm{id}_X \circ [f] = [e]^{-1} \circ [e] \circ [f] = [e]^{-1} \circ [e \circ f] = [e]^{-1} \circ [e] \simeq \mathrm{id}_X .$$

Thus every morphism is equivalent to $\mathrm{id}_X$ and so it follows that $[f] = [e]$ for all $f \in \mathscr{C}(X,X)$, showing that $[e]^{-1} = [e]$ and completing the proof that $\mathscr{C}(X,X)_{/\simeq} = \{[e]\}$. $\square$

**Corollary B.1.12.** *If $R$ is a ring and if $S \subseteq R$ contains a zero divisor, then $S^{-1}R \cong 0$.*

We will now move on from the general theory of localization to discuss situations in which we have more control over the morphisms that we produce in the category $S^{-1}\mathscr{C}$. For instance, the functor $\lambda : \mathscr{C} \to S^{-1}\mathscr{C}$ need not behave well and the category $S^{-1}\mathscr{C}$ can be quite wild; cf. Exercises B.1.3 and B.1.4 below, as well as many of the examples of sections 9.3 and 9.4 of [43]. Furthermore, we would also like to work with categories that we will see when we introduce and define the derived category; as such, we are obliged to move on to study categories that admit calculi of fractions.

## Exercises

**Exercise B.1.1.** Prove that a quotient category is a localization.

**Exercise B.1.2.** Prove that the localization functor $\lambda : \mathscr{C} \to S^{-1}\mathscr{C}$ is faithful if and only if there exists a faithful functor $F : \mathscr{C} \to \mathscr{D}$ and $\mathscr{D}$ is a category in which $Fs$ is an isomorphism for all $s \in S$.

**Exercise B.1.3.** Find an example of a category $\mathscr{C}$ and a set of morphisms $S$ for which the localization map $\lambda : \mathscr{C} \to S^{-1}\mathscr{C}$ is not faithful. Try to make the localization as nontrivial as possible (so in particular try not to use Corollary B.1.12).

**Exercise B.1.4.** Find an example of a locally small category $\mathscr{C}$ and a locally small class of morphisms $S$ (in the sense that for all objects $X$ and $Y$ of $\mathscr{C}$, the collection $\{f \in S \mid \mathrm{Dom}(f) = X, \mathrm{Codom}(f) = Y\}$ is a set) such that the category $S^{-1}\mathscr{C}$ is large.

**Exercise B.1.5.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor and let $S \subseteq \mathscr{C}_1$ and $T \subseteq \mathscr{D}_1$ such that $F(S) \subseteq T$. Prove that there exists a unique functor $\widetilde{F} : S^{-1}\mathscr{C} \to T^{-1}\mathscr{D}$ making the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\lambda_S"'] & \mathscr{D} \arrow[d, "\lambda_T"] \\ S^{-1}\mathscr{C} \arrow[r, dashed, "\exists ! \widetilde{F}"'] & T^{-1}\mathscr{D} \end{tikzcd}$$

commute.

**Exercise B.1.6.** Prove that if $\mathscr{C}$ and $\mathscr{D}$ are categories with subclasses of morphisms $S \subseteq \mathscr{C}_1$ and $T \subseteq \mathscr{D}_1$, then there is an isomorphism of categories

$$(S \times T)^{-1}(\mathscr{C} \times \mathscr{D}) \cong S^{-1}\mathscr{C} \times T^{-1}\mathscr{D}$$

Hint: Consider their universal properties.

**Exercise B.1.7.** Let $F \dashv G : \mathscr{C} \to \mathscr{D}$ be an adjunction and let $S \subseteq \mathscr{C}_1$ and $T \subseteq \mathscr{D}_1$ be morphism classes for which $F(S) \subseteq T$ and $G(T) \subseteq S$. Prove that there is an induced adjunction

$$F' \dashv G' : S^{-1}\mathscr{C} \to T^{-1}\mathscr{D}\,.$$

**Exercise B.1.8.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Prove that there is an equivalence relation $\simeq_{X,Y}$ on each $\mathscr{C}(X, Y)$ by saying that $\varphi \simeq \psi$ if and only if $F\varphi = F\psi$. Use this to show that there is a factorization of every functor as a quotient followed by a faithful functor.

## B.2 Localizations of Categories: Calculi of Fractions

We will now introduce and study those categories[10] whose localizations $S^{-1}\mathscr{C}$ admit simple descriptions.[11] Following the study of noncommutative rings[12], we would like to give some conditions in which the morphisms in $S^{-1}\mathscr{C}$ have the form either $s^{-1} \circ f$ or $f \circ s^{-1}$, depending on one's taste for sidedness of inversion. As in [43],[13] we are lead to study categories and morphisms classes which admit a left or right Ore condition, as this will ensure that the localization $S^{-1}\mathscr{C}$ has morphisms take one of the two forms described above.

**Definition B.2.1.** The system $S \subseteq \mathscr{C}$ admits a *calculus of left fractions* if the following conditions are satisfied:

1. $S$ is closed with respect to composition and $\mathrm{id}_X \in S$ for all $X \in \mathscr{C}_0$;

2. Given any span $Z \xleftarrow{s} X \xrightarrow{u} Y$ with $s \in S$, there exists a cospan $Z \xrightarrow{u'} W \xleftarrow{s'} Y$ with $s' \in S$ making the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "s"'] & Y \arrow[d, "s'"] \\ Z \arrow[r, "u'"'] & W \end{tikzcd}$$

commute;

3. If $s : W \to X$ is a morphism in $S$ and if there are parallel morphisms $f, g : X \to Y$ making the diagram

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

commute, then there exists a morphism $t : Y \to Z$ in $S$ such that the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "t"] & Z \end{tikzcd}$$

commute.

Moreover, we say that $S$ admits a *calculus of right fractions* if $S^{\mathrm{op}}$ admits a calculus of left fractions in $\mathscr{C}^{\mathrm{op}}$.

**Remark B.2.2.** Condition (1) is mild and assumed simply for the sake of convenience; if we take an arbitrary collection of morphisms $S$ that satisfy Conditions (2) and (3), we can close this with respect to composition and identities in the obvious way. However, it will be important to work with this "complete" set when we define the localization category.

**Remark B.2.3.** Condition (2) above is called the *left Ore condition*, in analog to the theory of Ore localization of rings; cf. Chapter 10 of [43] for details. Note that as in the theory of noncommutative rings, we should see the Ore condition as a way of ensuring that when we localize at $S$, we can replace fractions of the form $f \circ s^{-1}$ by fractions of the form $t^{-1} \circ g$ in the generated equivalence relation.

[10]And their specified classes of morphisms. Like any good scientific writer, it is important to omit arguably the most important detail whenever possible.

[11]Within reason. We're ignoring some set-theoretic technicalities here, and if I were a better author (or set-theorist) we'd really dive into those. However, by working in suitable Grothendieck universes we can sidestep these issues to a large degree, provided we be careful about hierarchies of universes and make sure that we're careful about when we jump universes. These discussions are not for this book, however, as they will distract us from seeing what's really making this stuff work. It's never a good idea to lose the forest for the trees, and set-theoretic difficulties sure are trees.

[12]Whether it is sensible to follow noncommutative ring theory is always debatable, but in this case the debate can be summarized as "yes."

[13]And hence *really* following the study of noncommutative rings.

**Remark B.2.4.** The terminology of whether something is left/right-sided as a calculus of fractions is not consistent in the literature (cf. the definitions in [**?**], [79], for instance). We take the conventions laid out in [**?**], which is the original reference for the subject material and is consistent with [36].

We will now show how to realize the localization $S^{-1}\mathscr{C}$ at a left Ore system as a category whose maps are all composites of the form $s^{-1} \circ f$. To see how to do this, we first will build a proto-representation for the fractions $s^{-1} \circ f$; that is, we will look at composites of the form

$$X \xrightarrow{f} Z \xleftarrow{s} Y$$

where $s \in S$. We should think of the cospan above as a roof (cf. [79]) which generates the fraction $s^{-1} \circ f$, as we will want to move against the morphism $s$ at some point. In what proceeds, we fix a category $\mathscr{C}$ and a left Ore system $S$. We now define the sets which we will quotient to generate the hom-sets in the localization of $\mathscr{C}$ at $S$, which will be the set of all roofs that begin at $X$ and end at $Y$.

**Definition B.2.5** ([**?**])**.** For any objects $X$ and $Y$ of $\mathscr{C}$, define the set $H(X,Y)$ by

$$H(X,Y) := \{(s,f) \in S \times \mathscr{C}_1 \mid \mathrm{Codom}(f) = \mathrm{Codom}(s), \mathrm{Dom}(f) = X, \mathrm{Dom}(s) = Y\}$$

Visually, $H$ is the set of all roofs

$$X \xrightarrow{f} Z \xleftarrow{s} Y$$

in $\mathscr{C}$.

**Definition B.2.6** ([**?**])**.** For any roofs $(s,f)$ and $(t,g)$ in $H(X,Y)$, we define a relation $\simeq$ on $H(X,Y)$ by declaring that $(s,f) \simeq (t,g)$ if and only if there exist morphisms $a$ and $b$ which make the diagram

$$\begin{tikzcd} & Z \arrow[d, "a"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & W & Y \arrow[ul, "s"'] \arrow[dl, "t"] \\ & Z' \arrow[u, "b"'] & \end{tikzcd}$$

commute in $\mathscr{C}$ with $a \circ s \in S$.

**Remark B.2.7** (A silly remark for intuition)**.** The idea for $(s,f) \simeq (t,g)$ is that while the roofs themselves may look different, we can tack on a weather vane to each tip of each roof, and after doing this we end up with the same roof.

**Proposition B.2.8.** *The relation $\simeq$ on $H(X,Y)$ is an equivalence relation.*

*Proof.* We first verify that $\simeq$ is reflexive. However, because $S$ contains all identity morphisms, for all $(s,f) \in S$ we can produce the commuting diagram

$$\begin{tikzcd} & Z \arrow[d, equal] & \\ X \arrow[ur, "f"] \arrow[dr, "f"'] & Z & Y \arrow[ul, "s"'] \arrow[dl, "s"] \\ & Z \arrow[u, equal] & \end{tikzcd}$$

which witnesses the fact that $(s,f) \simeq (s,f)$.

To verify symmetry, note that if $(s,f)\simeq(t,g)$, the diagram

$$\begin{tikzcd} & Z \arrow[d, "a"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & W & Y \arrow[ul, "s"'] \arrow[dl, "t"] \\ & Z' \arrow[u, "b"'] & \end{tikzcd}$$

commutes. However, this implies that

$$\begin{tikzcd} & Z' \arrow[d, "b"] & \\ X \arrow[ur, "g"] \arrow[dr, "f"'] & W & Y \arrow[ul, "t"'] \arrow[dl, "s"] \\ & Z \arrow[u, "a"'] & \end{tikzcd}$$

commutes and hence that $(t,g)\simeq(s,f)$.

Finally, in order to verify that $\simeq$ is transitive, assume that $(s,f)\simeq(t,g)$ and that $(t,g)\simeq(r,h)$ with resulting commuting diagrams

$$\begin{tikzcd} & Z \arrow[d, "a"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & W & Y \arrow[ul, "s"'] \arrow[dl, "t"] \\ & Z' \arrow[u, "b"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} & Z' \arrow[d, "c"] & \\ X \arrow[ur, "g"] \arrow[dr, "h"'] & W' & Y \arrow[ul, "t"'] \arrow[dl, "r"] \\ & Z'' \arrow[u, "d"'] & \end{tikzcd}$$

in $\mathscr{C}$. Now, using the Ore condition we take the span $W \xleftarrow{a\circ s} Y \xrightarrow{d\circ r} W'$ and find the cospan $W \xrightarrow{\varphi} A \xleftarrow{\psi} W'$ which makes the diagram

$$\begin{tikzcd} Y \arrow[r, "a\circ s"] \arrow[d, "d\circ r"'] & W \arrow[d, "\varphi"] \\ W' \arrow[r, "\psi"'] & A \end{tikzcd}$$

commute; note that because $a\circ s, d\circ r\in S$, it follows that $\varphi,\psi\in S$ as well. Now observe that because $a\circ s=b\circ t$ and $d\circ r=c\circ t$, we have that

$$\varphi\circ b\circ t=\varphi\circ a\circ s=\psi\circ d\circ r=\psi\circ c\circ t,$$

giving in turn that the diagram

$$\begin{tikzcd} Y \arrow[r, "t"] & Z' \arrow[r, shift left, "\varphi\circ b"] \arrow[r, shift right, "\psi\circ c"'] & A \end{tikzcd}$$

commutes in $\mathscr{C}$. Because $t \in S$, there exists a morphism $\alpha \in S$ making the diagram

$$\begin{tikzcd} Z' \arrow[r, shift left, "\varphi\circ b"] \arrow[r, shift right, "\psi\circ c"'] & A \arrow[r, "\alpha"] & W'' \end{tikzcd}$$

commute. However, from this it follows that

$$\alpha \circ \varphi \circ a \circ f = \alpha \circ \varphi \circ b \circ t = \alpha \circ \psi \circ c \circ t = \alpha \circ \psi \circ d \circ r$$

and that

$$\alpha \circ \varphi \circ a \circ s = \alpha \circ \psi \circ d \circ r.$$

Furthermore, because $a \circ s \in S$, $\varphi \in S$, and $\alpha \in S$, it follows that $\alpha \circ \varphi \circ a \circ s \in S$. Thus the diagram

$$\begin{tikzcd} & Z \arrow[d, "\alpha\circ\varphi\circ a"] & \\ X \arrow[ur, "f"] \arrow[dr, "h"'] & W'' & Y \arrow[ul, bend right, "s"'] \arrow[dl, bend left, "r"] \\ & Z'' \arrow[u, "\alpha\circ\psi\circ d"'] & \end{tikzcd}$$

commutes with the vertical arrows functions in $S$. Therefore it follows that $(s, f) \simeq (r, h)$. This concludes the proof that $\simeq$ is an equivalence relation. $\square$

We will now show that it is possible to make a category $\mathscr{H}$ whose objects coincide with $\mathscr{C}$ and whose morphisms $\mathscr{H}(X, Y)$ are given by $H(X, Y)_{/\simeq}$. However, to do this we first need to define a composition rule on the hom-sets[14] that defines a category. Before we do this, however, we make the following definition of convenience: Let $(s, f) \in H(X, Y)$. We write

$$s^{-1} \circ f := [s, f]$$

for the equivalence class of $(s, f)$ in $H(X, Y)_{/\simeq}$.

In order to give a composition of morphisms on $\mathscr{H}$, let $X, Y, Z \in \mathscr{C}_0$ be objects. Then for any two roofs

$$X \xrightarrow{f} W \xleftarrow{s} Y$$

and

$$Y \xrightarrow{g} W' \xleftarrow{t} Z$$

we generate morphisms $s^{-1} \circ f$ and $t^{-1} \circ g$ which should be composable in $\mathscr{H}$. In order to give a composition $(t^{-1} \circ g) \circ (s^{-1} \circ f)$, we need to find a roof which represents the composites. However, consider the zig-zag

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \arrow[dr, "g"'] & & Z \arrow[dl, "t"] \\ & W & & W' & \end{tikzcd}$$

in $\mathscr{C}$ generated by the two roofs at hand. By the Ore condition, there exists a cospan $W \xrightarrow{f'} W'' \xleftarrow{t'} W'$ making the square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s"'] & W' \arrow[d, "t'"] \\ W \arrow[r, "f'"'] & W'' \end{tikzcd}$$

[14]Once gain, we are being nebulous about what is a set and what is a proper class. Just pretend these are sets when we start defining functions and things. In the immortal words of Queen Victoria: “Close your eyes and think of ZFC set theory.”

commute with $t' \in S$. However, it then follows that we can produce the diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \arrow[dr, "g"'] & & Z \arrow[dl, "t"] \\
& W \arrow[dr, "f'"'] & & W' \arrow[dl, "t'"] & \\
& & W'' & &
\end{tikzcd}$$

and hence we can produce the roof

$$X \xrightarrow{f' \circ f} W'' \xleftarrow{t' \circ t} Z$$

in $\mathscr{C}$. Thus we define our composition

$$\circ_{X,Y,Z} : H(Y,Z)_{/\simeq} \times H(X,Y)_{/\simeq} \to H(X,Z)_{/\simeq}$$

as follows: Given $s^{-1} \circ f \in H(X,Y)_{/\simeq}$ and $t^{-1} \circ g \in H(Y,Z)_{/\simeq}$, we define

$$(t^{-1} \circ g) \circ (s^{-1} \circ f) := (t' \circ t)^{-1} \circ (f' \circ f).$$

We now must show that this is indeed a function and that it defines an associative composition law on $\mathscr{H}$.

**Lemma B.2.9.** *The functions* $\circ_{X,Y,Z} : H(Y,Z)_{/\simeq} \times H(X,Y)_{/\simeq} \to H(X,Z)_{/\simeq}$ *are well-defined.*

*Proof.* It suffices to prove that if $(s_1, f_1) \simeq (s_2, f_2)$ in $H(X,Y)$ and if $(t,g) \in H(Y,Z)$, then there is an equivalence $(t^{-1} \circ g) \circ (s_1^{-1} \circ f_1) \simeq (t^{-1} \circ g) \circ (s_2^{-1} \circ f_2)$, as the general case may be given by deducing that

$$(t_1^{-1} \circ g_1) \circ (s^{-1} \circ f) \simeq (t_2^{-1} \circ g_2) \circ (s^{-1} \circ f)$$

whenever $(t_1, g_1) \simeq (t_2, g_2) \in H(Y,Z)$ and $(s,f) \in H(X,Y)$ (which follows mutatis mutandis from the first case) and then composing the two cases together. So begin by letting $(s_1, f_1) \simeq (s_2, f_2)$ in $H(X,Y)$ and let $(t,g) \in H(Y,Z)$. Let $\alpha$ and $\beta$ be the morphisms making the diagram

$$\begin{tikzcd}
& W_1 \arrow[d, "\alpha"'] & \\
X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & W_3 & Y \arrow[ul, "s_1"'] \arrow[dl, "s_2"] \\
& W_2 \arrow[u, "\alpha"] &
\end{tikzcd}$$

commute and consider the diagram

$$\begin{tikzcd}
& & W_1'' & & \\
& W_1 \arrow[ur, "f_1'"] \arrow[d, "\alpha"'] & & W_1' \arrow[ul, "s_1'"'] & \\
X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & W_3 & Y \arrow[ul, "s_1"'] \arrow[ur, "g"] \arrow[dl, "s_2"] \arrow[dr, "g"'] & & Z \arrow[ul, "t"'] \arrow[dl, "t"] \\
& W_2 \arrow[u, "\beta"] \arrow[dr, "f_2'"'] & & W_2' \arrow[dl, "s_2'"] & \\
& & W_2'' & &
\end{tikzcd}$$

where $s_1', s_2', \alpha \circ s_1$, and $\beta \circ s_2$ are all members of $S$. We now need to construct a roof $W_1'' \xrightarrow{a} W_3'' \xleftarrow{b} W_2''$ that makes $a \circ f_1' \circ f_1 = b \circ f_2' \circ f_2$ and $a \circ s_1' \circ t = b \circ s_2' \circ t \in S$. To this end, consider that we have the span

$$\begin{tikzcd} W_1'' & W_1' \arrow[l, "s_1'"'] \arrow[r, "s_2'"] & W_2' \end{tikzcd}$$

with both $s_1', s_2' \in S$. We can then fill the diagram using the Ore condition

$$\begin{tikzcd} W_1' \arrow[r, "s_1'"] \arrow[d, "s_2'"'] & W_1'' \arrow[d, "r_1"] \\ W_2'' \arrow[r, "r_1"'] & A \end{tikzcd}$$

where $r_1, r_2 \in S$. Note that this diagram makes the equality

$$r_1 \circ s_1' \circ t = r_2 \circ s_2' \circ t$$

hold with the composite a member of $S$, but likely does not interact in the desired way with $f_1' \circ f_1$ and $f_2' \circ f_2$.

To make this work on the other side, begin by taking the span

$$\begin{tikzcd} W_3 & Y \arrow[l, "\alpha \circ s_1"'] \arrow[r, "f_1' \circ s_1"] & W_1'' \end{tikzcd}$$

and now use the Ore condition to construct the diagram

$$\begin{tikzcd} Y \arrow[r, "\alpha \circ s_1"] \arrow[d, "f_1' \circ s_1"'] & W_3 \arrow[d, "\psi_1"] \\ W_1'' \arrow[r, "\sigma_1"'] & B_1 \end{tikzcd}$$

with $\sigma_1 \in S$. Now, since the diagram

$$\begin{tikzcd} Y \arrow[r, "s_1"] & W_1 \arrow[r, shift left, "\psi_1 \circ \alpha"] \arrow[r, shift right, "\sigma_1 \circ f_1'"'] & B_1 \end{tikzcd}$$

commutes with $s_1 \in S$, there exists a morphism $\tau_1 \in S$ which makes the diagram

$$\begin{tikzcd} W_1 \arrow[r, shift left, "\psi_1 \circ \alpha"] \arrow[r, shift right, "\sigma_1 \circ f_1'"'] & B_1 \arrow[r, "\tau_1"] & B_1' \end{tikzcd}$$

commute. Finally, in this part of the construction, consider the span

$$\begin{tikzcd} B_1' & W_1'' \arrow[l, "\tau_1 \circ \sigma_1"'] \arrow[r, "r_1"] & A \end{tikzcd}$$

and construct the diagram

$$\begin{tikzcd} W_1'' \arrow[r, "r_1"] \arrow[d, "\tau_1 \circ \sigma_1"'] & A \arrow[d, "\rho_1"] \\ B_1' \arrow[r, "r_1'"'] & C_1 \end{tikzcd}$$

where $r_1'$ and $\rho_1$ are both members of $S$.

We now do the same construction, save now on the $f_2$ side of things. Proceeding mutatis mutandis to the prior case, we construct the commuting square

$$\begin{tikzcd} Y \arrow[r, "\beta\circ s_2"] \arrow[d, "f_2'\circ s_2"'] & W_3 \arrow[d, "\psi_2"] \\ W_2'' \arrow[r, "\sigma_2"'] & B_2 \end{tikzcd}$$

with $\sigma_2 \in S$. As before, we can find a $\tau_2 \in S$ making the diagram

$$\begin{tikzcd} W_2 \arrow[r, shift left, "\psi_2\circ\beta"] \arrow[r, shift right, "\sigma_2\circ f_2'"'] & B_2 \arrow[r, "\tau_2"] & B_2' \end{tikzcd}$$

commute. We then use this to produce the commuting diagram

$$\begin{tikzcd} W_2'' \arrow[r, "r_2"] \arrow[d, "\tau_2\circ\sigma_2"'] & A \arrow[d, "\rho_2"] \\ B_2' \arrow[r, "r_2'"'] & C_2 \end{tikzcd}$$

with $r_2', \rho_2 \in S$. This in turn gives us the span

$$C_1 \xleftarrow{\rho_1} A \xrightarrow{\rho_2} C_2$$

which we in turn complete to the diagram

$$\begin{tikzcd} A \arrow[r, "\rho_1"] \arrow[d, "\rho_2"'] & C_1 \arrow[d, "\eta"] \\ C_2 \arrow[r, "\gamma"'] & W_3'' \end{tikzcd}$$

with $\gamma, \eta \in S$. This in turn allows us to produce the cospan:

$$W_1'' \xrightarrow{\eta\circ\rho_1\circ r_1} W_3'' \xleftarrow{\gamma\circ\rho_2\circ r_2} W_2''.$$

A routine calculation shows that

$$\eta \circ \rho_1 \circ r_1 \circ s_1' \circ t = \gamma \circ \rho_2 \circ r_1 \circ s_1' \circ t = \gamma \circ \rho_2 \circ r_2 \circ s_2' \circ t$$

and, since each of $\gamma, \rho_2, r_2, s_2'$, and $t$ are members of $S$, we also have that $\gamma \circ \rho_2 \circ r_2 \circ s_2' \circ t \in S$.

In order to prove that the other compositions are equal, i.e., that

$$\eta \circ \rho_1 \circ r_1 \circ f_1' \circ f_1 \stackrel{?}{=} \gamma \circ \rho_2 \circ r_2 \circ f_2' \circ f_2,$$

we consider the commuting diagram:

$$
\begin{tikzcd}
 & & & B_2 \arrow[dr, "\tau_2"] & \\
 & & W_2'' \arrow[ur, "\sigma_2"] \arrow[rr, "\tau_2\circ\sigma_2"] \arrow[dd, "r_2"'] & & B_2' \arrow[dd, "r_2'"] \\
 & & & & \\
 & W_1'' \arrow[dl, "\sigma_1"'] \arrow[r, "r_1"] \arrow[dd, "\tau_1\circ\sigma_1"] & A \arrow[rr, "\rho_2"] \arrow[dd, "\rho_1"'] & & C_2 \arrow[dd, "\gamma"] \\
B_1 \arrow[dr, "\tau_1"'] & & & & \\
 & B_1' \arrow[r, "r_1'"'] & C_1 \arrow[rr, "\eta"'] & & W_3''
\end{tikzcd}
$$

It then follows form the construction that the diagram

$$
\begin{tikzcd}
X \arrow[rrr] \arrow[dr, "\beta\circ f_2"] \arrow[dddd, "\alpha\circ f_1"] & & & B_2 \arrow[dr, "\tau_2"] & \\
 & W_3 \arrow[urr, "\psi_2"'] \arrow[dddl, "\psi_1"'] & W_2'' \arrow[ur, "\sigma_2"] \arrow[rr, "\tau_2\circ\sigma_2"] \arrow[dd, "r_2"'] & & B_2' \arrow[dd, "r_2'"] \\
 & & & & \\
 & W_1'' \arrow[dl, "\sigma_1"'] \arrow[r, "r_1"] \arrow[dd, "\tau_1\circ\sigma_1"] & A \arrow[rr, "\rho_2"] \arrow[dd, "\rho_1"'] & & C_2 \arrow[dd, "\gamma"] \\
B_1 \arrow[dr, "\tau_1"'] & & & & \\
 & B_1' \arrow[r, "r_1'"'] & C_1 \arrow[rr, "\eta"'] & & W_3''
\end{tikzcd}
$$

commutes as well, which allows us to compute that

$$
\begin{aligned}
\eta \circ \rho_1 \circ r_1 \circ f_1' \circ f_1 &= \eta \circ r_1' \circ \tau_1 \circ \sigma_1 \circ f_1' \circ f_1 = \eta \circ r_1' \circ \tau_1 \circ \psi_1 \circ \alpha \circ f_1 = \eta \circ r_1' \circ \tau_1 \circ \psi_1 \circ \beta \circ f_2 \\
&= \gamma \circ r_2' \circ \tau_2 \circ \psi_2 \circ \beta \circ f_2 = \gamma \circ r_2' \circ \tau_2 \circ \sigma_2 \circ f_2' \circ f_2 = \gamma \circ \rho_2 \circ r_2 \circ f_2' \circ f_2.
\end{aligned}
$$

This establishes that the diagram

$$
\begin{tikzcd}
 & W_1'' \arrow[d, "\eta\circ\rho_1\circ r_1"] & \\
X \arrow[ur, bend left, "f_1'\circ f_1"] \arrow[dr, bend right, "f_2'\circ f_2"'] & W_3'' & Z \arrow[ul, bend right, "s_1'\circ t"'] \arrow[dl, bend left, "s_2'\circ t"] \\
 & W_2'' \arrow[u, "\gamma\circ\rho_2\circ r_2"'] &
\end{tikzcd}
$$

commutes with $\eta \circ \rho_1 \circ r_1 \circ s_1' \circ t \in S$. Together with the comments at the start of the proof, this establishes the lemma. $\square$

For the proof of associativity we will need one lemma. We provide it here for completeness, but states that any time we use the Ore condition to provide fillers of diagrams, any two choices of fillers define equivalent roofs. In particular, producing any two fillers from the Ore condition gives the same equivalence class in $H(X,Y)_{\simeq}$.

**Lemma B.2.10.** *Let $C \xleftarrow{s} A \xrightarrow{f} B$ be a span in $\mathscr{C}$ with $s \in S$. Then if $B \xrightarrow{f_1} D_1 \xleftarrow{s_1} C$ and $B \xrightarrow{f_2} D_2 \xleftarrow{s_2} C$ are two cospans in $\mathscr{C}$ making the squares*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & C \arrow[d, "s_1"] \\ B \arrow[r, "f_1"'] & D_1 \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & C \arrow[d, "s_2"] \\ B \arrow[r, "f_2"'] & D_2 \end{tikzcd}$$

*commute with $s_1$ and $s_2$ in $S$, then $(s_1, f_1) \simeq (s_2, f_2) \in H(B,C)$.*

*Proof.* Begin by producing the commuting square

$$\begin{tikzcd} C \arrow[r, "s_1"] \arrow[d, "s_2"'] & D_1 \arrow[d, "\sigma_1"] \\ D_2 \arrow[r, "\sigma_2"'] & E \end{tikzcd}$$

by using the Ore condition and note that $\sigma_1, \sigma_2 \in S$. Moreover, it follows from construction that

$$\sigma_1 \circ f_1 \circ s = \sigma_1 \circ s_1 \circ f = \sigma_2 \circ s_2 \circ f = \sigma_2 \circ f_2 \circ s.$$

This shows that the diagram

$$\begin{tikzcd} A \arrow[r, "s"] & B \arrow[r, shift left, "\sigma_1 \circ f_1"] \arrow[r, shift right, "\sigma_2 \circ f_2"'] & E \end{tikzcd}$$

commutes with $s \in S$; thus we can find a $t \in S$ such that the diagram

$$\begin{tikzcd} B \arrow[r, shift left, "\sigma_1 \circ f_1"] \arrow[r, shift right, "\sigma_2 \circ f_2"'] & E \arrow[r, "t"] & F \end{tikzcd}$$

commutes as well. From this we calculate (by construction) that

$$t \circ \sigma_1 \circ f_1 = t \circ \sigma_2 \circ f_2$$

and that

$$t \circ \sigma_1 \circ s_1 = t \circ \sigma_2 \circ s_2 \in S.$$

This shows that the diagram

$$\begin{tikzcd} & D_1 \arrow[d, "t \circ \sigma_1"] & \\ B \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & F & C \arrow[ul, "s_1"'] \arrow[dl, "s_2"] \\ & D_2 \arrow[u, "t \circ \sigma_2"'] & \end{tikzcd}$$

commutes with $t \circ \sigma_1 \circ s_1 \in S$. It thus follows that $(s_1, f_1) \simeq (s_2, f_2)$ in $H(B,C)$, as was to be shown. $\square$

**Proposition B.2.11.** *The composition maps* $\circ_{X,Y,Z} : H(Y,Z)_{/\simeq} \times H(X,Y)_{/\simeq} \to H(X,Z)_{\simeq}$ *define an associative composition. In particular, the diagram*

$$\begin{tikzcd}
(H(Z,W)_{/\simeq} \times H(Y,Z)_{/\simeq}) \times H(X,Y)_{/\simeq} \arrow[rr] \arrow[d, "\cong"'] & & H(Y,W)_{/\simeq} \times H(X,Y)_{/\simeq} \arrow[d] \\
H(Z,W)_{/\simeq} \times (H(Y,Z)_{/\simeq} \times H(X,Y)_{/\simeq}) \arrow[dr] & & H(X,W)_{/\simeq} \\
 & H(Z,W)_{/\simeq} \times H(X,Z)_{/\simeq} \arrow[ur] &
\end{tikzcd}$$

*commutes.*

*Proof.* Begin by assuming that $(s,f) \in H(X,Y), (t,g) \in H(Y,Z)$, and $(r,h) \in H(Z,W)$ are roofs which fit into the diagram:

$$\begin{tikzcd}
 & A_1 & & A_2 & & A_3 & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"'] \arrow[ur, "h"] & & W \arrow[ul, "r"']
\end{tikzcd}$$

The composition

$$(r^{-1} \circ h) \circ \big((t^{-1} \circ g) \circ (s^{-1} \circ f)\big)$$

is represented by the diagram

$$\begin{tikzcd}
 & & & C_1 & & & \\
 & & B_1 \arrow[ur, "h''"] & & & & \\
 & A_1 \arrow[ur, "g'"] & & A_2 \arrow[ul, "s'"] & & A_3 \arrow[uull, "(s'\circ t)'"'] & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"] \arrow[ur, "g"] & & Z \arrow[ul, "t"] \arrow[ur, "h"] & & W \arrow[ul, "r"']
\end{tikzcd}$$

while the composition

$$\big((r^{-1} \circ h) \circ (t^{-1} \circ g)\big) \circ (s^{-1} \circ f)$$

is represented by the diagram:

$$\begin{tikzcd}
 & & & C_2 & & & \\
 & & & & B_2 \arrow[ul, "s''"'] & & \\
 & A_1 \arrow[uurr, "(h'\circ g)'"] & & A_2 \arrow[ur, "h'"] & & A_3 \arrow[ul, "t'"'] & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"] \arrow[ur, "g"] & & Z \arrow[ul, "t"] \arrow[ur, "h"] & & W \arrow[ul, "r"']
\end{tikzcd}$$

We now use the Ore condition to produce the diagram

$$\begin{tikzcd}
W \arrow[r, "s''\circ t'\circ r"] \arrow[d, "(s'\circ t)'\circ r"'] & C_2 \arrow[d, "\sigma_2"] \\
C_1 \arrow[r, "\sigma_1"'] & D_1
\end{tikzcd}$$

where $\sigma_1, \sigma_2 \in S$. Then from the commutativity of the diagram

$$\begin{tikzcd}
W \arrow[r, "r"] & A_3 \arrow[r, shift left, "\sigma_2\circ s''\circ t'"] \arrow[r, shift right, "\sigma_1\circ(s'\circ t)'"'] & D_1
\end{tikzcd}$$

we can find a $\tau \in S$ making the diagram

$$\begin{tikzcd}
A_3 \arrow[r, shift left, "\sigma_2\circ s''\circ t'"] \arrow[r, shift right, "\sigma_1\circ(s'\circ t)'"'] & D_1 \arrow[r, "\tau"] & E_1
\end{tikzcd}$$

commute. We now calculate that

$$\tau \circ \sigma_1 \circ h'' \circ s' \circ t = \tau \circ \sigma_1 \circ (s' \circ t)' \circ h$$

and $\tau \circ \sigma_2 \circ s'' \circ t' \in S$. This shows that the diagram

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "t"'] & A_3 \arrow[d, "\tau\circ\sigma_1\circ(s'\circ t)'"] \\
A_2 \arrow[r, "\tau\circ\sigma_1\circ h''\circ s'"'] & E_1
\end{tikzcd}$$

commutes, and hence that $(t', h') \simeq (\tau \circ \sigma_1 \circ (s' \circ t)', \tau \circ \sigma_1 \circ s'' \circ s')$ by Lemma B.2.10. From here proceeding as in the proof of Proposition B.2.8 and Lemma B.2.9 proves the proposition. $\square$

We now have all the tools at hand to prove that our yet-to-be-defined category $\mathscr{H}_S$ is, in fact, a category. All that remains is to prove that composition has an identity, although this will be a straightforward calculation.

**Definition B.2.12.** Let $\mathscr{C}$ be a category and let $S$ be a left Ore system. Define the category $\mathscr{H}$ as follows:

- Objects: $X \in \mathscr{H}_0$ if and only if $X \in \mathscr{C}_0$;
- Morphisms: For all $X, Y \in \mathscr{C}_0$, we define

  $$\mathscr{H}(X, Y) := H(X, Y)_{/\simeq}$$

  so that each individual morphism is an equivalence class of a roof $X \xrightarrow{f} Z \xleftarrow{s} Y$;
- Composition is as in Lemma B.2.9, i.e., $(t^{-1} \circ g) \circ (s^{-1} \circ f)$ is defined to be the equivalence class of the large roof $X \xrightarrow{f'\circ f} W'' \xleftarrow{t'\circ t} Z$ constructed from the diagram:

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \arrow[dr, "g"'] & & Z \arrow[dl, "t"] \\
& W \arrow[dr, "\exists\, f'"'] & & W' \arrow[dl, "\exists\, t'"] & \\
& & W'' & &
\end{tikzcd}$$

- Identities: The roof $X = X = X$ is the identity in $\mathscr{H}(X, X)$.

**Proposition B.2.13.** *The object $\mathscr{H}$ is a category and there is a functor $\lambda_S : \mathscr{C} \to \mathscr{H}$ which is the identity on objects and sends morphisms $f$ to the roof $\mathrm{id}_{\mathrm{Codom}(f)}^{-1} \circ f$.*

*Proof.* We know that composition is well-defined by virtue of Lemma B.2.9 and associative by Proposition B.2.11; thus we need only prove that the claimed identities are indeed identity morphisms in the category. However, to see this let $s^{-1} \circ f \in \mathscr{H}(X, Y)$ be an arbitrary morphism/roof and consider the morphisms $\mathrm{id}_X^{-1} \circ \mathrm{id}_X$ and $\mathrm{id}_Y^{-1} \circ \mathrm{id}_Y$ represented by the roofs:

$$\begin{tikzcd} X \arrow[dr, equal] & & X \arrow[dl, equal] & Y \arrow[dr, equal] & & Y \arrow[dl, equal] \\ & X & & & Y & \end{tikzcd}$$

In the first composition, we note that the diagram

$$\begin{tikzcd} X \arrow[dr, equal] & & X \arrow[dl, equal] \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \\ & X & & Z & \end{tikzcd}$$

can be completed to the diagram

$$\begin{tikzcd} X \arrow[dr, equal] & & X \arrow[dl, equal] \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \\ & X \arrow[dr, "f"'] & & Z \arrow[dl, equal] & \\ & & Z & & \end{tikzcd}$$

which contracts to the roof:

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \\ & Z & \end{tikzcd}$$

This shows that

$$(s^{-1} \circ f) \circ (\mathrm{id}_X^{-1} \circ \mathrm{id}_X) = s^{-1} \circ f;$$

the calculation that $(\mathrm{id}_Y^{-1} \circ \mathrm{id}_Y) \circ (s^{-1} \circ f) = s^{-1} \circ f$ is similar and omitted. This completes the proof that $\mathscr{H}$ is a category.

We now prove the existence of the functor $\lambda$. First of all, it is certainly well-defined, so it suffices to prove that it is functorial. However, it is routine to verify that

$$\lambda_S(\mathrm{id}_X) = \mathrm{id}_X^{-1} \circ \mathrm{id}_X$$

is the identity in $\mathscr{H}$ at $X$, while for any two composable morphisms $g$ and $f$,

$$\lambda_S(g) \circ \lambda_S(f) = (\mathrm{id}_{\mathrm{Codom}(g)}^{-1} \circ g) \circ (\mathrm{id}_{\mathrm{Codom}(f)}^{-1} \circ f) = \mathrm{id}_{\mathrm{Codom}(g)}^{-1} \circ (g \circ f) = \lambda_S(g \circ f).$$

This proves that $\lambda_S$ is a functor and completes the proof of the proposition. $\square$

**Lemma B.2.14.** *For every $s \in S$, the roof $X \xrightarrow{s} Y \xleftarrow{\mathrm{id}_Y} Y$ is invertible in $S$.*

*Proof.* Begin by letting $s \in S$ and observing that the diagram

$$\begin{tikzcd}
& Y & \\
X \arrow[ur, "s"] & Y \arrow[u, equal] & X \arrow[ul, "s"'] \\
& X \arrow[ul, equal] \arrow[u, "s"] \arrow[ur, equal] &
\end{tikzcd}$$

commutes with $s \in S$. This shows that $s^{-1} \circ s = \mathrm{id}_X$ in $\mathscr{H}$. To see that $s \circ s^{-1} = \mathrm{id}_Y$, we note that

$$s \circ s^{-1} = (\mathrm{id}_Y^{-1} \circ s) \circ (s^{-1} \circ \mathrm{id}_Y),$$

so we can represent the composite $s \circ s^{-1}$ via the diagram:

$$\begin{tikzcd}
Y \arrow[dr, equal] & & X \arrow[dl, "s"'] \arrow[dr, "s"] & & Y \arrow[dl, equal] \\
& Y \arrow[dr, equal] & & Y \arrow[dl, equal] & \\
& & Y & &
\end{tikzcd}$$

However, because the induced composite is represented by the roof $Y \xrightarrow{\mathrm{id}_Y} Y \xleftarrow{\mathrm{id}_Y} Y$, it follows that $s \circ s^{-1} = \mathrm{id}_Y$ as well, which completes the proof of the lemma. $\square$

**Theorem B.2.15.** *There is an isomorphism of categories*

$$\mathscr{H} \cong S^{-1}\mathscr{C}.$$

*Proof.* In view of Proposition B.2.13 and Lemma B.2.14, we only need to prove that $\mathscr{H}$ has the universal property of a localization. To show this, assume that $F : \mathscr{C} \to \mathscr{D}$ is a functor such that for all morphisms $s \in S$, $F(s)$ is an isomorphism in $\mathscr{D}$. We now must construct a functor $\widetilde{F} : \mathscr{H} \to \mathscr{D}$ which factors $F$ first through $\lambda_S$ and then through $\widetilde{F}$. Define $\widetilde{F} : \mathscr{H} \to \mathscr{D}$ first by taking, for all $X \in \mathscr{H}_0$,

$$\widetilde{F}X := FX$$

and then by taking, for any morphism of $\mathscr{H}$, represented as a roof $X \xrightarrow{f} Z \xleftarrow{s} Y$,

$$\widetilde{F}(X \xrightarrow{f} Z \xleftarrow{s} Y) = F(s)^{-1} \circ F(f).$$

The assignment of $\widetilde{F}$ on objects is well-defined because $F$ is, so it suffices to show that $\widetilde{F}$ is well-defined on morphisms. Thus we must prove that for any morphisms $(s, f), (t, g) \in H(X, Y)$ with $(s, f) \simeq (t, g)$, then

$$F(s)^{-1} \circ F(f) = F(t)^{-1} \circ F(g).$$

To do this, let $\alpha$ and $\beta$ be the morphisms witnessing the fact that $(s, f) \simeq (t, g)$, i.e., let $\alpha$ and $\beta$ be given such that the diagram

$$\begin{tikzcd}
& A \arrow[d, "\alpha"] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] & C & Y \arrow[ul, "s"'] \arrow[dl, "t"] \\
& B \arrow[u, "\beta"'] &
\end{tikzcd}$$

commutes with $\alpha \circ s \in S$. We thus calculate that

$$\begin{aligned}F(s)^{-1} \circ F(f) = F(\alpha \circ s)^{-1} \circ F(\alpha \circ f) = \widetilde{F}(X \xrightarrow{\alpha \circ f} C \xleftarrow{\alpha \circ s} Y) = \widetilde{F}(X \xrightarrow{\beta \circ g} C \xleftarrow{\beta \circ t})\\ = F(\beta \circ t)^{-1} \circ F(\beta \circ g) = F(t)^{-1} \circ F(g),\end{aligned}$$

which shows that $\widetilde{F}$ is well-defined on morphisms.

Let us now proceed to verify that $\widetilde{F}$ is a functor. To see that $\widetilde{F}$ preserves identities, we note that

$$\widetilde{F}(X \xrightarrow{\mathrm{id}_X} X \xleftarrow{\mathrm{id}_X} X) = F(\mathrm{id}_X)^{-1} \circ F(\mathrm{id}_X) = \mathrm{id}_{FX}^{-1} \circ \mathrm{id}_{FX} = \mathrm{id}_{FX}\,.$$

Now let $s^{-1} \circ f \in \mathscr{H}(X, Y)$ and $t^{-1} \circ g \in \mathscr{H}(Y, Z)$ be represented by the roofs

$$X \xrightarrow{f} A \xleftarrow{s} Y$$

and

$$Y \xrightarrow{g} B \xleftarrow{t} Z,$$

respectively. Let the composite $(t^{-1} \circ g) \circ (s^{-1} \circ f)$ be represented by the roof

$$X \xrightarrow{f' \circ f} C \xleftarrow{t' \circ s} Y$$

where $t', f'$ are induced from the diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \arrow[dr, "g"'] & & Z \arrow[dl, "t"] \\
& A \arrow[dr, "\exists f'"'] & & B \arrow[dl, "\exists t'"] & \\
& & C & &
\end{tikzcd}$$

in $\mathscr{C}$. Then we find that

$$\begin{aligned}\widetilde{F}(Y \xrightarrow{g} B \xleftarrow{t} Z) \circ \widetilde{F}(X \xrightarrow{f} A \xleftarrow{s} Y) = F(t)^{-1} \circ F(g) \circ F(s)^{-1} \circ F(f) = F(t)^{-1} \circ F(t')^{-1} \circ F(f') \circ F(f)\\ = F(s' \circ t)^{-1} \circ F(f' \circ f) = \widetilde{F}(X \xrightarrow{f' \circ f} C \xleftarrow{t' \circ t} Z).\end{aligned}$$

This shows that $\widetilde{F}$ is a functor. Furthermore, we verify immediately that

$$(\widetilde{F} \circ \lambda_S)(X) = \widetilde{F}(\lambda_S(X)) = \widetilde{F}X = FX$$

and

$$(\widetilde{F} \circ \lambda_S)(f) = \widetilde{F}(\lambda_S(f)) = \widetilde{F}(X \xrightarrow{f} Y \xleftarrow{\mathrm{id}_Y} Y) = F(\mathrm{id}_Y)^{-1} \circ F(f) = F(f)$$

for all morphisms $f$ of $\mathscr{C}$ and for all objects $X$ of $\mathscr{C}$, which proves that the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "F"] \arrow[dr, "\lambda_S"'] & & \mathscr{D} \\
& \mathscr{H} \arrow[ur, "\exists \widetilde{F}"'] &
\end{tikzcd}$$

commutes.

It now suffices to prove that $\widetilde{F}$ is the unique such functor factoring the diagram above. To this end assume that $G : \mathscr{H} \to \mathscr{D}$ is a functor making the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "F"] \arrow[dr, "\lambda_S"'] & & \mathscr{D} \\
& \mathscr{H} \arrow[ur, shift left, "G"] \arrow[ur, shift right, "\widetilde{F}"'] &
\end{tikzcd}$$

commute. It then follows immediately that for all objects $X$ of $\mathscr{H}$, $GX = FX = \widetilde{F}X$, so in order to prove that $\widetilde{F}$ and $G$ coincide it suffices to show that they have the same output on morphisms. Let $f \in \mathscr{C}_1$ and observe that the commutativity of the diagram above gives that

$$G(\lambda_S(f)) = F(f) = \widetilde{F}(\lambda_S(f)),$$

from which we deduce that

$$G(X \xrightarrow{f} Y \xleftarrow{\mathrm{id}_Y} Y) = \widetilde{F}(X \xrightarrow{f} Y \xleftarrow{\mathrm{id}_Y} Y) = F(f).$$

In particular, for all $s \in S$, we have that $G(X \xrightarrow{s} Y \xleftarrow{\mathrm{id}_Y} Y) = F(s) = \widetilde{F}(X \xrightarrow{s} Y \xleftarrow{\mathrm{id}_Y} Y)$. From this it follows that

$$\widetilde{F}(Y \xrightarrow{\mathrm{id}_Y} Y \xleftarrow{s} X) = F(s)^{-1} = G(Y \xrightarrow{\mathrm{id}_Y} Y \xleftarrow{s} X)$$

by the uniqueness of inverses. It then follows that

$$G(X \xrightarrow{f} Z \xleftarrow{s} Y) = G(Z \xrightarrow{\mathrm{id}_Z} Z \xleftarrow{s} Y) \circ G(X \xrightarrow{f} Z \xleftarrow{\mathrm{id}_Z} Z) = \widetilde{F}(Z \xrightarrow{\mathrm{id}_Z} Z \xleftarrow{s} Y) \circ \widetilde{F}(X \xrightarrow{f} Z \xleftarrow{\mathrm{id}_Z} Z),$$

which allows us to conclude that $G = \widetilde{F}$. Thus whenever $\widetilde{F} : \mathscr{C} \to \mathscr{D}$ is a functor for which $F(s)$ is an isomorphism in $\mathscr{D}$ for $s \in S$, there exists a unique functor $\widetilde{F}$ making the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "F"] \arrow[dr, "\lambda_S"'] & & \mathscr{D} \\
& \mathscr{H} \arrow[ur, dashed, "\exists!\widetilde{F}"'] &
\end{tikzcd}$$

commute. This shows that $S^{-1}\mathscr{C} \cong \mathscr{H}$ and concludes the proof. $\square$

**Remark B.2.16.** If a system $S$ of a category $\mathscr{C}$ admits both a left and right calculus of fractions, then we can simultaneously describe morphisms as either left fractions $s^{-1} \circ f$ or right fractions $f \circ s^{-1}$, depending on our needs at the time.

**Remark B.2.17.** If $S$ is a right Ore system in a category $\mathscr{C}$, then we will write $\mathscr{C}S^{-1}$ to denote the category of right fractions. This will not come up much, and the notation is nonstandard[15], but we will use it because of the visual metaphor for having right fractions in $S$.[16]

**Example B.2.18.** Let $S$ be the class of identity morphisms in $\mathscr{C}$. Then $S$ admits a calculus of left and right fractions, and $S^{-1}\mathscr{C} \cong \mathscr{C}$.

**Example B.2.19.** Let $\mathscr{C} = [\mathscr{D}^{\mathrm{op}}, \mathbf{Set}]$ be a presheaf topos on a category $\mathscr{D}$ and let $J$ be a Grothendieck topology on $\mathscr{D}$. Then if $S$ is the class of local epimophisms with respect to $J$, $S$ admits a calculus of right fractions and

$$\mathscr{C}S^{-1} \cong \mathbf{Shv}(\mathscr{D}, J).$$

[15] As much as notation can be nonstandard, anyway.

[16] Notationally, this is analogous to writing $R$-**Mod** for the category of left modules over a ring and **Mod**-$R$ for the category of right modules over a ring.

**Example B.2.20.** If $\mathscr{A}$ is an Abelian category and $S$ is the class of quasi-isomorphisms in $\mathbf{Ch}(\mathscr{A})$, then $S$ does not admit a calculus of left or right fractions. This can be fixed, however, by first passing to the homotopy category of $\mathbf{Ch}(\mathscr{A})$ and then localizing at quasi-isomorphisms. This will all be explored explicitly below.

We now move to conclude this chapter[17] with an important theorem. One of the most important aspects of a calculus of fractions is that in this case, the canonical functor $\lambda_S : \mathscr{C} \to S^{-1}\mathscr{C}$ preserves finite colimits. We will use this later to prove that if we have an additive category with a calculus of left fractions, then its localization is additive as well.

**Theorem B.2.21.** *Assume $\mathscr{C}$ is a category and $S \subseteq \mathscr{C}_1$ admits a calculus of left fractions. Then the natural functor $\lambda_S : \mathscr{C} \to S^{-1}\mathscr{C}$ preserves finite colimits.*

*Proof.* Begin by letting $I$ be the indexing category (which is finite because the colimits with which we work in this proof are assumed to be finite) and assume that

$$X = \varinjlim X_i$$

is the colimit along the diagram functor $D : I \to \mathscr{C}$ in $\mathscr{C}$ with colimit diagrams

$$\begin{tikzcd} X_i \arrow[rr, "\alpha_{ij}"] \arrow[dr, "\alpha_i"'] & & X_j \arrow[dl, "\alpha_j"] \\ & X & \end{tikzcd}$$

for all $i, j \in I$ and connecting morphisms $\alpha_{ij} = D(i \to j)$. Now consider the diagram $\lambda_S \circ D : I \to S^{-1}\mathscr{C}$ and let $Y$ be a cocone to the resulting diagram. From this assumption, it follows that for all $i, j \in I$ and for all morphisms $\lambda_S(\alpha_{ij})$, which in turn ar represented by roofs $X_i \xrightarrow{\alpha_{ij}} X_{ij} \xleftarrow{\mathrm{id}_{X_j}} X_j$, since $Y$ is a cocone, there are roofs $X_i \xrightarrow{f_i} Z_i \xleftarrow{s_i} Y$ and $X_j \xrightarrow{f_j} Z_j \xleftarrow{s_j} Y$ which fit into a commuting diagram

$$\begin{tikzcd} & X_j \arrow[dr, equal] & \\ X_i \arrow[ur, "\alpha_{ij}"] \arrow[d, "f_1"'] & & X_j \arrow[d, "f_j"] \\ Z_i & Y \arrow[l, "s_i"] \arrow[r, "s_j"'] & Z_j \end{tikzcd}$$

where we interpret the diagram as composition of roofs in $S^{-1}\mathscr{C}$ and $s_i, s_j \in S$. Now, because this diagram commutes by virtue of $Y$ being a cocone, there exists an object $C_{ij}$ and morphisms $\gamma_{ij}^i : Z_i \to C_{ij}$ and $\eta_{ij}^j : Z_j \to C_{ij}$ which fit into a commuting diamond

$$\begin{tikzcd} & Z_j \arrow[d, "\eta_{ij}^j"] & \\ X_i \arrow[ur, "f_j \circ \alpha_{ij}"] \arrow[dr, "f_i"'] & C_{ij} & Y \arrow[ul, "s_j"'] \arrow[dl, "s_i"] \\ & Z_i \arrow[u, "\gamma_{ij}^i"'] & \end{tikzcd}$$

[17] Perhaps better referred to as a monster. There are some monstrous diagrams and pyramids throughout here, and I admire the reader who has gone through them, even if their eyes glazed over the entire time! If this is you, go have a drink; you've earned it!

in $\mathscr{C}$ with $\gamma^i_{ij} \circ s_i \in S$. We can now, by the finiteness of the colimit, do a finite number of Ore replacements using the axioms for what it means to be a left Ore system as we did in the proofs of Lemmas B.2.9 and B.2.10, as well as in the proofs of Propositions B.2.8 and B.2.11, in order to find a single object $A$ together with morphisms $\varphi^i_{ij} : Z_i \to A$ and $\psi^j_{ij} : Z_j \to A$ which make the diamond

$$\begin{tikzcd}
 & Z_j \arrow[d, "\psi^j_{ij}"] & \\
X_i \arrow[ur, "f_j \circ \alpha_{ij}"] \arrow[dr, "f_i"'] & A & Y \arrow[ul, "s_j"'] \arrow[dl, "s_i"] \\
 & Z_i \arrow[u, "\varphi^i_{ij}"'] &
\end{tikzcd}$$

commute in $\mathscr{C}$ with $\varphi^i_{ij} \circ s_i$. From this we see that $A$ is a cocone

$$\begin{tikzcd}
X_i \arrow[rr, "\alpha_{ij}"] \arrow[dr, "\varphi^i_{ij} \circ f_i"'] & & X_j \arrow[dl, "\psi^j_{ij} \circ f_j"] \\
 & A &
\end{tikzcd}$$

of the diagram $D : I \to \mathscr{C}$. This allows us to find a unique morphism $\theta : X \to A$ making the diagram

$$\begin{tikzcd}
X_i \arrow[rr, "\alpha_{ij}"] \arrow[dr, "\alpha_i"] \arrow[ddr, bend right, "\varphi^i_{ij} \circ f_i"'] & & X_j \arrow[dl, "\alpha_j"'] \arrow[ddl, bend left, "\psi^j_{ij} \circ f_j"] \\
 & X \arrow[d, dashed, "\exists ! \theta"] & \\
 & A &
\end{tikzcd}$$

commute in $\mathscr{C}$. Thus we produce the roof $X \xrightarrow{\theta} A \xleftarrow{\varphi^i_{ij} \circ s_i} Y$ in $S^{-1}\mathscr{C}$. Furthermore, it follows from construction that the diagrams

$$\begin{tikzcd}
X_i \arrow[dr, "\alpha_i"'] & & X \arrow[dl, equal] \arrow[dr, "\theta"] & & Y \arrow[dl, "\varphi^i_{ij} \circ s_i"] \\
 & X \arrow[dr, "\theta"'] & & A \arrow[dl, equal] & \\
 & & A & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_j \arrow[dr, "\alpha_j"'] & & X \arrow[dl, equal] \arrow[dr, "\theta"] & & Y \arrow[dl, "\psi^j_{ij} \circ s_j"] \\
 & X \arrow[dr, "\theta"'] & & A \arrow[dl, equal] & \\
 & & A & &
\end{tikzcd}$$

which represent the compositions

$$((\varphi^i_{ij} \circ s_i)^{-1} \circ \theta) \circ (\mathrm{id}_X^{-1} \circ \alpha_i)$$

and

$$((\psi^j_{ij} \circ s_j)^{-1} \circ \theta) \circ (\mathrm{id}_X^{-1} \circ \alpha_j)$$

in $S^{-1}\mathscr{C}$, respectively, contract to the roofs

$$X_i \xrightarrow{f_i} A \xleftarrow{\varphi^i_{ij} \circ s_i} Y$$

and

$$X_j \xrightarrow{f_j} A \xleftarrow{\psi^j_{ij} \circ s_j} Y$$

by the universal property of map $\theta$. From this we conclude that the diagram

$$\begin{tikzcd}
 & X_j \arrow[dr, equal] & \\
X_i \arrow[ur, "\alpha_{ij}"] \arrow[r, "\alpha_i"'] \arrow[dr, "\alpha_i"] \arrow[dddd, "f_i"'] & X & X_j \arrow[l, "\alpha_j"] \arrow[dl, "\alpha_j"] \arrow[dddd, "f_j"] \\
 & X \arrow[u, equal] \arrow[d, dashed, "\exists!\theta"] & \\
 & A & \\
Z_i \arrow[ur, "\varphi^i_{ij}"] & Y \arrow[l, "s_i"] \arrow[r, "s_j"'] \arrow[u, "\varphi^i_{ij}\circ s_i"] & Z_j \arrow[ul, "\psi^j_{ij}"']
\end{tikzcd}$$

of roofs commutes in $S^{-1}\mathscr{C}$, which shows that $X$ remains a cocone to the diagram, and that every cocone factors through $X$. Furthermore, the fact that this factorization is unique is straightforward from here and so is omitted. This concludes the proof fo the theorem. $\square$

**Corollary B.2.22.** *If $S$ is a right Ore system in $\mathscr{C}$, then the natural functor $\mathscr{C} \to \mathscr{C}S^{-1}$ preserves finite limits. In particular, if $S$ is both a left and right Ore system, then $\lambda_S : \mathscr{C} \to S^{-1}\mathscr{C}$ preserves finite limits and finite colimits.*

### Exercises

**Exercise B.2.1.** Prove or disprove: every category $\mathscr{C}$ has a two-sided Ore system $S$.

**Exercise B.2.2.** Find an example of a category $\mathscr{C}$ and a left Ore system $S$ such that:

1. The functor $\lambda_S : \mathscr{C} \to S^{-1}\mathscr{C}$ is faithful;
2. The functor $\lambda_S : \mathscr{C} \to S^{-1}\mathscr{C}$ is not faithful (for this one you may want to check out the section on derived categories first).

**Exercise B.2.3.** Let $\mathscr{C}$ be a category and assume that $S$ is a left and right Ore system in $\mathscr{C}$. If $S^{-1}\mathscr{C}$ is the category of left fractions and if $\mathscr{C}S^{-1}$ is the category of right fractions, prove that $S^{-1}\mathscr{C} \cong \mathscr{C}S^{-1}$.

**Exercise B.2.4.** Find an example of a category $\mathscr{C}$ and a left calculus of fractions $S$ on $\mathscr{C}$ and a limit in $\mathscr{C}$ such that the natural functor $\lambda_S$ does not preserve the limit.

**Exercise B.2.5.** Find an example of a calculus of left fractions such that the localization category $S^{-1}\mathscr{C}$ is large even when $\mathscr{C}$ is locally small. Hint: This is easy if $\mathscr{C}$ is large.

**Exercise B.2.6.** Let $R$ be a ring with identity and let $S$ be a left Ore set in $R$. If $\mathfrak{R}$ is the category on one object with hom-set $\mathfrak{R}(*,*) = R$ and composition given by multiplication in $R$, prove that the category $S^{-1}\mathfrak{R}$ has $S^{-1}\mathfrak{R}(*,*) = S^{-1}R$.

**Exercise B.2.7.** Find an example of a category $\mathscr{C}$ and a class of morphisms which is both a left and right Ore system. Can you do this in a nontrivial example?

**Exercise B.2.8.** Find a set theoretic condition on a category of left (or right) fractions $S^{-1}\mathscr{A}$, where $\mathscr{A}$ is an Abelian category, which makes $S^{-1}\mathscr{A}$ into a locally small category. In particular, what should it mean to be "locally small on the left?"

**Exercise B.2.9.** Let $\mathscr{C}$ be a $\mathscr{U}$-small category for some Grothendieck universe $\mathscr{U}$ and let $S \subseteq \mathscr{C}_1$ be a left Ore system in $\mathscr{C}$. Let $X \in \mathscr{C}_0$ be an object and define the category $X \downarrow S$ as follows:

- Objects: Arrows $s : X \to Y$ where $s \in S$;
- Morphisms, Composition, Identities: As in the coslice category $X/\mathscr{C}$.

1. Prove that $X\backslash S$ is the comma category of the functors $\mathbb{1} \xrightarrow{X} \mathscr{C} \xleftarrow{i_S} S$ (so $X\backslash S = X \downarrow i_S$), where $\mathbb{1}$ is the terminal category, $X : \mathbb{1} \to \mathscr{C}$ is the functor which picks out the object $X$, and $i_S : S \to \mathscr{C}$ is the inclusion of $S$, regarded as a category, into $\mathscr{C}$.
2. Prove that the category $X\backslash S$ is filtered for any object $X \in \mathscr{C}_0$.
3. Define a functor $F_X : X\backslash S \to \mathbf{Set}$ (where we mean the category of $\mathscr{U}$-sets, of course[18]) by, for any object $Y \in (X\backslash S)_0$,
$$F_X(Y) := \mathscr{C}(X,Y)$$
and for any morphism $f : Y \to Z \in (X\backslash S)_1$,
$$F_X(f) := f_* : \mathscr{C}(X,Y) \to \mathscr{C}(X,Z).$$
Fix objects $C, D \in \mathscr{C}_0$. Prove that there is a natural morphism of sets
$$\rho_{C,D} : \operatorname*{colim}_{E \in (D\backslash S)} F_C(E) \longrightarrow (S^{-1}\mathscr{C})(C,D).$$
4. Prove that the morphism $\rho_{C,D}$ is an isomorphism and conclude that the homsets in $S^{-1}\mathscr{C}$ are calculated by filtered colimits.
5. Show that if $S$ is a left and right Ore system then
$$(S^{-1}\mathscr{C})(C,D) \cong \operatorname*{colim}_{E \in (D\backslash S)} F_C(E) \cong \operatorname*{colim}_{B \in (S \downarrow C)} F_B(D) \cong \operatorname*{colim}_{B \in (S\downarrow C)} \operatorname*{colim}_{E \in (D\backslash S)} F_B(E).$$
Use this to give an alternate proof of the fact that $S^{-1}\mathscr{C} \cong \mathscr{C}S^{-1}$.

**Exercise B.2.10.** In this exercise we'll get to know what are frequently called Serre subcategories in the literature[19], as they allow us to describe "kernels" of exact functors between Abelian categories.[20] In particular, Serre subcategories also allow us to define quotients of Abelian categories by certain subcategories[21]

[18]This is an example of what I mean by losing the forest for the trees. Look at that set theory tree we have to pay attention to and be annoyed.

[19]As with many things, the way people use these terms is frequently different. I've chosen to follow the terminology used in the Stacks Project, as it seems to be the most commonly agreed upon usage of the term. Sometimes people use the term "Serre subcategories" to describe what we've called "weak Serre subcategories," but I find it helpful to distinguish between the two and begin by getting to know the stricter concept, as it helps us see what the whole weakening process is doing.

[20]As usual, I'm ignoring size issues here. Set-theoretic tyranny has no place in modern enlightened mathematics, as if it's not a set just throw a Grothendieck universe (or transfinite induction of Grothendieck universes) at it until it is a set in that universe.

[21]In some sense Serre subcategories are normal Abelian subcategories, as they allow us to define an Abelian category $\mathscr{A}/\mathscr{S}$ which is given via a strict localization. This "normal" business should be seen as just a convenient formal analogy, however, and nothing more (or at least I'm not claiming anything more — if you can develop a theory of normal Abelian subcategories that would be really cool and I'll be impressed/stoked).

Let $\mathscr{A}$ be an Abelian category. We say that a subcategory $\mathscr{S}$ of $\mathscr{A}$ is a Serre subcategory if:

- $\mathscr{S}$ is nonempty and full;
- If
$$A \longrightarrow B \longrightarrow C$$
is an exact sequence in $\mathscr{A}$ for which $A, C \in \mathscr{S}_0$ then $B \in \mathscr{S}_0$ as well.

We will study these in detail now.

1. Prove that a Serre subcategory $\mathscr{S}$ of an Abelian category $\mathscr{A}$ is an Abelian subcategory (i.e., $\mathscr{S}$ is Abelian) and the inclusion functor $\mathscr{S} \hookrightarrow \mathscr{A}$ is exact.
2. Prove that $\mathscr{S}$ is a Serre subcategory of $\mathscr{A}$ if and only if the following hold:
   - $0 \in \mathscr{S}_0$;
   - $\mathscr{S}$ is strictly full, i.e., $\mathscr{S}$ is closed under isomorphisms in $\mathscr{A}$;
   - If $A \in \mathscr{S}_0$ then every subobject of $A$ in $\mathscr{A}$ is in $\mathscr{S}$ and every object $X$ with an epimorphism $A \to X$ in $\mathscr{A}$ is also an object in $\mathscr{S}$;
   - Given a short exact sequence $0 \to A \to B \to C \to 0$ with $A, C \in \mathscr{S}_0$, $B \in \mathscr{S}_0$ as well.
3. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Prove that the full subcategory $\mathscr{K}$ of $\mathscr{A}$ generated by the objects $X \in \mathscr{A}_0$ for which $FX \cong 0$ is a Serre subcategory of $\mathscr{A}$. Such a category $\mathscr{K}$ is sometimes called a Kernel of the functor $F$ and denoted $\operatorname{Ker}(F)$.
4. Let $R$ be a unital ring, $S$ a two-sided Ore system in $R$ (treat $R$ as a category with one object $*$, $\operatorname{Hom}(*, *) = R$, enriched in Abelian groups, and whose multiplication is induced by $r \circ r' = rr'$), and let $(R\text{-}\mathbf{Mod})_S$ denote the subcategory of left $R$-modules for which $S^{-1}A \cong 0$. Prove that $(R\text{-}\mathbf{Mod})_S$ is a Serre subcategory of $R\text{-}\mathbf{Mod}$.

**Exercise B.2.11.** Let $\mathscr{A}$ be an Abelian category. We say that a full subcategory $\mathscr{W}$ of $\mathscr{A}$ is a weak Serre subcategory if the following hold:

- $\mathscr{W}$ is nonempty;
- For any exact sequence
$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$
in $\mathscr{A}$, if $A_0, A_1, A_3, A_4 \in \mathscr{W}_0$ then $A_2 \in \mathscr{W}_0$ as well.

1. Prove that if $\mathscr{W}$ is a weak Serre subcategory of $\mathscr{A}$ then $\mathscr{W}$ is an Abelian subcategory of $\mathscr{A}$ and that the inclusion $\mathscr{W} \to \mathscr{A}$ is exact.
2. Prove that $\mathscr{W}$ is a weak Serre subcategory of $\mathscr{A}$ if and only if the following hold:
   - $0 \in \mathscr{W}_0$;
   - $\mathscr{W}$ is a strictly full subcategory of $\mathscr{A}$;
   - If $f \in \mathscr{W}_1$ then $\operatorname{Ker}(f), \operatorname{Coker}(f) \in \mathscr{W}_0$.
   - If $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathscr{A}$ with $A, C \in \mathscr{W}_0$ then $B \in \mathscr{W}_0$ as well.

# Appendix C

# Additive and Abelian Categories: A Universal but Not-So-Quick Introduction

## C.1 What are Abelian Categories and Wherefore art they?

We begin our journey into understanding derived categories and perverse sheaves by learning about Abelian categories. Abelian categories should be seen as categories that "look like" module categories, for a given amount of "look like." In particular, if a category is to look like modules we should see the following things:

- For any two objects $A$ and $B$ of $\mathscr{C}$, the hom-set $\mathscr{C}(A,B)$ should be an Abelian group, i.e., $\mathscr{C}$ is **Ab**-enriched;
- $\mathscr{C}$ has a zero object (a zero module, if you will);
- Finite products and coproducts in $\mathscr{C}$ should coincide ($\mathscr{C}$ should have finite direct sums);
- $\mathscr{C}$ should be complete and cocomplete;
- $\mathscr{C}$ should have the property that every monic is a kernel and every epic is a cokernel ($\mathscr{C}$ should be a regular category).

We will now move through these properties and get to know them and get to know them, as well as their necessity to define an Abelian category, by seeing what happens if we do not require them. While we will certainly end up with useful categories if we do not require certain axioms (most notably with additive categories, as we will see below), we will not arrive at categories that have the properties we desire. For instance, one important property of Abelian categories is that the image of a morphism coincides with the coimage of that same morphism (which is to say that a First Isomorphism Theorem holds); this is not in general true if we don't have that every monic is a kernel.

Before examining the truth of my claims above, let us make some definitions of convenience

**Definition C.1.1.** An object $Z$ is a *zero object* in a category $\mathscr{C}$ if $Z$ is both initial and terminal.

**Definition C.1.2.** A category $\mathscr{C}$ is said to have *biproducts* if $\mathscr{C}$ admits finite products and finite coproducts, and if there is an isomorphism
$$A \times B \cong A \coprod B$$
for all objects $A$ and $B$ of $\mathscr{C}$.

The first necessary axiom of being an Abelian category, and one on which we will build everything, is that the category $\mathscr{A}$ is **Ab**-enriched. If it is impossible to add morphisms together, we have no real hope of getting things running. However, we will see that just because a category $\mathscr{C}$ is **Ab**-enriched, does *not* mean that has either zero objects *or* biproducts; in general zero objects do not imply the existence of biproducts.

**Example C.1.3** (Being **Ab**-enriched implies minimal structure)**.** Let $R$ be a ring and let $\mathfrak{R}$ be the category with one object whose morphism composition rule is given by multiplication in $R$, i.e., $\mathfrak{R}_0 = \{*\}$ and $\mathfrak{R}_1 = R$ with

$$f \circ g := fg$$

for all $f, g \in R$. Then $\mathfrak{R}$ is **Ab** enriched because $R$ is an Abelian group under addition and this plays well with composition by the distributive laws, but $\mathfrak{R}$ does not have a zero object if $|R| \geq 2$.

**Example C.1.4** (Having a zero object does not imply biproducts)**.** Consider the category $^{*}/\,\mathbf{Top} =: \mathbf{Top}_*$ of pointed topological spaces:

- Objects: Functions $x : \{*\} \to X$, where $X$ is a topological space;
- Morphisms: Continuous functions $f : X \to Y$ between pointed spaces making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & \{*\} \arrow[ul, "x"] \arrow[ur, "y"'] & \end{tikzcd}$$

  commute in **Top**;
- Composition and identities: As in **Top**.

Then it is easily seen that $\mathrm{id}_* : \{*\} \to \{*\}$ is the zero object in $\mathbf{Top}_*$, but $\mathbf{Top}_*$ does not admit biproducts because the coproduct in $\mathbf{Top}_*$ is not equal to the product in $\mathbf{Top}_*$.

While the above example is unfortunate, as it shows that zero objects and biproducts are distinct in general, it will not be particularly applicable to us for one big reason: $\mathbf{Top}_*$ is not **Ab**-enriched! We will now prove that if a category is **Ab**-enriched, has a zero object, and has finite products, then it has finite biproducts.

**Proposition C.1.5.** *Let $\mathscr{C}$ be an* **Ab***-enriched category with a zero object and with finite products. The category $\mathscr{C}$ then has finite biproducts.*

*Proof.* Write the zero object as 0 and let $X, Y \in \mathscr{C}_0$ be arbitrary. We must show that $A \times B$ is a coproduct, so in particular we must first find morphisms $X \to X \times Y$ and $Y \to X \times Y$. Consider the morphisms $0_{XY} : X \to Y$ and $0_{YX} : Y \to X$ which factor as in the diagrams

$$\begin{tikzcd} X \arrow[rr, "0_{XY}"] \arrow[dr] & & Y \\ & 0 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "0_{YX}"] \arrow[dr] & & X \\ & 0 \arrow[ur] & \end{tikzcd}$$

above. Then we can produce the spans $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{0_{XY}} Y$ and $X \xleftarrow{0_{YX}} Y \xrightarrow{\mathrm{id}_Y} Y$ to induce the maps $i_X : X \to X \times Y$ and $i_Y : Y \to Y \times X$ which make the diagrams

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[d, dashed, "\exists! i_X"'] \arrow[dr, "0_{XY}"] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dl, "0_{YX}"'] \arrow[d, dashed, "\exists! i_Y"] \arrow[dr, equal] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

commute, respectively.

We now assume that there is a cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathscr{C}$. Define the morphism $\varphi : X \times Y \to Z$ by

$$\varphi := f \circ \pi_1 + g \circ \pi_2.$$

Then we can check that

$$\varphi \circ i_X = (f \circ \pi_1 + g \circ \pi_2) \circ i_X = f \circ (\pi_1 \circ i_X) + g \circ (\pi_2 \circ i_X) = f \circ \mathrm{id}_X + g \circ 0 = f$$

and similarly

$$\varphi \circ i_Y = g.$$

This proves that the diagram

$$\begin{tikzcd} & Z & \\ X \arrow[ur, bend left, "f"] \arrow[r, "i_X"'] & X \times Y \arrow[u, "\varphi"] & Y \arrow[ul, bend right, "g"'] \arrow[l, "i_Y"] \end{tikzcd}$$

commutes; verifying the uniqueness of $\varphi$ is routine, and hence omitted. $\square$

The prior proposition is important for the following reason: It tells us that the **Ab**-enriched structure, together with the zero object of $\mathscr{C}$, imply that finite products are equivalent to finite coproducts. While this does not tell us anything about the existence, or lack thereof, of infinite products and coproducts, it does mean that giving the existence of finite products (or dually coproducts) in an **Ab**-enriched category with is enough to guarantee the existence of finite biproducts. This leads us to our next definition, which is that of an Additive category: An **Ab**-enriched category with finite biproducts (with the nullary biproduct assumed to be the zero object). While these categories will be shown to *not* have the full structure of a module category, and hence be missing the desiderata of an Abelian category, they do have a reasonable amount of structure to them, and one can *almost* do homological algebra within an additive category. However, many of the categories we meet in nature in derived algebraic geometry are not Abelian, but simply additive, so we should get comfortable with them now.

**Definition C.1.6.** A category $\mathscr{C}$ is an *additive category* if $\mathscr{C}$ is **Ab**-enriched and has all finite biproducts.

**Example C.1.7.** If $R$ is a ring, any category $R$-**Mod** of left $R$-modules is additive. In fact, the full subcategory $R$-$\mathbf{Mod}_{\text{free}}$ of $R$-**Mod** generated by free left $R$-modules is an additive subcategory of $R$-**Mod**.

**Example C.1.8.** The categories **Crng** and **Rng** of nonuntial (commutative) rings are both additive.

**Example C.1.9.** Let $R$ be a PID of characteristic zero which is not a field and consider the category $R$-$\mathbf{Mod}_{\text{free}}$ of free $R$-modules. Then choose $\mathfrak{p} \in |\operatorname{Spec} R|$ to be a nonzero prime ideal of $R$ for which $R/\mathfrak{p}$ has characteristic $p > 0$. Then $\mathfrak{p} \cong R$ as an $R$-module, and so $\mathfrak{p}$ is an object of $R$-$\mathbf{Mod}_{\text{free}}$. Now consider the embedding $\mathfrak{p} \to R$. This map does not have a cokernel in $R$-$\mathbf{Mod}_{\text{free}}$, as any cokernel would be a coequalizer with the zero morphism, and hence $\operatorname{Coker}(\mathfrak{p} \to R) = R/\mathfrak{p}$. However, since the characteristic of $R/\mathfrak{p}$ is distinct from that of $R$, $R/\mathfrak{p}$ is not free and so $R$-$\mathbf{Mod}_{\text{free}}$ cannot be a module category.

**Remark C.1.10.** The assumption that the pair $(R, R/\mathfrak{p})$ in the above example are of mixed characteristic is simply a sufficient condition, but not at all necessary. In fact, for any $R$ and any nonzero $\mathfrak{p}$, $R/\mathfrak{p}$ is not a free $R$-module by a dimension argument, as if $R/\mathfrak{p}$ were to be free, it would have to be the zero-module (which it is not). This fails, however, if $R$ is a field or a division ring, as all (left) modules over a field or division ring are free.

The above example shows that if we want to get to Abelian categories, it is not enough to stop at additive categories[1]. However, all is not lost: Being an additive category allows for a slick formulation of calculating limits and colimits, and an easy way to check if an additive category is (finitely) complete or cocomplete. We will lean on one well-known categorical result for this, which I will state here but without proof. The interested reader can find a proof of this result in any reasonable first text on category theory.

[1]This is sad for us, as it means we have to do more work.

**Theorem C.1.11.** *Let $\mathscr{C}$ be a category and assume that $\mathscr{C}$ admits all equalizers and all products of size less than or equal to some strongly inaccessible cardinal $\kappa$*[2]*. Then $\mathscr{C}$ has all limits of size less than or equal to $\kappa$.*

The reason we use this theorem is that, together with the fact that all additive categories have all finite biproducts, we can deduce the (co)completeness of any additive category by testing whether or not it has equalizers. Moreover, equalizers have a particularly nice form in additive categories: They are kernels! We will explain what this means precisely below, but the take-away is Proposition C.1.16. While we do not explicitly define cokernels, they are the categorical duals to kernels; their explication is left as Exercise C.1.6.

**Definition C.1.12.** A *kernel* of a morphism $f : A \to B$ in an additive category $\mathscr{A}$ is an object $\operatorname{Ker} f$ together with a morphism $\ker f : \operatorname{Ker} f \to A$ such that $f \circ \ker f = 0$ and if $g : Z \to A$ is any morphism such that $f \circ g = 0$, then there exists a unique morphism $k : Z \to \operatorname{Ker} f$ making the diagram

$$\begin{tikzcd} & B & \\ & A \arrow[u, "f"] & \\ \operatorname{Ker} f \arrow[uur, bend left, "0"] \arrow[ur, "\ker f"'] & & Z \arrow[ul, "g"] \arrow[uul, bend right, "0"'] \arrow[ll, dashed, "\exists! k"] \end{tikzcd}$$

commute in $\mathscr{A}$.

**Remark C.1.13.** Because kernels are technically a pair of information (an object $K$ and a morphism $k : K \to \operatorname{Dom} f$) we will denote the object of the kernel by $\operatorname{Ker} f$ and the morphism $\ker f$ of the kernel by $\ker f : \operatorname{Ker} f \to \operatorname{Dom} f$. Dually, we will write $\operatorname{Coker} f$ for the object of the cokernel of $f$ and $\operatorname{coker} f$ for the morphism $\operatorname{coker} f : \operatorname{Codom} f \to \operatorname{Coker} f$.

**Lemma C.1.14.** *An additive category has all equalizers if and only if it has all kernels.*

*Proof.* Since it is clear that a kernel is an equalizer of $f$ and $0$ (check this!), it suffices to prove that an equalizer can be reduced to a kernel. Assume that $E(f, g)$ is the equalizer of $f$ and $g$ with equalizing morphism $e : E(f, g) \to A$. Now if $h : Z \to A$ is any morphism for which $f \circ h = g \circ h$, consider that

$$(f - g) \circ h = f \circ h - g \circ h = 0 = 0 \circ h$$

and similarly, if $(f - g) \circ h = 0 \circ h = 0$, then $f \circ h = g \circ h$. It is routine from here to show that the universal property of the equalizer $E(f, g)$ is the same as the universal property of $\operatorname{Ker}(f - g)$ using this identification. This proves that $E(f, g) \cong \operatorname{Ker}(f - g)$ and completes the lemma. $\square$

**Corollary C.1.15.** *The kernel morphism* $\ker f : \operatorname{Ker} f \to A$ *is monic.*

**Proposition C.1.16.** *An additive category $\mathscr{A}$ has all (finite) (co)limits if and only if it admits all (co)kernels.*

**Remark C.1.17.** Example C.1.9 and Exercise C.1.4 show that additive categories do not necessarily have finite limits or finite colimits. In particular, they can fail to have (co)equalizers.

The above remark shows that if we want Abelian categories to have finite limits and finite colimits, and we do, we need to axiomatize that property, i.e., it is not sufficient to simply work with additive categories. We could then ask if it is sufficient to stop by assuming we have all (co)kernels; however, the example below shows this is not the case. While we could separate the categories that have these properties, which has

[2]If you are uncomfortable with this, do not worry. The choice of using a strongly inaccessible cardinal is to be in line with the formalism of Grothendieck universes; it would really suffice to use any regular or limit cardinal for this theorem. Or you can think of this as admitting all limits indexed by all sets, for whatever that means, and ignore the foundational issues.

been done in the past[3], it will not be useful for us to do so. Instead we will simply show that we need one final axiom. The last place for Abelian categories to fail is that monics and epics need not behave sufficiently "regularly" in the sense that either monics or epics can fail to be kernels or cokernels. For the reader familiar with category theory, this is to say that (co)complete additive categories need not be regular categories. This will be an issue when we want to have things like diagram chasing lemmas and canonical image factorizations (cf. Theorem C.2.1, Lemma C.2.18, and Lemma C.2.22, amongst other results), so we will need to axiomatize this away as well.

**Example C.1.18.** Let $K$ be a locally compact complete metric field (for instance, $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$, or finite extensions $F/\mathbb{Q}_p$) and let $\mathbf{Ban}_K$ be the category of Banach spaces over $K$. It is easy to check that this category is additive and (co)complete, but let us see that epimorphisms can fail to be cokernels. Let $X$ be a compact subspace of $K$ and consider the Banach algebra $C(X)$ of continuous functions on $X$. This is a proper dense subalgebra of $L^1(X)$, and the embedding $\varepsilon : C(X) \to L^1(X)$ is easily seen to be epic by virtue of the universal property of completions. However, this is not a cokernel, as the cokernel of a morphism $f : V \to W$ is $V/\overline{\operatorname{Im} f}$, where $\overline{\operatorname{Im} f}$ is the closure of the image of $f$ in $W$, and by density $L^1(X)/\overline{\operatorname{Im} \varepsilon} = 0$ while $\varepsilon$ is not surjective.

This leads us finally to Abelian categories! We will present the axioms that define Abelian categories and then spend the remainder of this section on examples.

**Definition C.1.19.** An Abelian category is a category $\mathscr{A}$ such that:

- $\mathscr{A}$ is **Ab**-enriched;
- $\mathscr{A}$ has a zero object and finite biproducts;
- $\mathscr{A}$ is (finitely) complete and cocomplete;
- $\mathscr{A}$ has the property that every monomorphism is a kernel and every epimorphism is a cokernel.

**Example C.1.20.** If $R$ is a ring, the category $R$-**Mod** of left $R$-modules is an Abelian category with all **Set**-indexed (co)limits.

**Example C.1.21.** If $\mathcal{E}$ is an elementary topos, the category $\mathcal{E}(\mathbf{Ab})$ of internal Abelian groups is an Abelian category with (co)limits of whatever size admitted within $\mathcal{E}$.

**Example C.1.22.** Let $X = (|X|, \mathcal{O}_X)$ be a locally ringed space. Then the category $\mathcal{O}_X$-**Mod** of modules over $\mathcal{O}_X$ is an Abelian category.

**Example C.1.23** (This is a subexample of Example C.1.21)**.** Let $X$ be a topological space and let $\mathbf{Shv}_{\mathbf{Ab}}(X)$ denote the category of sheaves of Abelian groups on $X$. Then $\mathbf{Shv}_{\mathbf{Ab}}(X)$ is Abelian. More generally, if $R$ is a ring the category $\mathbf{Shv}_{R\text{-}\mathbf{Mod}}(X)$ is an Abelian category.

**Example C.1.24.** If $\mathscr{A}$ is an Abelian category and $\mathscr{C}$ is a category such that $[\mathscr{C}, \mathscr{A}]$[4] is a locally small category, then $[\mathscr{C}, \mathscr{A}]$ is Abelian. In particular, this holds for small categories $\mathscr{C}$.

**Example C.1.25** (This is an important example)**.** If $\mathscr{A}$ is an Abelian category, define the category $\mathbf{Ch}(\mathscr{A})$ as follows:

- Objects: Integer graded pairs $A^\bullet = (A^n, \partial_n)_{n\in\mathbb{Z}}$, where the $A^n$ are all objects in $\mathscr{A}$ and the $\partial_n$ are morphisms $\partial_n \in \mathscr{A}(A^n, A^{n+1})$ such that for all $n \in \mathbb{Z}$, $\partial_{n+1} \circ \partial_n = 0$. These are visualized as (co)chain complexes

$$\cdots \longrightarrow A^{n-1} \xrightarrow[\partial_{n-1}]{} A^n \xrightarrow[\partial_n]{} A^{n+1} \longrightarrow \cdots$$

  in which any composite of morphisms is the zero map.

[3]Such categories have been called pre-Abelian. This terminology is like a teenager with access to beer, a social media platform, and a public declaration of love: awkward.

[4]The category $[\mathscr{C}, \mathscr{A}]$ is the functor category $\mathbf{Fun}(\mathscr{C}, \mathscr{A})$; we use the $[-,-]$ notation to emphasize that it is the internal hom-functor in **Cat**, and hence to abuse the Cartesian closure of **Cat**.

- Morphisms: Integer graded maps $\varphi = (\varphi_n)_{n\in\mathbb{Z}} : A^\bullet \to B^\bullet$ where $\varphi_n \in \mathscr{A}(A^n, B^n)$ for all $n \in \mathbb{Z}$ such that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & A^{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\varphi_{n-1}"'] & A^n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"'] & A^{n+1} \arrow[r] \arrow[d, "\varphi_{n+1}"] & \cdots \\ \cdots \arrow[r] & B^{n-1} \arrow[r, "\delta_{n-1}"'] & B^n \arrow[r, "\delta_n"'] & B^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

  commutes.

- Composition: As in $\mathscr{A}$.

- Identities: $\mathrm{id}_{A^\bullet} = (\mathrm{id}_{A^n})_{n\in\mathbb{Z}}$.

The category $\mathbf{Ch}(\mathscr{A})$ is an Abelian category. This is the category of (co)chain complexes in $\mathscr{A}$. We will see this category as a crucial tool with which to understand the base category $\mathscr{A}$ (especially in Appendix D).

We present a short construction/definition about chain complex categories here for later use, as it is extremely important in practice.

**Proposition C.1.26.** *Let $\mathscr{A}$ and $\mathscr{B}$ be additive categories with $F : \mathscr{A} \to \mathscr{B}$ a functor. Then there is a unique functor, by abuse of notation also called $F : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{B})$ for which the diagrams*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "\mathrm{incl}_n"'] & \mathscr{B} \arrow[d, "\mathrm{incl}_n"] \\ \mathbf{Ch}(\mathscr{A}) \arrow[r, "F"'] & \mathbf{Ch}(\mathscr{B}) \end{tikzcd}$$

*commute for all $n \in \mathbb{Z}$ where the functors $\mathrm{incl}_n : \mathscr{A} \to \mathbf{Ch}(\mathscr{A})$, $\mathrm{incl}_n : \mathscr{B} \to \mathbf{Ch}(\mathscr{B})$ are the degree $n$-inclusion, i.e., $\mathrm{incl}_n$ are given by sending objects and morphisms to complexes and morphisms concentrated in degree $n$.*

**Definition C.1.27.** If $F : \mathscr{A} \to \mathscr{B}$ is a functor then the functor $F : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{B})$ Prolongment

We now would like to close this section by giving some definitions and proving some basic, but useful, results about Abelian categories. While the first two results we give are corollaries of Definition C.1.19, it is worth writing them down in order to emphasize their importance. After this we will move to define what it means to be the image and coimage of a morphism in an Abelian category, and then try to prove two results: The first is that the image and coimage of a morphism coincide, and the second is that a morphism is an isomorphism in an Abelian category if and only if it is epic and monic. While the second property is true in any regular category, we will give the proof here to both illustrate how to reason about Abelian categories, but also to give some foreshadowing towards the next topic we will cover: The Epic/Monic factorization system.

The next proposition we cover will need one basic assumption: We assume that the morphism $f$ we discuss is nonzero. While there are certainly analogous results, the statement of the proposition is much cleaner if we take $f$ to be nonzero and worry about the zero case later.

**Proposition C.1.28.** *Let $f \in \mathscr{A}(A, B)$ be a nonzero morphism. The following are equivalent:*

1. *$f$ is monic;*

2. *post-composition by $f$ sends nonzero maps to nonzero maps, i.e., $f \circ g = 0$ if and only if $g = 0$;*

3. *The kernel of $f$ is zero, i.e., $\operatorname{Ker} f = 0$ and $\ker f : \operatorname{Ker} f \to A$ is the map $0 : 0 \to A$.*

*Proof.* (1) $\implies$ (2): Assume that $f$ is monic and let $g : U \to A$ be a morphism in $A$ such that $f \circ g = 0$. However, since $\varphi \circ 0 = 0 = 0 \circ \varphi$ for any morphism $\varphi \in \mathscr{A}_1$, it follows that $f \circ g = 0 = f \circ 0$; using that $f$ is monic gives $g = 0$ and hence proves (2).[5]

(2) $\implies$ (3): Assume that there exists a morphism $g : U \to A$ making the diagram

$$\begin{tikzcd} U \arrow[r, "g"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

commute, i.e., $f \circ g = 0$. Then since $f$ sends nonzero maps to nonzero maps, it must be the case that $g$ is the zero map and hence factors uniquely through the diagram:

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr, dashed, "\exists!"'] & & A \arrow[r, "f"] & B \\ & 0 \arrow[ur, dashed, "\exists!"'] & & \end{tikzcd}$$

This in turn implies that $\operatorname{Ker} f = 0$ and proves (3).

(3) $\implies$ (1): Find $g, h \in \mathscr{A}(U, A)$ such that $f \circ g = f \circ h$. Using the bilinear composition of morphisms, it then follows that $f \circ (g - h) = 0$ and so $g - h$ must factor through the kernel of $f$. However, since $\operatorname{Ker} f = 0$, it follows that $g - h = 0$ and hence $g = h$. This proves (1). $\square$

Dually we get the following proposition (whose proof is left as an exercise in explicating dual statements involving cokernels).

**Proposition C.1.29.** *Let $f \in \mathscr{A}(A, B)$ be a nonzero morphism. The following are equivalent:*

1. *$f$ is epic;*
2. *pre-composition by $f$ sends nonzero morphisms to nonzero morphisms, i.e., $g \circ f = 0$ if and only if $g = 0$;*
3. $\operatorname{Coker} f = 0$.

**Remark C.1.30.** The statements for Propositions C.1.28 and C.1.29 only require the assumption of $f$ being nonzero for the second claims (about pre-composition and post-composition preserving nonzero maps), respectively. If $f$ is allowed to be the zero morphism, then points (1) and (3) hold in both cases (save now the only way they can be monic and epic is if either $\operatorname{Dom} f = 0$ or $\operatorname{Codom} f = 0$, respectively).

**Proposition C.1.31.** *Let $\mathscr{A}$ be an Abelian category and consider the pullback diagram:*

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

*Then:*

1. *The object and morphism pair* $(\operatorname{Ker} p_2, p_1 \circ \ker p_2)$ *is a kernel of $f$;*
2. *The map $f$ is monic if and only if $p_2$ is.*

[5] We've technically shown $f \circ 0 = 0$ by the remark involving $\varphi$, giving the other direction of the "if and only if" statement.

*Proof.* We first prove (1). Let $X$ be an object in $\mathscr{A}$ together with a morphism $h : X \to A$ for which $f \circ h = 0$. Then the diagram

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

commutes in $\mathscr{A}$, so there exists a unique morphism $\theta : X \to P$ making the diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\theta"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "h"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

commute. However, since $p_2 \circ \theta = 0$, there exists a unique map $\eta : X \to \operatorname{Ker} p_2$ making the diagram

$$\begin{tikzcd} \operatorname{Ker} p_2 \arrow[r, "\ker p_2"] & P \arrow[r, shift left, "p_2"] \arrow[r, shift right, "0"'] & C \\ X \arrow[u, dashed, "\exists!\eta"] \arrow[ur, "\theta"'] & & \end{tikzcd}$$

commute. However, since this factorizes the unique map $\theta$, we derive from the pullback square that

$$h = p_1 \circ \theta = p_1 \circ \ker p_2 \circ \eta.$$

In particular, this allows us to deduce that

$$0 = f \circ h = f \circ p_1 \circ \ker(p_2) \circ \eta$$

and hence provides the commuting diagram

$$\begin{tikzcd} \operatorname{Ker} p_2 \arrow[r, "p_1 \circ \ker p_2"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & C \\ X \arrow[u, dashed, "\exists!\eta"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Because the map $\eta$ was constructed uniquely, it follows that $(\operatorname{Ker} p_2, p_1 \circ \ker p_2)$ is a kernel to $p_1$ as well. This proves (1).

For (2), we consider that in any category, if $f$ is monic, so is $p_2$ automatically. On the other hand, if $p_2$ is monic, then $\operatorname{Ker} p_2 = 0$ and so $p_1 \circ \ker p_2 = p_1 \circ 0 = 0$. Because the morphism 0 is monic if and only if $\operatorname{Dom} 0 = 0$ and the kernel map is always monic, it follows that $\operatorname{Ker} f = 0$ as well. $\square$

In an Abelian category we have notions of an image of a morphism. Let us see how to define this. In the category $R$-**Mod**, it is easy to define the image: For modules $M$ and $N$ and a homomorphism $f : M \to N$, the image of $f$ is $\operatorname{Im}(f) = \{f(m) \mid m \in M\}$, and this is canonically a submodule of $N$. More generally, we cannot do this trick, as objects in a generic Abelian category need not be literal sets. For instance, given an arbitrary elementary topos $\mathcal{E}$ (such as even the category of left $G$-sheaves for a group sheaf $G$ in the category of sheaves on a site), the objects of $\mathcal{E}(\mathbf{Ab})$ are not sets and the image is much more difficult to describe. However, let us consider the $R$-**Mod** case again: For any morphism $f : M \to N$, the cokernel

Coker $f = N/f(M)$ exists and is defined in terms of the image. Moreover, the image $\operatorname{Im} f = f(M)$ is completely determined as the kernel of this morphism[6]. Thus we can define images by using only kernels and cokernels, and both those are objects/morphisms which exist in any Abelian category!

**Definition C.1.32.** Let $\mathscr{A}$ be an Abelian[7] category and let $f \in \mathscr{A}_1$. Then the *image of* $f$ is defined to be the object $\operatorname{Im} f := \operatorname{Ker}(\operatorname{coker} f)$. Dually, the *coimage of* $f$ is defined to the object $\operatorname{Coim} f := \operatorname{Coker}(\ker f)$.

Immediate from the fact that kernels are equalizers we get the following proposition and its dual:

**Proposition C.1.33.** *The image* $\operatorname{Ker}(\operatorname{coker} f)$ *of a morphism* $f$ *in an Abelian category is a subobject of* $\operatorname{Codom} f$*.*

*Proof.* Because $\operatorname{Ker}(\operatorname{coker} f)$ is a kernel and hence an equalizer of $\operatorname{coker} f : \operatorname{Codom} f \to \operatorname{Coker} f$, there is a canonical monomorphism $\ker(\operatorname{coker} f) : \operatorname{Ker}(\operatorname{coker} f) \to \operatorname{Codom} f$ which realizes the image of $f$ as a subobject of $\operatorname{Codom} f$. ☐

**Proposition C.1.34.** *If* $f : A \to B$ *is a morphism in an Abelian category, there is a canonical epimorphism* $A \to \operatorname{Coim} f$*.*

This gets us close to factorizing maps in an arbitrary Abelian category, as we have shown that given any morphism $f$ there is a canonical diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ \operatorname{Coker}(\ker f) & \operatorname{Ker}(\operatorname{coker} f) \arrow[u] \end{tikzcd}$$

where the arrow $A \to \operatorname{Coker}(\ker f)$ is epic and the arrow $\operatorname{Ker}(\operatorname{coker} f)$ is monic. If we just had an isomorphism $\operatorname{Coker}(\ker f) \to \operatorname{Ker}(\operatorname{coker} f)$ we would be able to factorize arbitrary morphisms! This is the subject of our next section, and we will explore this factorization business in detail there.

## Exercises

**Exercise C.1.1** (An easy warm-up exercise)**.** Prove that any **Ab**-enriched category is locally small. Find an example of a small **Ab**-enriched category and an **Ab**-enriched category that is not small.

**Exercise C.1.2.** Prove that the coproduct and product in $\mathbf{Top}_*$ do not agree. Hint: The product is the product of $X$ and $Y$ in $\mathbf{Top}_*$ is the product of the maps $x : \{*\} \to X$ and $y : \{*\} \to Y$, while the coproduct $X \vee Y$ (often called the pinch product of $X$ and $Y$) is the disjoint union of $X$ and $Y$ glued along the minimal equivalence relation $\sim$ making $x = y$ in $(X \sqcup Y)/\sim$.

**Exercise C.1.3.** Prove the dual of Proposition C.1.5: If $\mathscr{C}$ is an **Ab**-enriched category with a zero object 0, then if $A \coprod B$ exists in $\mathscr{C}$, $A \coprod B \cong A \times B$.

**Exercise C.1.4.** Prove that a category $\mathscr{A}$ is additive if and only if $\mathscr{A}^{\mathrm{op}}$ is additive as well.

**Exercise C.1.5.** Find an example of an additive category which does not have arbitrary biproducts. Hint: Proposition C.1.5 says that finite products are biproducts, so you'll need to look at infinite products. You should already be familiar with many categories that have this property, but you should come to terms with it explicitly.

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[6]In fact, every subobject of $N$ is determined as the kernel of an epimorphism with domain $N$. There is a Yoneda-style argument hiding here.

[7]Technically, one can define images in additive categories as well; however, there is no reason a priori to assume they exist as objects in the additive category, so one needs to make sure that the morphism $f$ admits both a kernel and a cokernel.

**Exercise C.1.6.** Write out explicitly what it means to be a cokernel in an additive category, and prove that cokernels and coequalizers coincide.

**Exercise C.1.7.** Find an Abelian category which has finite (co)limits but not **Set**-indexed (co)limits.

**Exercise C.1.8** (For those who know some topos theory and internal algebra)**.** Prove that if $\mathcal{E}$ is an elementary topos, and if $R$ is an internal ring to $\mathcal{E}$, then the category of internal left $R$-modules $\mathcal{E}(R\text{-}\mathbf{Mod})$ is an Abelian category.

**Exercise C.1.9.** Prove that $\mathbf{Ch}(\mathscr{A})$ is an Abelian category.

**Exercise C.1.10.** Prove or disprove: An additive category $\mathscr{A}$ for which every morphism admits images and coimages is an Abelian category.

**Exercise C.1.11.** Prove that in any Abelian category, the initial morphism $0 \to A$ is monic and that the terminal morphism $A \to 0$ is epic for any object $A$.

**Exercise C.1.12.** Prove that a morphism $f$ in an Abelian category is an isomorphism if and only if it is both monic and epic. Hint: Use Propositions C.1.28 and C.1.29.

**Exercise C.1.13.** Prove that there are functors $D_n : \mathscr{A} \to \mathbf{Ch}(\mathscr{A})$ for all $n \in \mathbb{Z}$ given by embedding an object $A$ into the complex whose objects are given by

$$A^k := \begin{cases} A & \text{if } k = n, n-1; \\ 0 & \text{else;} \end{cases}$$

and whose sequence maps are the relevant zero maps; $D_n$ is defined on morphisms analogously. Prove that for $n \neq m$, there are no natural transformations $\sigma_{n,m} : D_n \to D_m$. Hint: Morphisms in $\mathbf{Ch}(\mathscr{A})$ are of degree zero.

**Exercise C.1.14.** Let $f : A \to B$ be a morphism in an Abelian category and let $m : B \to C$ be monic. Then

$$\mathrm{Ker}(f) \cong \mathrm{Ker}(m \circ f).$$

**Exercise C.1.15.** Let $\mathscr{A}$ be an Abelian category with infinite products and infinite coproducts. Show that there is a canonical map

$$\coprod_{i \in I} A_i \longrightarrow \prod_{i \in I} A_i$$

and find an example of an Abelian category for which the map you've constructed is not an isomorphism. Hint: Try looking in **Ab** with the infinite direct sum of $\mathbb{Z}$.

**Exercise C.1.16.** Prove or disprove each statement: This is an example of an Abelian category $\mathscr{A}$ and an Abelian category $\mathscr{B}$ together with an immersion $\mathscr{B} \to \mathscr{A}$ such that:

1. The essential image of the category $\mathscr{B}$ is an Abelian subcategory of $\mathscr{A}$;
2. The essential image of the category $\mathscr{B}$ is *not* an Abelian subcategory of $\mathscr{A}$.

**Exercise C.1.17.** Let $\mathscr{A}$ be an **Ab**-enriched category with a zero object and assume that given objects $X$ and $Y$ of $\mathscr{C}$ there exists an object $Z$ with the following properties:

- There exist morphisms $f : X \to Z, g : Y \to Z, h : Z \to X$, and $k : Z \to Y$;
- The equations $h \circ f = \mathrm{id}_X$ and $k \circ g = \mathrm{id}_Y$ hold;
- The equations $k \circ f = 0$ and $h \circ g = 0$ hold;
- The identity $f \circ h + g \circ k = \mathrm{id}_Z$ holds.

Prove that $Z$ is a biproduct for $X$ and $Y$.

**Exercise C.1.18.** If $\mathscr{A}$ is an additive category, prove that there is a chain category $\mathbf{Ch}(\mathscr{A})$ as in the case of Abelian categories. Show further that $\mathbf{Ch}(\mathscr{A})$ is additive, but not in general Abelian.

## C.2 The Epic-Monic Factorization System and Basic Diagram Chasing

We now would like to collect some basic results on the Epic-Monic factorization system in an Abelian category, as well as some theorems/lemmas involving chasing diagrams in Abelian categories. Many of these results should be known at least in the module case, but may be new in the Abelian categorical case. While we can prove most (if not all) of the results in this section by using the Mitchell-Freyd Embedding Theorem, I will try to avoid this as much as possible for at least two reasons: First, there should be an intrinsic proof of these results that only uses the categorical structure of $\mathscr{A}$ and does not rely on the magic of modules; Second, the proofs should be instructive of how to use kernels and cokernels when you have them, and how to avoid getting into trouble by assuming things are sets. In either case, however, I will at least point to the proof invoking the Mitchell-Freyd Embedding Theorem so that the reader uninterested with doing Abelian category theory from universal properties[8] can at least see an easy proof of the theorems in question.

We begin this section by considering how to factor an arbitrary morphism by using the image and coimage. The first ingredient of this will be to show that in this factorization, the first morphism is epic and the second is monic.

**Theorem C.2.1.** *Let $f$ be a morphism in an Abelian category $\mathscr{A}$. Then $f$ factors through its image $\operatorname{Im} f$ as an epimorphism followed by a monomorphism.*

*Proof.* We prove the isomorphism by showing that the image $\operatorname{Ker}(\operatorname{coker} f)$ is isomorphic to the coimage $\operatorname{Coker}(\ker f)$ via a canonical comparison map $\operatorname{Coker}(\ker f) \to \operatorname{Ker}(\operatorname{coker} f)$. To do this, we must first show that there is a canonical map from $A$ to $\operatorname{Ker}(\operatorname{coker} f)$. With this in mind, first observe that since $\operatorname{coker} f$ is the universal coequalizing map of $f$ and 0, we have that

$$\operatorname{coker} f \circ f = 0.$$

Similarly, $\operatorname{coker} f \circ \ker(\operatorname{coker} f) = 0$. Thus there exists a unique morphism $\varepsilon : A \to \operatorname{Ker}(\operatorname{coker} f)$ which makes the diagram

$$\begin{tikzcd} \operatorname{Ker}(\operatorname{coker} f) \arrow[r, "\ker(\operatorname{coker} f)"] & B \arrow[r, shift left, "\operatorname{coker} f"] \arrow[r, shift right, "0"'] & \operatorname{Coker} f \\ A \arrow[u, dashed, "\exists! \varepsilon"] \arrow[ur, "f"'] & & \end{tikzcd}$$

commute by the universal property of the kernel as an equalizer. Observe also that $f = \ker(\operatorname{coker} f) \circ \varepsilon$, which is our desired factorization:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & B \\ & \operatorname{Ker}(\operatorname{coker} f) \arrow[ur, "\ker(\operatorname{coker} f)"'] & \end{tikzcd}$$

In order to complete the proof of the theorem, we now must prove that $\varepsilon$ is epic. To do this, assume that there are two morphisms $\varphi, \psi : \operatorname{Ker}(\operatorname{coker} f) \to X$ for which the diagram

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] & \operatorname{Ker}(\operatorname{coker} f) \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & X \end{tikzcd}$$

commutes, i.e., for which $\varphi \circ \varepsilon = \psi \circ \varepsilon$. Then we have that $(\varphi - \psi) \circ \varepsilon = 0$, so $\varepsilon$ must factor uniquely through the kernel $\operatorname{Ker}(\varphi - \psi)$ as in the diagram

$$\begin{tikzcd} \operatorname{Ker}(\varphi - \psi) \arrow[r, "\ker(\varphi - \psi)"] & \operatorname{Ker}(\operatorname{coker} f) \arrow[r, shift left, "\varphi - \psi"] \arrow[r, shift right, "0"'] & \operatorname{Coker} f \\ A \arrow[u, dashed, "\exists! \rho"] \arrow[ur, "\varepsilon"'] & & \end{tikzcd}$$

[8] Or for those people who have no issue with invoking the tyranny of set theory on modern mathematics.

where the unique map $\rho$ is induced by the universal property of the kernel. We then derive the factorization

$$f = \ker(\operatorname{coker} f) \circ \varepsilon = \ker(\operatorname{coker} f) \circ \ker(\varphi - \psi) \circ \rho;$$

because the composite $\ker(\operatorname{coker} f) \circ \ker(\varphi - \psi)$ is monic and all monics are kernels, there exists a morphism $\alpha \in \mathscr{A}_1$ for which

$$\ker(\operatorname{coker} f) \circ \ker(\varphi - \psi) = \ker \alpha.$$

Now, mimicking the first part of this proof involving the construction of $\varepsilon$, we can find a unique morphism $\varepsilon'$ such that

$$\ker(\operatorname{coker} f) = \ker \alpha \circ \varepsilon' = \ker(\operatorname{coker} f) \circ \ker(\varphi - \psi) \circ \varepsilon'.$$

Using the fact that $\ker(\operatorname{coker} f)$ is monic allows us to cancel $\ker(\operatorname{coker} f)$ and get that

$$\mathrm{id}_{\operatorname{Ker}(\operatorname{coker} f)} = \ker(\varphi - \psi) \circ \varepsilon'.$$

Because $\ker(\varphi - \psi)$ is a monic with a pre-compositional inverse, it then follows that $\ker(\varphi - \psi)$ is an isomorphism; however, since it is a kernel map, we then derive that $\varphi - \psi = 0$. This in turn gives that $\varphi = \psi$ and hence shows that $\varepsilon$ is epic.

□

*Mitchell-Freyd Embedding proof of Theorem C.2.1.* The diagrams we consider are only ever finite, so we can use the Mitchell-Freyd Embedding. The result now follows from the First Isomorphism Theorem for left $R$-modules over a unital ring $R$. □

**Remark C.2.2.** The statement of this for the category $R$-**Mod**, as I have stated in the second proof, is the First Isomorphism Theorem for modules. To see this, note that the the image of $f \in R\text{-}\mathbf{Mod}(M, N)$ is the module $f(M) = \{f(m) \in N \mid m \in M\}$ together with the inclusion $f(M) \to N$, while the coimage is the module $M/\operatorname{Ker} f$, together with the projection $M \to M/\operatorname{Ker} f$. The isomorphism of the coimage with the image then is the isomorphism

$$\operatorname{Im} f = f(M) \cong \frac{M}{\operatorname{Ker} f} = \operatorname{Coim} f,$$

which is exactly the First Isomorphism Theorem; this gives the factorization

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi_{\operatorname{Ker} f}"'] & N \\ M/\operatorname{Ker} f \arrow[r, "\cong"'] & f(M) \arrow[u, "i_{f(M)}"'] \end{tikzcd}$$

where the map $\pi_{\operatorname{Ker} f}$ is epic by construction and $i_{f(M)}$ is trivially monic.

It is our goal to show that this really induces an Epic-Monic factorization system on $\mathscr{A}$. Thus we should tell you what it means to be a factorization system in a category:

**Definition C.2.3.** A *factorization system* (often called an orthogonal factorization system by the precise, although it is standard for a factorization system without any ugly adjectives hanging to the left like ugly capes billowing in the wind[9] obscuring the scenery to be assumed orthogonal) for a category $\mathscr{C}$ is a pair of classes[10] $(\mathcal{L}, \mathcal{R})$ (the left and right classes, respectively), where $\mathcal{L}, \mathcal{R} \subseteq \mathscr{C}_1$ such that:

[9] The movie *The Incredibles* has a nice description as to why capes are bad.

[10] For the reader worried about foundational/set-theoretic issues: Don't be! We simply remark that these classes may or may not be proper classes, but if they are, we will do our best to not have to quantify over these objects. If we do have to quantify over these objects, we will simply assume that we have enriched our Grothendieck universe to a stronger one in which we have a theory of quantification over proper classes. This will not cause any issues save for ones caused by staring at sets for too long, much like how staring at the sun for too long hurts one's eyes, and so can be safely ignored unless you really want that game of Sun-Stare.

(FS1). Both $\mathcal{L}$ and $\mathcal{R}$ are closed with respect to composition, and if $\mathcal{I}$ is the class of isomorphisms in $\mathscr{C}$, then $\mathcal{I} \subseteq \mathcal{L}$ and $\mathcal{I} \subseteq \mathcal{R}$;

(FS2). Every map $f \in \mathscr{C}(X, Y)$ factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\ell"'] & & Y \\ & A \arrow[ur, "r"'] & \end{tikzcd}$$

where $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$;

(FS3). For any two $(\mathcal{L}, \mathcal{R})$ factorizations $X \xrightarrow{\ell} A \xrightarrow{r} Y$ and $X \xrightarrow{\ell'} B \xrightarrow{r'}$ there is a unique isomorphism $A \xrightarrow{\cong} B$ making the diagram

$$\begin{tikzcd} & A \arrow[dr, "r"] \arrow[dd, dashed, "\exists! \,\cong"] & \\ X \arrow[ur, "\ell"] \arrow[dr, "\ell'"'] & & Y \\ & B \arrow[ur, "r'"'] & \end{tikzcd}$$

commute in $\mathscr{C}$.

**Remark C.2.4.** We will explicitly define it later for the sake of formal completeness, but when we write the classes $\mathcal{E}$ and $\mathcal{M}$ of morphisms in a category $\mathscr{C}$, we mean the classes of epic and monic morphisms in $\mathscr{C}$, respectively.

**Example C.2.5.** In any category $\mathscr{C}$, the pairs $(\mathcal{I}, \mathscr{C}_1)$ and $(\mathscr{C}_1, \mathcal{I})$ are factorization systems on $\mathscr{C}$.

**Example C.2.6.** In **Set**, there is a factorization system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is the class of all epics in **Set** and $\mathcal{M}$ is the class of all monics in **Set**. The factorization works by taking a map $f : X \to Y$ and first projecting $X$ onto the image of $f$ and then embedding the image into $Y$.

**Example C.2.7.** In **Set**, the pair $(\mathcal{M}, \mathcal{E})$ is *not* a factorization system because there are not canonical isomorphisms between factorizing objects. It is, however, a weak factorization system; such things are useful in the study of pure category theory, homotopy type theory, and in general homotopy theory. These appear in the study of Quillen model categories and homotopy type theory; cf. [**?**] and [**?**], for instance, for details.

Because we are trying to define the Epic-Monic factorization system, we know that we must verify Axioms (FS1) − (FS3) with the definitions $\mathcal{E} := \{f \in \mathscr{A}_1 \mid f \text{ is epic}\}$ and $\mathcal{M} := \{f \in \mathscr{A}_1 \mid f \text{ is monic}\}$. The verification of the first axiom is immediate and holds in any category (not just Abelian categories):

**Lemma C.2.8.** *In any category $\mathscr{C}$, the class of isomorphisms $\mathcal{I}$ is a subclass of the class of epimorphisms and a subclass of the class of monomorphisms, and $\mathcal{E}$ and $\mathcal{M}$ are both closed with respect to composition.*

*Proof.* This is immediate because every isomorphism is both epic and monic. The second statement is a basic categorical fact about epic and monic morphisms that we omit proving; proofs may be found in any introductory book on category theory[11] such as [53]. $\square$

The verification of Axiom (FS2) has technically already been done and is implicit[12] in Theorem C.2.1. The only difference is that in Theorem C.2.1 we have not explicitly defined $\mathcal{E}$ or $\mathcal{M}$, but the identification is immediate.

[11]This is also found in my notes on category theory and algebraic geometry [**?**], but I will generically try to avoid citing myself explicitly because that is an activity that should be best left behind closed doors without the presence of children.

[12]If not outright explicitly stated. Sometimes my life choices of writing without coffee and over the course of many days is a problem from which everyone suffers.

It is the last factorization system axiom, (FS3), that we will have to work to prove. However, this will show us why we need diagram chasing lemmas, as well as show how certain classical tools in homological algebra are really revealing categorical structure about Abelian categories and regular categories.

To show how to prove (FS3), we will present an argument involving pushouts, kernels, and cokernels. This argument is strictly more general than the one in which we reside; it holds generally in regular categories and does not need the additive structure we have at hand. However, it is convenient to use it to make certain identifications and keep our intuition and context grounded. The interested reader should see, for example, [**?**] for the definition, and some basic properties, of a regular category.

**Theorem C.2.9.** *In an Abelian category $\mathscr{A}$, if $(\varepsilon : A \to I, \mu : I \to B)$ and $(\varepsilon' : A \to I', \mu' : I' \to B)$ are two epic-monic factorizations of a morphism $f : A \to B$, then there is a canonical isomorphism $\alpha : I \to I'$ making the diagram*

$$\begin{tikzcd}
A \arrow[r, "\varepsilon'"] \arrow[d, "\varepsilon"'] & I' \arrow[d, "\mu'"] \\
I \arrow[r, "\mu"'] \arrow[ur, dashed, "\exists ! \alpha", "\cong"'] & B
\end{tikzcd}$$

*commute.*

*Proof.* We prove this by showing that any two epic-monic factorizations of $f$ are isomorphic to their pushout; composing these isomorphisms will give the result. To this end, consider the pushout

$$\begin{tikzcd}
A \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon'"'] & I \arrow[d, "i_1"] \arrow[ddr, bend left, "\mu"] & \\
I' \arrow[r, "i_2"'] \arrow[drr, bend right, "\mu'"'] & I \coprod_A I' \arrow[dr, dashed, "\exists ! \theta"] & \\
& & B
\end{tikzcd}$$

in $\mathscr{A}$; this exists because $\mathscr{A}$ is finitely complete. It is now our goal to show that $i_1$ and $i_2$ are isomorphisms.

We first show that the pushout injections are epic. However, this is immediate: The categorical dual to the fact that pullbacks against monics are monic, since both $\varepsilon$ and $\varepsilon'$ are epic, so are $i_1$ and $i_2$.

We now show that the pushout injections are monic. Consider that from the construction of the pushout we have that

$$\mu = \theta \circ i_1$$

and

$$\mu = \theta \circ i_2.$$

Because $\mu$ is monic, it is straightforward to check that post-composition by $i_1$ and $i_2$ send nonzero morphisms to nonzero morphisms (in the case that $i_1$ and $i_2$ are nonzero — otherwise the result is immediate, so we can wolog reduce to this case) which, by Proposition C.1.28 gives that both $i_1$ and $i_2$ are monic. Now, since in an Abelian category a morphism is an isomorphism if and only if it is epic and monic (cf. Exercise C.1.12) we get that both $i_1$ and $i_2$ are isomorphisms. This in turn allows us to observe that the map $\theta$ is monic, as it can be written

$$\theta = \mu \circ i_1^{-1} = \mu \circ i_2^{-1}$$

and all three maps appearing to the right of $\theta$ are monic. This in turn shows that

$$\theta \circ (i_1 \circ \varepsilon) = \mu \circ i_1^{-1} \circ i_1 \circ \varepsilon = \mu \circ \varepsilon = f$$

and dually. Therefore we can conclude that the two factorizations are isomorphic in the sense described in the statement of the theorem; the map $\alpha : I \to I'$ in this case is $i_2^{-1} \circ i_1$. $\square$

**Remark C.2.10.** This theorem is not necessarily true in a category $\mathscr{C}$ for which morphisms that are both epic and monic need not be isomorphisms.

**Corollary C.2.11.** *In an Abelian category $\mathscr{A}$, if $f \in \mathscr{A}_1$ then* $\operatorname{Im} f \cong \operatorname{Coim} f$.

*Proof.* The dual to Theorem C.2.1 gives an epic-monic factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\operatorname{coker}(\ker f)"'] & & B \\ & \operatorname{Coker}(\ker f) \arrow[ur, "m"'] & \end{tikzcd}$$

through the coimage. Invoking Theorem C.2.1 to give the image factorization of $f$ then shows that

$$\operatorname{Coker}(\ker f) = \operatorname{Coim} f \cong \operatorname{Im} f = \operatorname{Ker}(\operatorname{coker} f)$$

by Theorem C.2.9. ☐

**Theorem C.2.12.** *There is an Epic-Monic factorization system $(\mathcal{E}, \mathcal{M})$ on any Abelian category $\mathscr{A}$.*

*Proof.* Axiom (FS1) is verified by Lemma C.2.8. Axiom (FS2) is verified by Theorem C.2.1, and Axiom (FS3) is verified by Theorem C.2.9. ☐

Now that we have the Epic-Monic factorization system on an arbitrary Abelian category, let us move on to consider and see some basic diagram chasing lemmas in a more general context than simply modules. To do this, however, we will need to define what it means to be a sequence in an Abelian category, which will in turn allow us to reason about the morphisms of $\mathscr{A}$ and how they relate to other objects.

**Definition C.2.13.** A sequence in an Abelian category is an object $A^\bullet$ in $\mathbf{Ch}(\mathscr{A})$. Moreover, a sequence is said to be finite if at most finitely many $A^n$ are nonzero.

**Remark C.2.14.** Usually a sequence is defined as: A collection of objects $\{A_k \in \mathscr{A} \mid k \in S\}$, where $S \subseteq \mathbb{Z}$ is a subset such that if $k, \ell \in S$ with $k \leq \ell$, then $j \in S$ for all $k \leq j \leq \ell$; A collection of morphisms $\partial_k : A^k \to A^{k+1}$ whenever $k, k+1 \in S$ such that for all successive pairs of morphisms $\partial_k$ and $\partial_{k+1}$, $\partial_{k+1} \circ \partial_k = 0$. This ad hoc definition may be made into an object in $\mathbf{Ch}(\mathscr{A})$ by simply pre-and-post-pending zeros with appropriate zero morphisms on the left and right hand side of the sequence, respectively. In this way we never need to mention short exact sequences as if they are special objects, and simply understand that if we talk about a finite part of a sequence, it is the only part that is particularly relevant.

**Definition C.2.15.** Let $A^\bullet$ be a sequence in $\mathscr{A}$. The sequence $A^\bullet = (A^k, \partial_k)$ is said to be *exact at* $A^k$ if there is an isomorphism

$$\operatorname{Ker} \partial_k \cong \operatorname{Im} \partial_{k-1} = \operatorname{Ker}(\operatorname{coker} \partial_{k-1}).$$

The sequence $A^\bullet$ is said to be exact if and only if it is exact at $A^k$ for all $k \in \mathbb{Z}$.

**Remark C.2.16.** As an abuse of notation, we will say that a finite sequence

$$A^k \longrightarrow \cdots \longrightarrow A^n$$

in an Abelian category is exact if it is exact at all $A^j$ with $k < j < n$. This is not exactly the same as saying the induced infinite sequence is exact, but it is an abuse common in the literature and in order to state things succinctly, it is necessary.

Exact sequences are an extremely important part of homological algebra and Abelian category theory. They allow us to do many things involving testing whether or not objects are (or fail to be) isomorphic, as well as prove surprising results that would be otherwise false in other categories. For instance, the lemma we prove below will show that epimorphisms in Abelian categories are stable under pullback, which is in (potentially) stark contrast to the general situation.

**Lemma C.2.17.** *Let $\mathscr{A}$ be an Abelian category with pullback square*

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

*and assume that $f$ is an epimorphism. Then $p_2$ is an epimorphism as well.*

*Proof.* We will prove that the square above is a pushout square. Begin by considering that

$$f_2 \circ g \circ p_2 = 0 \circ p_2 = 0$$

and note furthermore that the diagram

$$0 \longrightarrow P \xrightarrow{[p_1,p_2]} A \oplus B \xrightarrow{\langle f,-g \rangle} C$$

is a sequence of objects in $\mathscr{A}$ because

$$\langle f, -g \rangle \circ [p_1, p_2] = f \circ p_1 - g \circ p_2 = 0.$$

We will now show this sequence is exact by using Part (3) of Exercise C.2.11. First, we will verify that $[p_1, p_2]$ is monic.

It is straightforward to check using the universal property of $P$ as a pullback that $\operatorname{Ker}[p_1, p_2] = 0$; this follows because any map into $P$ equalizing $[p_1, p_2]$ and the zero morphism must equalize $p_1$ and $p_2$, which in turn implies that there is exactly one such map by the universal property, implying further that the kernel must be zero. Since $\operatorname{Ker}[p_1, p_2] = 0$, it follows that $[p_1, p_2]$ is monic by Proposition C.1.28. The explicit details of this argument (and hence why it suffices to prove the next case as well) are asked of the reader in Exercise C.2.9.

We now must verify that the map $\langle f, -g \rangle : A \oplus B \to C$ is epic; we do this by using Proposition C.1.29. In particular, let $\alpha : C \to D$ be any morphism; if we can show that this implies that $\alpha \circ \langle f, -g \rangle = 0$ if and only if $\alpha = 0$ we are done. So consider the following deduction, using $\iota_1 : A \to A \oplus B$ and $\iota_2 : B \to A \oplus B$ as the coproduct injections:

$$\frac{\dfrac{\alpha \circ \langle f, -g \rangle = 0}{\alpha \circ f \circ \iota_1 - \alpha \circ g \circ \iota_2 = 0}}{\alpha \circ f \circ \iota_1 = \alpha \circ g \circ \iota_2}$$

Because the coproduct injections $A$ and $B$ are monic, either of these composites are going to be nonzero if and only if $\alpha \circ f$ and $\alpha \circ g$ are both nonzero; however, because $f$ is an epimorphism, this implies that $\alpha \circ f = 0$ if and only if $\alpha = 0$. Thus we can assume that $\alpha \circ f \circ \iota_1 = \alpha \circ g \circ \iota_2$ with $\alpha \neq 0$. However, from the fact that $f$ is epic, it is clear that

$$\alpha \circ f \circ \iota_1 = \alpha \circ g \circ \iota_2$$

if and only if $\alpha = 0$, which in turn implies that $\langle f, -g \rangle$ is epic. By Exercise C.2.9 we have that $C$ is a pushout. However, since $C$ is a pushout it follows that $p_2$ is epic because $f$ is, proving the lemma. $\square$

The first diagram chasing lemma that we will prove is the Five Lemma. This is a very important lemma, as many times when you wish to prove that two objects are isomorphic in an Abelian category, it is often easier to reduce to a statement involving the Five Lemma; for instance, I use this liberally in [**?**] to prove various difficult-to-prove-directly isomorphisms. Let us now state the Five Lemma:

**Lemma C.2.18** (Five Lemma)**.** *Assume that the diagram*

$$\begin{tikzcd} A^0 \arrow[r, "\partial_0"] \arrow[d, "f_0"'] & A^1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & A^2 \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & A^3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & A^4 \arrow[d, "f_4"] \\ B^0 \arrow[r, "\delta_0"'] & B^1 \arrow[r, "\delta_1"'] & B^2 \arrow[r, "\delta_2"'] & B^3 \arrow[r, "\delta_3"'] & B^4 \end{tikzcd}$$

*commutes in an Abelian category $\mathscr{A}$ with the properties:*

- *The top and bottom rows are exact sequences (so $\delta_i \circ \delta_{i-1} = 0 = \partial_i \circ \partial_{i-1}$ and $\operatorname{Ker} \partial_i \cong \operatorname{Im} \partial_{i-1}$ and $\operatorname{Ker} \delta_i \cong \operatorname{Im} \partial_{i-1}$ for all $i$ where both morphisms indexed at $i$ and $i-1$ appear in the diagram);*
- *$f_0$ is epic;*
- *$f_4$ is monic;*
- *$f_1$ and $f_3$ are isomorphisms.*

*Then $f_2$ is an isomorphism.*

This lemma is difficult and frustrating to prove directly, so we prove two Four Lemmas that will combine (by a gluing process) into the Five Lemma. The first we will prove is a lemma that I call the Monic Four Lemma, and it gives a diagrammatic condition for a middle morphism to be a monic.

**Lemma C.2.19** (Monic Four Lemma)**.** *Assume that the diagram*

$$\begin{tikzcd} A^0 \arrow[r, "\partial_0"] \arrow[d, "f_0"'] & A^1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & A^2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & A^3 \arrow[d, "f_3"] \\ B^0 \arrow[r, "\delta_0"'] & B^1 \arrow[r, "\delta_1"'] & B^2 \arrow[r, "\delta_2"'] & B^3 \end{tikzcd}$$

*commutes in $\mathscr{A}$ with the properties that:*

- *Both rows are exact sequences;*
- *$f_0$ is epic;*
- *$f_1$ and $f_3$ are monic.*

*Then the map $f_2$ is monic.*

*Proof.* We must show that $f_2$ is monic, which is equivalent to showing that any morphism that makes $f_2 \circ h = 0$ implies that $h = 0$ by Proposition C.1.28. With this in mind, begin by assuming that there is an object $C$ in $\mathscr{A}$ together with a morphism $g \in \mathscr{A}(C, A^2)$ such that $f_2 \circ g = 0$. Since $f_2 \circ g = 0$, it follows that $\delta_2 \circ f_2 \circ g = 0$, so using the commutativity of the diagram in the hypotheses of the lemma, we derive that

$$0 = \delta_2 \circ 0 = \delta_2 \circ f_2 \circ b = f_3 \circ \partial_2 \circ g.$$

Thus, since $f_3$ is monic, it follows that $\partial_2 \circ g = 0$ and so the diagram

$$\begin{tikzcd} C \arrow[d, "g"'] \arrow[dr, "0"] & \\ A^2 \arrow[r, "\partial_2"'] \arrow[d, "f_2"'] & A^3 \arrow[d, "f_3"] \\ B^2 \arrow[r, "\delta_2"'] & B^3 \end{tikzcd}$$

commutes. However, since $\partial_2 \circ g = 0$, it follows that $g$ factors through the kernel of $\partial_2$. Thus there exists a unique morphism $\gamma : C \to \operatorname{Ker} \partial_2$ making the diagram

$$\begin{tikzcd}
\operatorname{Ker} \partial_2 \arrow[r, "\ker \partial_2"] & A^2 \arrow[r, shift left, "\partial_2"] \arrow[r, shift right, "0"'] & A^3 \\
C \arrow[u, dashed, "\exists! \gamma"] \arrow[ur, "g"'] & &
\end{tikzcd}$$

commute in $\mathscr{A}$. Using now that the top row is exact at $A^2$ implies that there is a unique[13] isomorphism $\theta : \operatorname{Ker} \partial_2 \to \operatorname{Ker}(\operatorname{coker} \partial_1)$. In particular, from the factorization

$$g = \ker(\partial_2) \circ \gamma$$

it follows that there is a unique morphism $\gamma' : C \to \operatorname{Ker}(\operatorname{coker} \partial_1)$ making the diagram

$$\begin{tikzcd}
\operatorname{Ker}(\operatorname{coker} \partial_1) \arrow[r, "\ker(\operatorname{coker} \partial_1)"] & A^2 \arrow[r, shift left, "\operatorname{coker} \partial_1"] \arrow[r, shift right, "0"'] & \operatorname{Coker} \partial_1 \\
C \arrow[u, dashed, "\exists! \gamma'"] \arrow[ur] & &
\end{tikzcd}$$

commute; note that $\gamma' = \theta \circ \gamma$. Now by the Epic-Monic Factorization System (cf. Theorem C.2.12) it follows that there is a unique epimorphism $\varepsilon : A^1 \to \operatorname{Ker}(\operatorname{coker} \partial_1)$ (cf. the proof of Theorem C.2.1) such that

$$\begin{tikzcd}
A^1 \arrow[rr, "\partial_1"] \arrow[dr, "\varepsilon"'] & & A^2 \\
& \operatorname{Ker}(\operatorname{coker} \partial_1) \arrow[ur, "\ker(\operatorname{coker} \partial_1)"'] &
\end{tikzcd}$$

commutes. Now consider the pullback

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\gamma'"] \\
A^1 \arrow[r, "\varepsilon"'] & \operatorname{Ker}(\operatorname{coker} \partial_1)
\end{tikzcd}$$

in $\mathscr{A}$; we will use this to classify whether or not $g$ must be the zero morphism, as the map $p_2$ is epic by the fact that $\varepsilon$ is and Lemma C.2.17. Note that it is valid to use this to test for the nonzero-ness of $g$ by Proposition C.1.29. In particular, the map $p_2$ may be wolog taken to be nonzero.

Observe now that from this pullback, because it gives a place from which to realize $g$, we have that

$$\begin{aligned}
0 &= 0 \circ p_2 = f_2 \circ g \circ p_2 = f_2 \circ \ker(\operatorname{coker} \partial_1) \circ \gamma' \circ p_2 = f_2 \circ \ker(\operatorname{coker} \partial_1) \circ \varepsilon \circ p_1 \\
&= f_2 \circ \partial_1 \circ p_1 = \delta_1 \circ f_1 \circ p_1,
\end{aligned}$$

[13]The uniqueness of this isomorphism follows from the fact that both objects have the same universal property; a reader unconvinced can do Exercise C.2.8 instead.

which shows that the diagram

$$
\begin{tikzcd}
P \arrow[rr, "p_2"] \arrow[dd, "p_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & C \arrow[dl, "\gamma'"] \arrow[dd, "g"] \\
& \operatorname{Im}\partial_1 \arrow[dr, "\operatorname{im}\partial_1"'] & \\
A^1 \arrow[ur, "\varepsilon"] \arrow[rr, "\partial_1"'] \arrow[d, "f_1"'] & & A^2 \arrow[d, "f_2"] \\
B^1 \arrow[rr, "\delta_1"'] & & B^2
\end{tikzcd}
$$

commutes. Moreover, since $\delta_1 \circ f_1 \circ p_1 = 0$, it follows that $f_1 \circ p_1$ factors (uniquely) through the kernel of $\delta_1$; explicitly there is a unique morphism $\eta : P \to \operatorname{Ker}\delta_1$ making the diagram

$$
\begin{tikzcd}
\operatorname{Ker}\delta_1 \arrow[r, "\ker\delta_1"] & B^1 \arrow[r, shift left, "\delta_1"] \arrow[r, shift right] & B^2 \\
P \arrow[u, dashed, "\exists!\eta"] \arrow[ur, "f_1 \circ p_1"'] & &
\end{tikzcd}
$$

commute. Using the exactness of the bottom row then gives a unique isomorphism $\theta' : \operatorname{Ker}\delta_1 \to \operatorname{Im}\delta_0$ together with a unique map $\eta' : P \to \operatorname{Im}\delta_0$ making the diagram

$$
\begin{tikzcd}
\operatorname{Ker}(\operatorname{coker}\delta_0) \arrow[r, "\ker(\operatorname{coker}\delta_0)"] & B^1 \arrow[r, shift left, "\operatorname{coker}\delta_0"] \arrow[r, shift right, "0"'] & \operatorname{Coker}\delta_0 \\
P \arrow[u, dashed, "\exists!\eta'"] \arrow[ur, "f_1 \circ p_1"'] & &
\end{tikzcd}
$$

commute.

Playing the same game as before with exactness, first factorize the map $\delta_0$ by

$$
\begin{tikzcd}
B^0 \arrow[rr, "\delta_0"] \arrow[dr, "\varepsilon'"'] & & B^1 \\
& \operatorname{Ker}(\operatorname{coker}\delta_0) \arrow[ur, "\ker(\operatorname{coker}\delta_0)"'] &
\end{tikzcd}
$$

with $\varepsilon'$ epic and pull back to form the commuting square

$$
\begin{tikzcd}
Q \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & P \arrow[d, "\eta'"] \\
B^0 \arrow[r, "\varepsilon'"'] & \operatorname{Ker}(\operatorname{coker}\delta_0)
\end{tikzcd}
$$

and observe that since $\varepsilon'$ is epic, so is $q_1$. Moreover, by construction these composites satisfy the relation

$$
f_1 \circ p_1 \circ q_1 = \ker(\operatorname{coker}\delta_0) \circ \eta' \circ q_1 = \ker(\operatorname{coker}\delta_0) \circ \varepsilon' \circ q_2 = \delta_0 \circ q_2
$$

so we can use $q_1$ as a wolog nonzero epimorphism to test the non-zeroness of our situation (as with $p_1$; the fact that $p_1 \circ q_1$ remains epic shows that using the composite as a test does not change things by a transitivity argument).

We now consider the epimorphism $f_0 : A^0 \to B^0$. Note that there is a canonical epimorphism

$$\varepsilon' \circ f_0 : A^0 \to \operatorname{Im} \delta_0;$$

furthermore, we can produce the commuting diagram

$$\begin{tikzcd} A^0 \arrow[r, "\partial_0"] \arrow[d, "0"'] & A^1 \arrow[d, "\partial_1"] \\ C \arrow[r, "g"'] & A^2 \end{tikzcd}$$

to induce the unique morphism $\alpha : A^0 \to P$ making the diagram

$$\begin{tikzcd} A^0 \arrow[dr, dashed, "\exists!\alpha"] \arrow[drr, bend left, "\partial_0"] \arrow[ddr, bend right, "0"'] & & \\ & P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A^1 \arrow[d, "\partial_1"] \\ & C \arrow[r, "g"'] & A^2 \end{tikzcd}$$

commute. We will now use this unique map to induce a morphism $\beta : A^0 \to Q$. To see how to do this we must show that the square

$$\begin{tikzcd} A^0 \arrow[r, "\alpha"] \arrow[d, "f_0"'] & P \arrow[d, "\eta'"] \\ B^0 \arrow[r, "\varepsilon'"'] & \operatorname{Ker}(\operatorname{coker} \delta_0) \end{tikzcd}$$

commutes. To see this consider the following chain of equalities:

$$\ker(\operatorname{coker} \delta_0) \circ \eta' \circ \alpha = f_1 \circ p_1 \circ \alpha = f_1 \circ \partial_0 = \delta_0 \circ f_0 = \ker(\operatorname{coker} \delta_0) \circ \varepsilon' \circ f_0.$$

Using now that $\ker(\operatorname{coker} \delta_0)$, as a kernel map, is monic gives that

$$\eta' \circ \alpha = \varepsilon' \circ f_0$$

and hence establishes the commutativity of the square

$$\begin{tikzcd} A^0 \arrow[r, "\alpha"] \arrow[d, "f_0"'] & P \arrow[d, "\eta'"] \\ B^0 \arrow[r, "\varepsilon'"'] & \operatorname{Ker}(\operatorname{coker} \delta_0) \end{tikzcd}$$

in $\mathscr{A}$. Furthermore, this allows us to induce a unique map $\beta : A^0 \to Q$ making the diagram

$$\begin{tikzcd} A^0 \arrow[dr, dashed, "\exists!\beta"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "f_0"'] & & \\ & Q \arrow[r, "q_1"] \arrow[d, "q_2"'] & P \arrow[d, "\eta'"] \\ & B^0 \arrow[r, "\varepsilon'"'] & \operatorname{Ker}(\operatorname{coker} \delta_0) \end{tikzcd}$$

commute. Furthermore, from basic categorical structure, it is straightforward to check that $\beta$ is an epimorphism and that the composites $f_1 \circ \partial_0$ and $\delta_0 \circ f_0$ factor through $\beta$. Explicitly we see this through

$$\begin{aligned} f_1 \circ \partial_0 &= f_1 \circ p_1 \circ \alpha = f_1 \circ p_1 \circ q_1 \circ \beta = \ker(\operatorname{coker} \delta_0) \circ \eta' \circ q_1 \circ \beta \\ &= \ker(\operatorname{coker} \delta_0) \circ \varepsilon' \circ q_2 \circ \beta = \delta_0 \circ f_0. \end{aligned}$$

This then gives the factorization

$$\partial_0 = p_1 \circ q_1 \circ \beta$$

by the fact that $f_1$ is monic. Proceeding from here, we find that

$$0 = \partial_1 \circ \partial_0 = \partial_1 \circ p_1 \circ q_1 \circ \beta = g \circ p_2 \circ q_1 \circ \beta$$

so $0 = \beta \circ p_2 \circ q_1 \circ \beta$. Moreover, since $p_2 \circ q_1 \circ \beta$ is nonzero and epic, it follows from Proposition C.1.29 that it must be the case that $g = 0$. However, this shows that any morphism which equalizes 0 and $f_2$ must be identically the zero morphism, and hence proves that $\operatorname{Ker} f_2 = 0$, finally giving us that $f_2$ is monic by Proposition C.1.28. $\square$

**Remark C.2.20.** This proof is significantly more involved than the classical Mitchell-Freyd Embedding proof of the lemma. However, a careful analysis of this proof shows that the assumptions of being in an Abelian category are simply convenient. As long as we know that the epimorphisms we construct throughout the proof are all *regular* and that we have these pullbacks present in any additive category, the proof will carry over to show *exactly* where a Monic Four Lemma holds.

The next lemma is the Epic Four Lemma, which is a lemma which allows us to test if a middle morphism is epic. This is dual to the Monic Four Lemma, but we prove how this is dual explicitly.

**Lemma C.2.21** (Epic Four Lemma)**.** *Assume that the diagram*

$$\begin{tikzcd} A^1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & A^2 \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & A^3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & A^4 \arrow[d, "f_4"] \\ B^1 \arrow[r, "\delta_1"'] & B^2 \arrow[r, "\delta_2"'] & B^3 \arrow[r, "\delta_3"'] & B^4 \end{tikzcd}$$

*commutes in $\mathscr{A}$ with the following properties:*

- *Both rows are exact sequences;*
- *$f_1$ and $f_3$ are epic;*
- *$f_4$ is monic.*

*Then the map $f_2$ is epic.*

*Proof.* This is categorically dual to Lemma C.2.19, but we will prove how this is the case explicitly. Begin by observing that a morphism $\varphi$ in a category $\mathscr{C}$ is monic if and only if $\varphi^{\mathrm{op}}$ is epic in $\mathscr{C}^{\mathrm{op}}$. Thus if the diagram

$$\begin{tikzcd} A^1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & A^2 \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & A^3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & A^4 \arrow[d, "f_4"] \\ B^1 \arrow[r, "\delta_1"'] & B^2 \arrow[r, "\delta_2"'] & B^3 \arrow[r, "\delta_3"'] & B^4 \end{tikzcd}$$

satisfies the hypotheses of the lemma, then in $\mathscr{A}^{\mathrm{op}}$, the diagram

$$\begin{tikzcd}
B^4 \arrow[r, "\delta_3^{\mathrm{op}}"] \arrow[d, "f_4^{\mathrm{op}}"'] & B^3 \arrow[r, "\delta_2^{\mathrm{op}}"] \arrow[d, "f_3^{\mathrm{op}}"'] & B^2 \arrow[r, "\delta_1^{\mathrm{op}}"] \arrow[d, "f_2^{\mathrm{op}}"] & B^1 \arrow[d, "f_1^{\mathrm{op}}"] \\
A^4 \arrow[r, "\partial_3^{\mathrm{op}}"'] & A^3 \arrow[r, "\partial_2^{\mathrm{op}}"'] & A^2 \arrow[r, "\partial_1^{\mathrm{op}}"'] & A^1
\end{tikzcd}$$

commutes. Moreover, $f_4^{\mathrm{op}}$ is epic, $f_1^{\mathrm{op}}$ and $f_3^{\mathrm{op}}$ are monic, and both rows are exact by Exercise C.2.7. Thus Lemma C.2.19 holds in $\mathscr{A}^{\mathrm{op}}$, and $f_2^{\mathrm{op}}$ is monic. However it then follows that $f_2$ is epic in $\mathscr{A}$ and the lemma is proved. $\square$

This allows us to prove the Five Lemma, which we will do below. The Mitchell-Freyd proof relies on the proof of the $R$-**Mod** version of the same lemmas, which are well-known and whose proofs may be done by element chasing.

*Proof of Lemma C.2.18.* Consider the diagram

$$\begin{tikzcd}
A^0 \arrow[r, "\partial_0"] \arrow[d, "f_0"'] & A^1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & A^2 \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & A^3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & A^4 \arrow[d, "f_4"] \\
B^0 \arrow[r, "\delta_0"'] & B^1 \arrow[r, "\delta_1"'] & B^2 \arrow[r, "\delta_2"'] & B^3 \arrow[r, "\delta_3"'] & B^4
\end{tikzcd}$$

where both rows are exact, $f_0$ is epic, $f_4$ is monic, and $f_1$ and $f_3$ are isomorphisms. Then we can apply Lemmas C.2.19 and C.2.21 to deduce that $f_2$ is both epic and monic. Using Exercise C.1.12 we derive that $f_2$ is an isomorphism, which concludes the proof. $\square$

We now move to the last diagram chasing lemma we would like to cover in this section: The Snake Lemma. This lemma is fundamental for doing cohomology theory, as it allows one to produce connecting homomorphisms between different degrees of cohomology groups $H^n(X,C) \to H^{n+1}(X,A)$, where the $X$ is there to emphasize that generically we care about doing a specific type of cohomology theory such as sheaf cohomology, group cohomology, or Galois cohomology.

**Lemma C.2.22** (The Snake Lemma)**.** *Let $\mathscr{A}$ be an Abelian category and let*

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X \arrow[r, "h"'] & Y \arrow[r, "k"'] & Z &
\end{tikzcd}$$

*be a commuting diagram with exact rows. Then there is a morphism $\delta : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$, called a* connecting morphism *such that the sequence*

$$\operatorname{Ker}\alpha \xrightarrow{\overline{f}} \operatorname{Ker}\beta \xrightarrow{\overline{g}} \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \xrightarrow{\overline{h}} \operatorname{Coker}\beta \xrightarrow{\overline{k}} \operatorname{Coker}\gamma$$

*is exact in $\mathscr{A}$.*

*Proof.* The proof of the exactness of $\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma$ and $\operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma$ is straightforward and left as Exercise C.2.12; thus we need only construct the connecting map $\delta$ and then prove that it makes the resulting sequence exact. So first consider kernel map $\operatorname{Ker}\gamma \xrightarrow{\ker\gamma} C$ and pullback against the epimorphism $g : B \to C$ to produce the commuting diagram

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Ker}\gamma \arrow[d, "\ker\gamma"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

with $p_1$ an epimorphism by Lemma C.2.17 and $p_2$ a monomorphism by abstract nonsense. Post-composing by the square into $Z$ then allows us to compute that

$$0 = \gamma \circ \ker\gamma \circ p_1 = \gamma \circ g \circ p_2 = k \circ \beta \circ p_2$$

and verify the commutativity of the diagram

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[r, "\beta"] \arrow[d, "g"] & Y \arrow[d, "k"] \\ \operatorname{Ker}\gamma \arrow[r, "\ker\gamma"'] & C \arrow[r, "\gamma"'] & Z \end{tikzcd}$$

in $\mathscr{A}$. Since $k \circ \beta \circ p_2 = 0$, $\beta \circ p_2$ factors uniquely through $\operatorname{Ker} k$, i.e., there exists a unique morphism $\rho : P \to \operatorname{Ker} k$ making

$$\begin{tikzcd} \operatorname{Ker} k \arrow[r, "\ker k"] & Y \arrow[r, shift left, "k"] \arrow[r, shift right, "0"'] & Z \\ P \arrow[u, dashed, "\exists!\rho"] \arrow[ur, "\beta\circ p_2"'] & & \end{tikzcd}$$

commute. Now, using the exactness at $Y$ gives that there is an isomorphism

$$\operatorname{Ker} k \cong \operatorname{Im} h = \operatorname{Ker}(\operatorname{coker} h)$$

and hence the existence of a unique morphism $\rho' : P \to \operatorname{Ker}(\operatorname{coker} h)$ making

$$\begin{tikzcd} \operatorname{Ker} k \arrow[r, "\ker k"] & Y \arrow[r, shift left, "k"] \arrow[r, shift right, "0"'] & Z \\ P \arrow[u, dashed, "\exists!\rho'"] \arrow[ur, "\beta\circ p_2"'] & & \end{tikzcd}$$

commute. Now give the map $h : X \to Y$ the Epic-Monic factorization

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "\varepsilon"'] & & Y \\ & \operatorname{Ker}(\operatorname{coker} h) \arrow[ur, "\ker(\operatorname{coker} h)"'] & \end{tikzcd}$$

and pull $\rho'$ back against $\varepsilon$ to produce the pullback

$$\begin{tikzcd} Q \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\rho'"] \\ X \arrow[r, "\varepsilon"'] & \operatorname{Ker}(\operatorname{coker} h) \end{tikzcd}$$

in $\mathscr{A}$. Note that $\varepsilon$ is epic by Lemma C.2.17. Furthermore, by Exercise C.1.14 it follows that

$$\operatorname{Ker}(\varepsilon) \cong \operatorname{Ker}(\ker(\operatorname{coker} h) \circ \varepsilon) \cong \operatorname{Ker}(h) \cong 0,$$

giving that $\varepsilon$ is monic is well, and hence an isomorphism by Exercise C.1.12.

We now need only to reason about the morphism $p_1$ in order to complete the construction of the connecting homomorphism. Consider the pullback diagram

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Ker}\gamma \arrow[d, "\ker\gamma"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

in $\mathscr{A}$. Because this is a pullback, by Exercise C.2.9 it follows that the sequence

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "{[p_1, p_2]}"] & \operatorname{Ker}\gamma \oplus B \arrow[r, "{\langle \ker\gamma, -g \rangle}"] & C
\end{tikzcd}$$

is exact. This in turn implies that the morphism $[p_1, p_2] : P \to \operatorname{Ker}\gamma \oplus B$ is monic, which further implies that each of the component maps $p_1$ and $p_2$ are monic. This, however, shows that $p_1$ is an isomorphism.

To construct the desired connecting morphism we first consider the following commuting diagram

$$\begin{tikzcd}
 & & & P \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[ddl, dashed, "\rho'"'] & \operatorname{Ker}\gamma \arrow[d, "\ker\gamma"] & \\
 & A \arrow[rr, "f"] \arrow[dd, "\alpha"'] & & B \arrow[r, "g"] \arrow[dd, "\beta"'] & C \arrow[r] \arrow[dd, "\gamma"] & 0 \\
 & & \operatorname{Ker}(\operatorname{coker} h) \arrow[dr] & & & \\
0 \arrow[r] & X \arrow[rr, "h"'] \arrow[ur, "\varepsilon"] \arrow[d, "\operatorname{coker}\alpha"'] & & Y \arrow[r, "k"'] & Z & \\
 & \operatorname{Coker}\alpha & & & &
\end{tikzcd}$$

in which both $\varepsilon$ and $p_1$ are isomorphisms. Thus we define $\delta : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ to be the following composite:

$$\begin{tikzcd}
\operatorname{Ker}\gamma \arrow[r, "p_1^{-1}"] \arrow[dr, "\delta"'] & P \arrow[r, "\rho'"] & \operatorname{Ker}(\operatorname{coker} h) \arrow[d, "\varepsilon^{-1}"] \\
 & \operatorname{Coker}\alpha & X \arrow[l, "\operatorname{coker}\alpha"]
\end{tikzcd}$$

We now must prove that the map $\delta$ makes the sequence

$$\begin{tikzcd}
\operatorname{Ker}\alpha \arrow[r, "\overline{f}"] & \operatorname{Ker}\beta \arrow[r, "\overline{g}"] & \operatorname{Ker}\gamma \arrow[r, "\delta"] & \operatorname{Coker}\alpha \arrow[r, "\overline{h}"] & \operatorname{Coker}\beta \arrow[r, "\overline{k}"] & \operatorname{Coker}\gamma
\end{tikzcd}$$

exact. In order to work towards this goal, let us first prove that the above diagram is indeed a sequence. To see this we first consider the composite $\delta \circ \overline{g}$. Note that since the diagram

$$\begin{tikzcd}
\operatorname{Ker}\beta \arrow[r, "\overline{g}"] \arrow[d, "\ker\beta"'] & \operatorname{Ker}\gamma \arrow[d, "\ker\gamma"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

commutes, there is a unique map $\theta : \operatorname{Ker}\beta \to P$ making

$$\begin{tikzcd}
\operatorname{Ker}\beta \arrow[dr, dashed, "\exists!\theta"] \arrow[drr, bend left, "\overline{g}"] \arrow[ddr, bend right, "\ker\beta"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] & \operatorname{Ker}\gamma \arrow[d, "\ker\gamma"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

commute. It then follows that

$$\delta \circ \overline{g} = \operatorname{coker}\alpha \circ \varepsilon^{-1} \circ \rho' \circ p_1^{-1} \circ \overline{g} = \operatorname{coker}\alpha \circ \varepsilon^{-1} \circ \rho' \circ p_1^{-1} \circ p_1 \circ \theta = \operatorname{coker}\alpha \circ \varepsilon^{-1} \circ \rho' \circ \theta.$$

Now consider moreover that

$$h \circ \varepsilon^{-1} \circ \rho' \circ \theta = \ker(\operatorname{coker} h) \circ \varepsilon \circ \varepsilon^{-1} \circ \rho' \circ \theta = \ker(\operatorname{coker} h) \circ \rho' \circ \theta = \beta \circ p_2 \circ \theta = \beta \circ \ker\beta = 0 = h \circ 0;$$

it thus follows that $\varepsilon^{-1} \circ \rho' \circ \theta = 0$, in turn implying that $\delta \circ \overline{g} = 0$.

We now will show that $\overline{h} \circ \delta = 0$. To see this, we note that the diagram

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\operatorname{coker}\alpha"'] & Y \arrow[d, "\operatorname{coker}\beta"] \\
\operatorname{Coker}\alpha \arrow[r, "\overline{h}"'] & \operatorname{Coker}\beta
\end{tikzcd}$$

commutes. Then we can calculate that

$$\begin{aligned}
\overline{h} \circ \delta &= \overline{h} \circ \operatorname{coker}\alpha \circ \varepsilon^{-1} \circ \rho' \circ p_1^{-1} = \operatorname{coker}\beta \circ h \circ \varepsilon^{-1} \circ \rho' \circ p_1^{-1} = \operatorname{coker}\beta \circ \ker(\operatorname{coker} h) \circ \rho' \circ p_1^{-1} \\
&= \operatorname{coker}\beta \circ \beta \circ p_2 \circ p_1^{-1} = 0.
\end{aligned}$$

This verifies that the diagram is indeed a sequence.

Let us now move to proving the exactness of the sequence by beginning with showing the exactness at $\operatorname{Ker}\gamma$. Begin by considering the Epic-Monic factorization

$$\begin{tikzcd}
\operatorname{Ker}\beta \arrow[rr, "\overline{g}"] \arrow[dr] & & \operatorname{Ker}\gamma \\
& \operatorname{Ker}(\operatorname{coker}\overline{g}) \arrow[ur, "\ker(\operatorname{coker}\overline{g})"'] &
\end{tikzcd}$$

of $\overline{g}$. It then follows from the fact that $\delta \circ \overline{g} = 0$ that there is a further factorization:

$$\begin{tikzcd}
\operatorname{Ker}(\operatorname{coker}\overline{g}) \arrow[r, dashed, "\exists!\xi"] \arrow[dr, "\ker(\operatorname{coker}\overline{g})"'] & \operatorname{Ker}\delta \arrow[d, "\ker\delta"] \\
& \operatorname{Ker}\gamma
\end{tikzcd}$$

It then follows that

$$\operatorname{Ker}(\xi) \cong \operatorname{Ker}(\ker\delta \circ \xi) = \operatorname{Ker}(\ker(\operatorname{coker}\overline{g})) = 0$$

so $\operatorname{Ker}(\operatorname{coker}\overline{g})$ is a subobject of $\operatorname{Ker}\delta$. To see the isomorphism, we now prove that $\xi$ is epic by using the Epic Four Lemma; cf. Lemma C.2.21. Begin by considering that since $\delta \circ \overline{g} = 0$, the diagram

$$\begin{tikzcd} \operatorname{Ker}\beta \arrow[r, shift left, "\overline{g}"] \arrow[r, shift right, "0"'] & \operatorname{Ker}\gamma \arrow[r, "\delta"] & \operatorname{Coker}\alpha \end{tikzcd}$$

commutes. Consequently, by the universal property of the cokernel, there is a unique morphism $\tau : \operatorname{Coker}\overline{g} \to \operatorname{Coker}\alpha$ making the diagram

$$\begin{tikzcd} \operatorname{Ker}\beta \arrow[r, shift left, "\overline{g}"] \arrow[r, shift right, "0"'] & \operatorname{Ker}\gamma \arrow[r, "\operatorname{coker}\overline{g}"] \arrow[dr, "\delta"'] & \operatorname{Coker}\overline{g} \arrow[d, dashed, "\exists!\tau"] \\ & & \operatorname{Coker}\alpha \end{tikzcd}$$

commute. Let us now consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & \operatorname{Ker}(\operatorname{coker}\overline{g}) \arrow[r, "\ker(\operatorname{coker}\overline{g})"] \arrow[d, "\xi"'] & \operatorname{Ker}\gamma \arrow[r, "\operatorname{coker}\overline{g}"] \arrow[d, equal] & \operatorname{Coker}\overline{g} \arrow[d, "\tau"] \\ 0 \arrow[r] & \operatorname{Ker}\delta \arrow[r, "\ker\delta"'] & \operatorname{Ker}\gamma \arrow[r, "\delta"'] & \operatorname{Coker}\alpha \end{tikzcd}$$

It remains to show that the diagram commutes; however, the commutativity of the first cell is immediate, and the commutativity of the next two cells follows from the universal constructions, i.e., from the universal factorizations

$$\ker(\operatorname{coker}\overline{g}) = \ker\delta \circ \xi$$

and

$$\tau \circ \operatorname{coker}\overline{g} = \delta.$$

Thus the diagram commutes. Moreover, the exactness of the rows of the diagram follows immediately from construction. With this it suffices to prove that $\tau$ is monic in order to apply Lemma C.2.21; however, this is a routine argument which is sketched in (and left as) Exercise C.2.15. Thus appealing to Lemma C.2.21 gives that $\xi$ is epic and hence an isomorphism

$$\xi : \operatorname{Ker}(\operatorname{coker}\overline{g}) \xrightarrow{\cong} \operatorname{Ker}\delta.$$

This establishes the exactness at $\operatorname{Ker}\gamma$.

We now show how to deduce the exactness of the sequence at $\operatorname{Coker}\alpha$ by using duality arguments. First observe that the diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & X \arrow[r, "h"'] & Y \arrow[r, "k"'] & Z \end{tikzcd}$$

becomes, in $\mathscr{A}^{\mathrm{op}}$,

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "g^{\mathrm{op}}"] & B \arrow[r, "f^{\mathrm{op}}"] & A \\ & Z \arrow[r, "k^{\mathrm{op}}"'] \arrow[u, "\gamma^{\mathrm{op}}"] & Y \arrow[r, "h^{\mathrm{op}}"'] \arrow[u, "\beta^{\mathrm{op}}"] & X \arrow[r] \arrow[u, "\alpha^{\mathrm{op}}"] & 0 \end{tikzcd}$$

Note that both rows are exact by Exercise C.2.7. Furthermore, since $\operatorname{Ker}\gamma \cong \operatorname{Coker}\gamma^{\mathrm{op}}$ and $\operatorname{Coker}\alpha \cong \operatorname{Ker}\alpha^{\mathrm{op}}$, as well as for the other kernel/cokernel operations for all morphisms present in the diagram, we can deduce that the sequence

$$\operatorname{Ker}\alpha \xrightarrow{\overline{f}} \operatorname{Ker}\beta \xrightarrow{\overline{g}} \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \xrightarrow{\overline{h}} \operatorname{Coker}\beta \xrightarrow{\overline{k}} \operatorname{Coker}\gamma$$

gives rise to the sequence

$$\operatorname{Ker}\gamma^{\mathrm{op}} \xrightarrow{\overline{k}^{\mathrm{op}}} \operatorname{Ker}\beta^{\mathrm{op}} \xrightarrow{\overline{h}^{\mathrm{op}}} \operatorname{Ker}\alpha^{\mathrm{op}} \xrightarrow{\delta^{\mathrm{op}}} \operatorname{Coker}\gamma^{\mathrm{op}} \xrightarrow{\overline{f}^{\mathrm{op}}} \operatorname{Coker}\beta^{\mathrm{op}} \xrightarrow{\overline{g}^{\mathrm{op}}} \operatorname{Coker}\alpha^{\mathrm{op}}$$

in $\mathscr{A}^{\mathrm{op}}$. The exactness at $\operatorname{Coker}\alpha$ can thus be deduced from exactness at $\operatorname{Ker}\alpha^{\mathrm{op}}$. However, the exactness of $\delta^{\mathrm{op}}$ at $\operatorname{Ker}\alpha^{\mathrm{op}}$ can be deduced from the exactness at $\operatorname{Ker}\gamma$ by following the same process, save now in the opposite category. This gives the desired exactness at $\operatorname{Coker}\alpha$ and hence proves the Snake Lemma. $\square$

This ends the section on basic diagram chasing in Abelian categories through universal properties. We now move on to consider and study injective and projective objects in Abelian categories (and categories in general, whenever possible), which will be important when we study cohomology and actually ask to *compute* cohomology objects.

## Exercises

**Exercise C.2.1.** Prove that in any finitely complete and cocomplete additive category $\mathscr{A}$, there is a canonical map

$$\varphi : \operatorname{Coim} f \to \operatorname{Im} f$$

for any morphism $f \in \mathscr{A}_1$. Show that this is an isomorphism if and only if every monic is a kernel and every epic is a cokernel, and conclude that an alternative definition for an Abelian category is a complete and cocomplete additive category with isomorphisms $\operatorname{Im} f \cong \operatorname{Coim} f$.

**Exercise C.2.2.** This is a categorical exercise on factorization systems. Let $\mathcal{C}$ be a class of morphisms in a category $\mathscr{C}$. Say that a map $f : X \to Y$ has the *right lifting property* with respect to $\mathcal{C}$ if for all morphisms $\varphi : A \to B$ in $\mathcal{C}$, there exists a unique map $\rho : B \to X$ making the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed, "\exists!\rho"] & Y \end{tikzcd}$$

commute; the dual statement is for a map $f$ to have the *left lifting property* against $\mathcal{C}$, which asks for the commuting diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[d, "\varphi"] \\ Y \arrow[r] \arrow[ur, dashed, "\exists!\rho"] & B \end{tikzcd}$$

for all $\varphi \in \mathcal{C}$. The class of all morphisms with the left-lifting property against $\mathcal{C}$ is denoted by ${}^{\perp}\mathcal{C}$ and the class of all morphisms with the right-lifting property against $\mathcal{C}$ is denoted by $\mathcal{C}^{\perp}$.

Prove the following: The pair $(\mathcal{L}, \mathcal{R})$ is a factorization system on $\mathscr{C}$ if and only if every morphism admits an $(\mathcal{L}, \mathcal{R})$ factorization, $\mathcal{L}^{\perp} = \mathcal{R}$, and ${}^{\perp}\mathcal{R} = \mathcal{L}$.

**Exercise C.2.3.** For the sake of completeness: Give a module-theoretic proof of the Five Lemma by proving the Four Lemmas by element chasing and then deducing the Five Lemma in the same way, i.e., by combining the Four Lemmas. This completes the Mitchell-Freyd proof of the Five Lemma.

**Exercise C.2.4.** Examine the proof of Theorem C.2.1. Rewrite it in a format that does not use the **Ab**-enrichment by identifying kernels with equalizers and cokernels with coequalizers and using that every monic is the equalizer map of something, and dually for epics. This is the image factorization for regular categories.

**Exercise C.2.5.** Prove that for any sequence $A^\bullet = (A^k, \partial_k)$ in an Abelian category $\mathscr{A}$, for all $k \in \mathbb{Z}$, $\operatorname{Im}\partial_{k-1} = \operatorname{Ker}(\operatorname{coker}\partial_{k-1})$ is a subobject of $\operatorname{Ker}\partial_k$.

**Exercise C.2.6.** Prove that in an Abelian category, every monic is the kernel of its cokernel and every epimorphism is the cokernel of its kernel. Hint: Use Corollary C.2.11.

**Exercise C.2.7.** Prove that for any Abelian category $\mathscr{A}$, the sequence

$$\cdots \longrightarrow A^{k-1} \xrightarrow{\partial_{k-1}} A^k \xrightarrow{\partial_k} A^{k+1} \longrightarrow \cdots$$

is exact if and only if

$$\cdots \longrightarrow A^{k+1} \xrightarrow{\partial_k^{\mathrm{op}}} A^k \xrightarrow{\partial_{k-1}^{\mathrm{op}}} A^{k-1} \longrightarrow \cdots$$

is exact in $\mathscr{A}^{\mathrm{op}}$.

**Exercise C.2.8.** If $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at $B$, prove that the isomorphism $\operatorname{Ker} g \cong \operatorname{Im} f$ is unique.

**Exercise C.2.9** (Proposition 2.53 of [25])**.** Prove the following proposition of [25]: Consider a commuting square

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

in an Abelian category $\mathscr{A}$. Then:

- The sequence

$$0 \longrightarrow P \xrightarrow{[p,q]} A \oplus B \xrightarrow{\langle f,-g\rangle} C$$

  is exact if and only if $P$ is a pullback;

- The sequence

$$P \xrightarrow{[p,q]} A \oplus B \xrightarrow{\langle f,-g\rangle} C \longrightarrow 0$$

  is exact if and only if $C$ is a pushout;

- The sequence

$$0 \longrightarrow P \xrightarrow{[p,q]} A \oplus B \xrightarrow{\langle f,-g\rangle} C \longrightarrow 0$$

  is exact if and only if $P$ is a pulllback and $C$ is a pushout.

**Exercise C.2.10** (Exercise 9.3 of [35])**.** We say that square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

is exact if and only if the sequence

$$A \xrightarrow{[f,g]} B \oplus C \xrightarrow{\langle -h,k\rangle} D$$

is exact. Prove that if the two inner squares in the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "h"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"] \\ X \arrow[r, "h"'] & Y \arrow[r, "k"'] & Z \end{tikzcd}$$

are exact, then so is the outer square. For even more fun, do this *without* the Mitchell-Freyd Embedding Theorem!

**Exercise C.2.11.** Let $\mathscr{A}$ be an Abelian category. Prove the following about things about exact sequences:

1. The sequence
$$0 \longrightarrow A \longrightarrow 0$$
is exact at $A$ if and only if $A = 0$.

2. The sequence
$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$
is exact if and only if the map $f$ is an isomorphism.

3. The sequence
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
is exact if and only if $f$ is monic and $g$ is epic.

**Exercise C.2.12.** Let $\mathscr{A}$ be an Abelian category and assume that the diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & X \arrow[r, "h"'] & Y \arrow[r, "k"'] & Z & \end{tikzcd}$$

commutes with top and bottom rows exact sequences. Prove that the induced sequences

$$\operatorname{Ker}\alpha \xrightarrow{\overline{f}} \operatorname{Ker}\beta \xrightarrow{\overline{g}} \operatorname{Ker}\gamma$$

and

$$\operatorname{Coker}\alpha \xrightarrow[\widetilde{h}]{} \operatorname{Coker}\beta \xrightarrow[\widetilde{k}]{} \operatorname{Coker}\gamma$$

are also exact. Furthermore, show that if the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & X \arrow[r, "h"'] & Y \arrow[r, "k"'] & Z \arrow[r] & 0 \end{tikzcd}$$

commutes with exact rows, then the induced exact sequences take the form

$$0 \longrightarrow \operatorname{Ker}\alpha \xrightarrow{\overline{f}} \operatorname{Ker}\beta \xrightarrow{\overline{g}} \operatorname{Ker}\gamma$$

and

$$\operatorname{Coker}\alpha \xrightarrow[\widetilde{h}]{} \operatorname{Coker}\beta \xrightarrow[\widetilde{k}]{} \operatorname{Coker}\gamma \longrightarrow 0$$

instead.

**Exercise C.2.13.** Give an elementary diagram-chasing proof of the Snake Lemma in the category $R$-**Mod**.

**Exercise C.2.14** (The $3 \times 3$ Lemma)**.** Consider a commuting diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"'] & A_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"'] & A_2 \arrow[r] \arrow[d, "\alpha_2"'] & 0 \\
0 \arrow[r] & B_0 \arrow[r, "g_0"] \arrow[d, "\beta_0"'] & B_1 \arrow[r, "g_1"] \arrow[d, "\beta_1"'] & B_2 \arrow[r] \arrow[d, "\beta_2"'] & 0 \\
0 \arrow[r] & C_0 \arrow[r, "h_0"] \arrow[d] & C_1 \arrow[r, "h_1"] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in an Abelian category $\mathscr{A}$ in which the columns are all exact. Prove the following:

1. If the bottom two rows are exact, then so is the top row;
2. If the top two rows are exact, then so is the bottom row;
3. If the top and bottom rows are exact and if $g_1 \circ g_0 = 0$, then the middle row is exact.

**Exercise C.2.15.** Prove that the unique map $\tau : \operatorname{Coker} \overline{g} \to \operatorname{Coker} \alpha$ constructed in the proof of Lemma C.2.22 is monic. Hint: Assume that the diagram

$$\begin{tikzcd}
W \arrow[r, shift left, "\omega"] \arrow[r, shift right, "\sigma"'] & \operatorname{Coker} \overline{g} \arrow[r, "\tau"] & \operatorname{Coker} \alpha
\end{tikzcd}$$

commutes and consider the pullback

$$\begin{tikzcd}
Q \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & W \arrow[d, "\omega - \sigma"] \\
\operatorname{Ker} \gamma \arrow[r, "\operatorname{coker} \overline{g}"'] & \operatorname{Coker} \overline{g}
\end{tikzcd}$$

in $\mathscr{A}$. Prove that $q_1$ is epic, and then use this to prove that $(\omega - \sigma) \circ q_1 = 0$. Conclude that $\omega - \sigma = 0$ and hence that $\omega = \sigma$.

**Exercise C.2.16.** Prove that the Snake Lemma is natural in the following sense: Consider a commuting diagram

$$\begin{tikzcd}
 & A \arrow[rr, "f"] \arrow[dr] \arrow[dd, "\alpha"'] & & B \arrow[rr, "g"] \arrow[dr] \arrow[dd, "\beta", near start] & & C \arrow[rr] \arrow[dr] \arrow[dd, "\gamma", near start] & & 0 & \\
 & & A' \arrow[rr, crossing over, "f'", near start] & & B' \arrow[rr, crossing over, "g'", near start] & & C' \arrow[rr] & & 0 \\
0 \arrow[r] & X \arrow[rr, "h"', near start] \arrow[dr] & & Y \arrow[rr, "k"', near start] \arrow[dr] & & Z \arrow[dr] & & & \\
 & 0 \arrow[r] & X' \arrow[rr, "h'"'] \arrow[from=uu, crossing over, "\alpha'", near start] & & Y' \arrow[rr, "k'"'] \arrow[from=uu, crossing over, "\beta'", near start] & & Z' \arrow[from=uu, crossing over, "\gamma'", near start] & &
\end{tikzcd}$$

in $\mathscr{A}$ for which each of the horizontal rows are exact. Show that there is an induced map between the resulting kernel/cokernel sequences from the back face to the front face, i.e., show that there exist morphisms making the diagram

$$\begin{tikzcd}
\operatorname{Ker}\alpha \arrow[r] \arrow[d] & \operatorname{Ker}\beta \arrow[r] \arrow[d] & \operatorname{Ker}\gamma \arrow[r] \arrow[d] & \operatorname{Coker}\alpha \arrow[r] \arrow[d] & \operatorname{Coker}\beta \arrow[r] \arrow[d] & \operatorname{Coker}\gamma \arrow[d] \\
\operatorname{Ker}\alpha' \arrow[r] & \operatorname{Ker}\beta' \arrow[r] & \operatorname{Ker}\gamma' \arrow[r] & \operatorname{Coker}\alpha' \arrow[r] & \operatorname{Coker}\beta' \arrow[r] & \operatorname{Coker}\gamma'
\end{tikzcd}$$

commute.

**Exercise C.2.17.** Let $\mathscr{A}$ be an Abelian category. Define a complex $A^\bullet \in \mathbf{Ch}(\mathscr{A})_0$ to be *split* if there exists a set of morphisms $\{s_n \in \mathscr{A}(A^{n+1}, A^n) \mid n \in \mathbb{Z}\}$ for which the identity

$$\partial_n \circ s_n \circ \partial_n = \partial_n$$

holds. That is, for all $n \in \mathbb{Z}$ the diagram

$$\begin{tikzcd}
A^n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"'] & A^{n+1} \arrow[d, "s_n"] \\
A^{n+1} & A^n \arrow[l, "\partial_n"]
\end{tikzcd}$$

commutes in $\mathscr{A}$. Moreover, we say that $A^\bullet$ is *split exact* if $A^\bullet$ splits and is exact.

1. Find an example of a complex $A^\bullet$ that splits.
2. Find an example of a complex $A^\bullet$ that does not split.
3. Prove that the sequence $X^\bullet$ defined via the diagram

   $$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\iota_1} A \oplus B \xrightarrow{\pi_2} B \longrightarrow 0 \longrightarrow \cdots$$

   is split exact for any $A, B \in \mathscr{A}_0$.
4. Fix an arbitrary complex $A^\bullet$ in $\mathbf{Ch}(\mathscr{A})$. Prove that we can produce a split exact complex $C(A^\bullet)$ as follows: Define the objects $C(A^\bullet)^n$ as

   $$C(A)^n := A^{n+1} \oplus A^n$$

   for all $n \in \mathbb{Z}$. Define the differentials $\partial_n^{C(A)}$ via:

   $$\partial_n^{C(A)} := \langle -\partial_{n+1}^A \circ \pi_1, \partial_n^A \circ \pi_2 - \pi_1 \rangle : A^{n+1} \oplus A^n \longrightarrow A^{n+2} \oplus A^{n+1}.$$

   Show that $(C(A), \partial_n^{C(A)})$ is indeed a complex in $\mathbf{Ch}(\mathscr{A})$. Now prove that $C(A)$ is split exact with splitting maps

   $$s_n := \langle -\operatorname{id}_{A^{n+1}} \circ \pi_2, 0 \rangle$$

   for all $n \in \mathbb{Z}$. This is what is called the mapping cone of the identity morphism on $A^\bullet$; see Definition **??** for the introduction of mapping cones in generality and some of their basic properties.

## C.3 Injectives and Projectives: Can we have Enough?

We now move to further our study of Abelian categories, faiseaux pervers, and algebraic geometry by learning about projective and injective objects. These should be seen as some sort of epimorphism classifying objects, or those whose contravariant hom-functors behave particularly well with respect to monics and epics. The presence of these objects is indispensable for the study of cohomology and the resolution of singularities of objects, as it allows us to say that we can reasonably compute (co)homology objects via resolutions; it also will give us canonical ways to do the homotopy theory that is intrinsic to the construction of derived categories.

Whenever possible in this section we will present the notions of projective and injective objects for all categories, as opposed to simply for Abelian categories. While we will likely not use explicitly injective and projective objects beyond the context additive categories, we will present some exercises showing that injectives *do* in fact arise in categories outside of the usual additive ones.

This section will be a little shorter than the last two, as it serves primarily as a location to discuss and introduce special classes of objects that will be important later on in this article when we learn about cohomology and how to actually compute cohomology.

**Definition C.3.1.** An object $P$ in a category $\mathscr{C}$ is *projective* if and only if whenever there is a morphism $f : P \to B$ and an epimorphism $k : A \to B$, then there exists a morphism $g : P \to A$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "g"'] \arrow[d, "f"] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$

commutes.

An important consequence of the above definition is that projective objects behave particularly well with the coYoneda embedding $\mathbf{y}^\vee : \mathscr{C}^{\mathrm{op}} \to [\mathscr{C}, \mathbf{Set}]$.

**Proposition C.3.2.** *Let $P$ be an object in a category $\mathscr{C}$. The following are equivalent:*

1. *$P$ is projective;*
2. *The functor $\mathscr{C}(P, -)$ preserves epimorphisms.*

*Proof.* (1) $\implies$ (2): To see that that $\mathscr{C}(P, -)$ preserves epimorphisms whenever $P$ is projective, assume that $e : A \to B$ is epic in $\mathscr{C}$ and consider the pushforward morphism

$$\mathscr{C}(P, e) := e_* : \mathscr{C}(P, A) \to \mathscr{C}(P, B)$$

in **Set**.[14] In order to prove that $e_*$ is epic, we must show that it is surjective, i.e., that for all $h \in \mathscr{C}(P, B)$ there exists a morphism $f : P \to A$ for which $h = e \circ f = e_*(f)$. However, this is immediate: Because $P$ is projective and $e$ is epic, we can form the diagram

$$\begin{tikzcd} A \arrow[r, "e"] & B \\ & P \arrow[ul, dashed, "\exists f"] \arrow[u, "h"'] \end{tikzcd}$$

which in turn implies that $h = e \circ f = e_*(f)$.

[14] Note that while I have formalized this proof in an implicit "locally small" type language, this is done wolog. If the category $\mathscr{C}$ is instead large, we can simply enrich our universe to a strictly larger Grothendieck universe $\mathscr{V}$ (taken to be sufficiently large so that every object $\mathscr{C}(X, Y)$, for all $X, Y \in \mathscr{C}_0$, is a $\mathscr{V}$-set) whose set theory satisfies an internal $\mathscr{V}$ version of the ZFC axioms and proceed mutatis mutandis in the enriched setting.

(2) $\implies$ (1): This is simply a restatement of the above argument. If $e_* : \mathscr{C}(P, A) \to \mathscr{C}(P, B)$ is epic whenever $e : A \to B$ is epic, then for any map $h : P \to B$ we can find an $f : P \to A$ for which

$$h = e_*(f) = e \circ f,$$

i.e., the diagram

$$\begin{tikzcd} A \arrow[r, "e"] & B \\ & P \arrow[u, "h"'] \arrow[ul, dashed, "\exists f"] \end{tikzcd}$$

commutes. However, this is exactly what it means to be projective. □

If $\mathscr{C}$ has pullbacks and if epimorphisms are reflected by pullbacks in $\mathscr{C}$, we can make the following adaptation to the above proposition. This alternative characterization of projectives shows that instead of checking that *every* cospan $A \xrightarrow{f} B \xleftarrow{h} P$ with $f$ epic admits a factorization through $P$, it suffices to prove that every epimorphism with codomain $P$ splits.

**Proposition C.3.3.** *For a category $\mathscr{C}$ with pullbacks that reflect epimorphisms, the following are equivalent:*

1. *$P$ is projective;*
2. *Every epimorphism $e : A \to P$ splits;*
3. *The functor $\mathscr{C}(P, -)$ preserves epimorphisms.*

*Proof.* (1) $\implies$ (2): This is immediate; If there is an epimorphism $g : A \to P$ and if $P$ is projective, using the map $\mathrm{id}_P : P \to P$ allows us to find a morphism $s : P \to A$ making the diagram

$$\begin{tikzcd} & P \arrow[dl, "s"'] \arrow[d, equal] \\ A \arrow[r, "g"'] & P \end{tikzcd}$$

commute. However, the equation $g \circ s = \mathrm{id}_P$ shows that $s$ is a section of $g$.

(2) $\implies$ (1): Assume $e : A \to B$ is an epimorphism in $\mathscr{C}$ and that there is a morphism $h : P \to B$. Now pullback against $e$ and $h$ to produce the diagram

$$\begin{tikzcd} Q \arrow[r, "q_1"] \arrow[d, "q_2"'] & P \arrow[d, "h"] \\ A \arrow[r, "e"'] & B \end{tikzcd}$$

and note that $q_1$ is epic. Consequently, there is a section $s : P \to Q$ for which $q_1 \circ s = \mathrm{id}_P$. Now consider that

$$e \circ q_2 \circ s = h \circ q_1 \circ s = h,$$

so setting $g := q_2 \circ s$ gives the desired map filling the diagram

$$\begin{tikzcd} & P \arrow[d, "h"] \arrow[dl, dashed, "\exists g"'] \\ A \arrow[r, "e"] & B \end{tikzcd}$$

and proving that $P$ is projective. □

**Corollary C.3.4.** *In an Abelian category, an object $P$ is projective if and only if all epimorphisms into $P$ split.*

The dual concept of a projective object is key to our study of perverse sheaves, so we will make it explicit here. These objects, which we will see are called injective, are also very important to algebraic geometry. We will see later in this chapter that it is not always possible for categories to admit the existence of nonzero projective objects (cf. Examples C.3.17 and C.3.18), but it is the case that the categories $\mathcal{O}_X$-**Mod** and **QCoh**$(X)$ (for a scheme $X = (|X|, \mathcal{O}_X)$) do admit nonzero injective objects; these are the well-known *flasque*[15] sheaves. As such, we will make at least the definition of injective objects and the dual proposition involving them explicit; however, proofs involving injective objects will generally be left to the reader as an exercise in duality.

**Definition C.3.5.** An object $I$ in a category is said to be *injective* if for every monic $m : A \to B$ and for every map $f : A \to I$, there exists a map $g : B \to I$ making the diagram

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists g"] \\ I & \end{tikzcd}$$

commute.

**Remark C.3.6.** It is immediate from the definitions that an object $I$ in a category $\mathscr{C}$ is injective if and only if $I$ is projective in $\mathscr{C}^{\mathrm{op}}$.

**Example C.3.7.** The Abelian group $\mathbb{Q}/\mathbb{Z}$ is injective in **Ab**, while the groups $\mathbb{Z}/n\mathbb{Z}$ are injective in the category of $\mathbb{Z}/n\mathbb{Z}$-modules. However, the groups $\mathbb{Z}/n\mathbb{Z}$ are *not* injective in **Ab**.

We now record the basic dual statements of the two alternative perspectives we gave on projective objects. One will be true for general categories, and the other in categories with pushouts for which the pushout against a monomorphism is again a monomorphism.

**Proposition C.3.8.** *The following are equivalent for an object $I$ in a category $\mathscr{C}$:*

1. *$I$ is injective;*
2. *The functor $\mathscr{C}(-, I) : \mathscr{C}^{\mathrm{op}} \to$ **Set** sends epimorphisms in $\mathscr{C}^{\mathrm{op}}$ to epimorphisms in **Set**.*

**Proposition C.3.9.** *Let $\mathscr{C}$ be a category with pushouts such that the pushout against a monomorphism is a monomorphism. The following are equivalent for an object $I$:*

1. *$I$ is injective;*
2. *Every monomoprhism $m : I \to A$ has a retract $r : A \to I$;*
3. *The functor $\mathscr{C}(-, I) : \mathscr{C}^{\mathrm{op}} \to$ **Set** sends epimorphisms in $\mathscr{C}^{\mathrm{op}}$ to epimorphisms in **Set**.*

We now move on to give a characterization of how adjoints interact with projective objects. Recall that an adjunction of functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ with unit/counit pair $(\eta, \varepsilon)$ is written as $F \dashv G : \mathscr{C} \to \mathscr{D}$ whenever $F$ is the left adjoint to $G$. Furthermore, we assume some basic familiarity with the triangle identities satisfied by adjoint functors; the reader unfamiliar can either trust me[16] or see an introductory work on category theory, such as [53] or Section 1.4 of my notes [**?**].

[15] Also known as flabby, but this is much less fun to say. It is also vaguely insulting to walk up to a sheaf in the middle of a busy party and call it "flabby" in front of all the popular kids, like the Serre twisting sheaves $\mathcal{O}_X(n)$; at least there's a chance that they don't know French (they all know French, courtesy of Serre and Grothendieck), so calling them flasque can provide you and your sheafy friends with social grace.

[16] Don't trust me.

**Proposition C.3.10.** *Let $F \dashv G : \mathscr{C} \to \mathscr{D}$ be an adjunction such that $G$ preserves epimorphisms. Then $F$ sends projective objects to projective objects.*

*Proof.* Begin by assuming that $P$ is a projective object in $\mathscr{C}$ and that there is a diagram of the form

$$\begin{tikzcd} & FP \arrow[d, "h"] \\ X \arrow[r, "e"'] & Y \end{tikzcd}$$

in $\mathscr{D}$, where $e : X \to Y$ is an epimorphism. Applying the functor $G$ to the diagram above, using that $Ge$ is an epimorphism by assumption, and using the unit of adjunction $\eta_P$ then gives the diagram:

$$\begin{tikzcd} P \arrow[r, "\eta_P"] & G(FP) \arrow[d, "Gh"] \\ GX \arrow[r, "Ge"'] & GY \end{tikzcd}$$

Because $P$ is projective, we can find a $\varphi : P \to GX$ making the diagram

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[d, dashed, "\exists\varphi"'] & G(FP) \arrow[d, "Gh"] \\ GX \arrow[r, "Ge"'] & GY \end{tikzcd}$$

commute in $\mathscr{C}$. Now applying the functor $F$ and pasting the counit naturality square at $F(GX) \xrightarrow{F(ge)} F(GY)$ and at $F(G(FP))$ gives the diagram

$$\begin{tikzcd} FP \arrow[r, "F\eta_P"'] \arrow[d, "F\varphi"'] \arrow[rr, equal, bend left] & F(G(FP)) \arrow[r, "\varepsilon_{FP}"'] \arrow[d, "F(Gh)"] & FP \arrow[d, "h"] \\ F(GX) \arrow[r, "F(Gh)"'] \arrow[d, "\varepsilon_X"'] & F(GY) \arrow[r, "\varepsilon_Y"'] \arrow[d, "\varepsilon_Y"] & Y \arrow[dl, equal] \\ X \arrow[r, "e"'] & Y & \end{tikzcd}$$

in $\mathscr{D}$. It then follows that

$$e \circ \varepsilon_X F\varphi = \varepsilon_Y \circ F(Ge) \circ F\varphi = \varepsilon_Y \circ F(Gh) \circ F\eta_P = h \circ \varepsilon_{FP} \circ F\eta_P = h \circ \mathrm{id}_{FP} = h.$$

The map $g := \varepsilon_X \circ F\varphi : FP \to X$ thus makes the diagram

$$\begin{tikzcd} & FP \arrow[d, "h"] \arrow[dl, "g"'] \\ X \arrow[r, "e"'] & Y \end{tikzcd}$$

commute, proving that $FP$ is projective. $\square$

**Corollary C.3.11.** *If $F \dashv G : \mathscr{C} \to \mathscr{D}$ is an adjunction such that $F$ preserves monomorphisms, then $G$ sends injectives to injectives.*

We now recall that an object $R$ is said to be a *retract* of $X$ if there exists a morphism $r : X \to R$ and a morphism $s : R \to X$ such that $r \circ s = \mathrm{id}_R$. The reason we introduce retracts is because projective objects play particularly well with retracts; in fact, we will prove that retracts of projective objects are projective!

**Proposition C.3.12.** *If $P$ is projective in a category $\mathscr{C}$ and if $R$ is a retract of $P$, then $R$ is projective.*

*Proof.* Let $e : A \to B$ be an epimorphism, let $h : R \to B$ be a morphism, and let $r : P \to R$ and $s : R \to P$ be the section/retraction pair, i.e.,

$$r \circ s = \mathrm{id}_R .$$

We can then produce the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists g"'] \arrow[d, shift right, "r"'] \\ & R \arrow[u, shift right, "s"'] \arrow[d, "h"] \\ A \arrow[r, "e"'] & B \end{tikzcd}$$

in $\mathscr{C}$ with the map $g : P \to A$ making $e \circ g = h \circ r$ existing because $P$ is projective. However, it then follows that

$$e \circ g \circ s = h \circ r \circ s = h \circ \mathrm{id}_R = h$$

implying that the diagram

$$\begin{tikzcd} & R \arrow[dl, "g \circ s"'] \arrow[d, "h"] \\ A \arrow[r, "e"'] & B \end{tikzcd}$$

commutes. Thus $R$ is projective and we are done. □

Now that we have met injective and projective objects in categories, we would like to discuss when we have "enough[17]" injectives or projectives. This should be seen as a homotopy-like condition asking for every object to be a subobject of an injective object or quotient of a projective object[18], or asking if this is or is not possible. The connection to homotopy theory will be seen later in the chapter on model categories, but for the moment is not necessary.

**Definition C.3.13.** A category $\mathscr{C}$ is said to have *enough projectives* if for every object $X$ of $\mathscr{C}$, there exists a projective $P$ and an epimorphism $P \to X$. Dually, $\mathscr{C}$ is said to have *enough injectives* if for every object $A$ of $\mathscr{C}$, there exists an injective $I$ and a monomorphism $A \to I$.

We will end this section with some examples. In particular, we will show three categories that have enough projectives/injectives and then show a two categories that have enough injectives but not enough projectives. The final two examples also justify, in that regard, the use of cohomology to study algebraic geoemtric objects, while simultaneously showing that sheaf homology is not helpful.

**Example C.3.14.** If $R$ is a ring with identity, then the categories $R$-**Mod** and **Ch**($R$-**Mod**) both have enough projectives and enough injectives.

**Example C.3.15.** Every object in the terminal category $\mathbb{1} = (\{*\}, \{\mathrm{id}_*\})$ and in the initial category $\varnothing = (\varnothing, \varnothing)$ is both injective and projective.

[17]Just like potato chips, you can never have enough.

[18]Or the analogous generalization of for all objects $X$ there existing a projective $P_X$ for which there is an epimorphism $P_X \to X$.

**Example C.3.16.** If $R$ is a nonunital ring and $\mathscr{A}$ is the category of nondegenerate left $R$-modules, then $\mathscr{A}$ has enough injectives and projectives.

**Example C.3.17** ([33], Exercise III.6.2.a)**.** If $K$ is an infinite field, and if $X = \mathbb{P}^1_K$ (as a scheme, of course), then there does not exist a projective object $\mathscr{P}$ in the category $\mathcal{O}_X$-**Mod** with an epimorphism $\mathscr{P} \to \mathcal{O}_X \to 0$. In particular, the only projective object in $\mathcal{O}_X$-**Mod** is then the zero object.

To see this, first let $\mathfrak{a} \in |\mathbb{P}^1_K|$ be a closed point and let $V$ be a nontrivial open containing $\mathfrak{a}$. Then any epimorphism $\mathscr{P} \to \mathcal{O}_{\mathbb{P}^1_K}$ in the exact sequence $\mathscr{P} \to \mathcal{O}_{\mathbb{P}^1_K} \to 0$ factors as

$$\begin{tikzcd} \mathscr{P} \arrow[r] & \mathcal{O}_{\mathbb{P}^1_K} \arrow[rr] \arrow[dr] & & 0 \\ & & i_* \mathcal{O}_{\kappa(\mathfrak{a})} \arrow[ur] & \end{tikzcd}$$

where $i : \{\mathfrak{a}\} \to \mathbb{P}^1_K$ is the inclusion of the closed point and $\kappa(\mathfrak{a})$ is the residue field of the local ring $\mathcal{O}_{\mathbb{P}^1_K,\mathfrak{a}}$. Now consider a nontrivial open $U$ of $\mathfrak{a}$ and define $V := |\mathbb{P}^1_K| \setminus \{\mathfrak{a}\}$ with open immersions $j : U \to \mathbb{P}^1_K$ and $j' : V \to \mathbb{P}^1_K$. Then $\{U, V\}$ is a cover of $\mathbb{P}^1_K$, and so there is an epimorphism of $\mathcal{O}_{\mathbb{P}^1_K}$-modules

$$j_!(\mathcal{O}_{\mathbb{P}^1_K}\big|_U) \oplus j'_!(\mathcal{O}_{\mathbb{P}^1_K}\big|_V) \to \mathcal{O}_{\mathbb{P}^1_K},$$

which can be seen by checking at the level of stalks. Now, if $\mathscr{P}$ is a projective in the scheme-module category, since the map $\mathcal{O}_{\mathbb{P}^1_K} \to i_* \mathcal{O}_{\kappa(\mathfrak{a})}$ is an epimorphism, there is a factorization

$$\begin{tikzcd} & \mathscr{P} \arrow[dl, dashed, "\exists\lambda"'] \arrow[d] & \\ j_!(\mathcal{O}_{\mathbb{P}^1_K}\big|_U) \oplus j'_!(\mathcal{O}_{\mathbb{P}^1_K}\big|_V) \arrow[r] & \mathcal{O}_{\mathbb{P}^1_K} \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{O}_X$-**Mod**. However, upon calculating the global sections we find that

$$\left(j_!\,\mathcal{O}_{\mathbb{P}^1_K}\big|_U \oplus j'_!\,\mathcal{O}_{\mathbb{P}^1_K}\big|_V\right)(|\mathbb{P}^1_K|) = j_!\,\mathcal{O}_U(|\mathbb{P}^1_K|) \oplus j'_!\,\mathcal{O}_V(|\mathbb{P}^1_K|) = 0$$

and hence that the epimorphism itself must be zero. This in turn implies that the map $\mathscr{P} \to \mathcal{O}_{\mathbb{P}^1_K}$ is also zero, which shows that there can be no projective object $\mathscr{P}$ with a nonzero epimorphism to $\mathcal{O}_{\mathbb{P}^1_K}$.

**Example C.3.18** ([33], Exercise III.6.2.b)**.** If $K$ is an infinite field, and if $X = \mathbb{P}^1_K$, then there does not exist a projective object $\mathscr{P}$ in the category $\mathbf{QCoh}(X)$ with an epimorphism $\mathscr{P} \to \mathcal{O}_X \to 0$. In particular, this shows that the same is true of $\mathbf{Coh}(X)$.

To see why this holds, note first that it suffices to argue using the Serre twisting sheaves; that is, if we can show that now $\mathcal{O}_X(n)$ is projective, then we will be done. So first let $\mathscr{F}$ be a nonzero quasicoherent sheaf on $X$. Because $\dim \mathbb{P}^1_K > 0$, there exists an $n \in \mathbb{N}$ strictly positive so that there is an epimorphism $\varepsilon : \mathcal{O}_X(n) \to \mathscr{F}$. Moreover, by construction of the twisting sheaves of Serre, there are no nonzero maps from $\mathcal{O}_X(n) \to \mathcal{O}_X(m)$ whenever $m < n$; that is, $\varphi \in \mathbf{QCoh}(\mathbb{P}^1_K)(\mathcal{O}_X(n), \mathcal{O}_X(m))$ for $m < n$ implies that $\varphi = 0$. Now, since $\mathscr{F}$ is quasicoherent and there is an epimorphism from $\mathcal{O}_X(n)$ on to $\mathscr{F}$, we can find an index set $I$ and a collection $\{m_i \mid i \in I, 0 < m_i < n\}$ for which there is an epimorphism

$$\bigoplus_{i \in I} \mathcal{O}_X(m_i) \xrightarrow{\varepsilon'} \mathscr{F}.$$

However, because each of the $m_i < n$, there is no way that the epimorphism $\varepsilon'$ can split through $\varepsilon$, and so $\mathcal{O}_X(n)$ cannot be projective.

## Exercises

**Exercise C.3.1.** Prove that a category $\mathscr{C}$ has enough injectives if and only if $\mathscr{C}^{\mathrm{op}}$ has enough projectives.

**Exercise C.3.2.** Assume that $\mathscr{C}$ has enough projectives and that $\mathbf{y} : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding. Prove or disprove: $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ has enough projectives.

**Exercise C.3.3** (For those who know some topos theory; cf. [54], section IV.10 for details)**.** Let $\mathcal{E}$ be a topos. We will prove that in $\mathcal{E}$, every object $X$ is a subobject of some injective object.

1. Prove that the subobject classifier $\Omega$ of $\mathcal{E}$ is injective.

2. Prove that for any object $X$, the internal hom-space $[X, \Omega]$ is injective.

3. Prove that the canonical map $X \to [X, \Omega]$ is monic. Hint: The canonical map $\Delta_X : X \to X \times X$ is monic and the classifying map $\chi_\Delta : X \times X \to \Omega$ fits into the diagram:

$$\begin{tikzcd} X \arrow[r, dashed, "!_X"] \arrow[d, "\Delta_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & \top \arrow[d, "\mathrm{true}"] \\ X \times X \arrow[r, "\chi_\Delta"'] & \Omega \end{tikzcd}$$

The map $X \to [X, \Omega]$ is then the exponential transpose:

$$\frac{\chi_\Delta : X \times X \to \Omega}{X \to [X, \Omega]}$$

Prove that this map is monic by adjunction-pushing.

**Exercise C.3.4.** Show that every object $X$ in $\mathbf{Set}$ is both injective and projective.

**Exercise C.3.5.** Let $\mathscr{C}$ be a category with a zero object. Prove that 0 is both projective and injective.

**Exercise C.3.6.** Let $F \dashv G : \mathscr{C} \to \mathscr{D}$ be an adjunction such that $F$ preserves monomorphisms. Prove that $G$ sends injective objects to injective objects.

**Exercise C.3.7** (A useful exercise for algebraic geometers)**.** Let $f : X \to Y$ be a closed immersion of schemes and consider the corresponding functors $f^* : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X), f_* : \mathbf{QCoh}(X) \to \mathbf{QCoh}(Y)$, and $f^! : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$ together with induced adjunctions $f^* \dashv f_*$ and $f_* \dashv f^!$. Prove that $f_*$ preserves both injectives and projectives. Hint: This is easy for affine schemes, so try it there first.

**Exercise C.3.8.** Find an example of a category with enough projectives, but not enough injectives.

**Exercise C.3.9.** Prove or disprove: An object $P$ is projective if and only if the map $0 \to P$ satisfies the left-lifting property against epimorphisms, i.e., if whenever there exists a commuting diagram of the form

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ P \arrow[r, "h"'] & B \end{tikzcd}$$

with $f$ an epimorphism, there exists a morphism $g$ making

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ P \arrow[r, "h"'] \arrow[ur, dashed, "\exists g"] & B \end{tikzcd}$$

commute. Similarly, prove that an object $I$ is injective if and only if the morphism $I \to 0$ satisfies the right lifting property against monomoprhisms.

**Exercise C.3.10.** Prove or disprove: If $\mathscr{C}$ is a category with enough injectives and enough projectives, and if $\mathscr{D}$ is a full subcategory of $\mathscr{C}$, is it true that $\mathscr{D}$ has enough injectives or projectives.

**Exercise C.3.11.** Prove that if $\mathscr{C} \simeq \mathscr{D}$, then $\mathscr{C}$ has enough injectives if and only if $\mathscr{D}$ has enough injectives.

**Exercise C.3.12.** Assume that $\mathscr{A}$ is an Abelian category and assume that the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathscr{A}$ is short exact. Prove that if there are monics $i : A \to I$ and $k : C \to K$, where $I$ and $K$ are injective objects in $\mathscr{A}$, then there exists an injective object $J$ and a monomorphism $j : B \to K$ such that there are morphisms $\iota : I \to J$ and $\pi : J \to K$ which make the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[r, "g"] \arrow[d, "j"'] & C \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & I \arrow[r, "\iota"'] & J \arrow[r, "\pi"'] & K \arrow[r] & 0
\end{tikzcd}$$

commute with exact rows. Hint: Take $J = I \oplus K$.

**Exercise C.3.13.** Let $\mathscr{C}$ be a category with finite limits. Prove or disprove: If $I$ and $J$ are injective objects, then $I \times J$ is injective.

## C.4 Additive Functors and Exact Functors

We begin this section by defining additive functors and studying how they behave with respect to exactness. These functors will be the central tool we use to compare Abelian and additive categories, as they are the functors which preserve the additive structure we require[19] and those that we will use to produce cohomology. For references to additive functors, see, for example [35], [53], [52], or [39].

**Definition C.4.1.** Let $\mathscr{A}$ and $\mathscr{B}$ be additive categories. A functor $F : \mathscr{A} \to \mathscr{B}$ is said to be *additive* if and only if for all $X, Y \in \mathscr{A}_0$, the induced map

$$\mathscr{A}(X,Y) \to \mathscr{B}(FX, FY)$$

is a morphism of Abelian groups.

An important property of additive functors is that they preserve biproducts. This will be fundamental when we start to look at cohomology and do computations with cohomology.

**Proposition C.4.2.** *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Then $F$ preserves biproducts.*

*Proof.* We first show that $F$ preserves the zero object. Observe that an object $A$ in any category with a zero object is a zero object if and only if $\mathrm{id}_A = 0_A$, where $0_A$ is the endomorphism:

$$\begin{tikzcd} A \arrow[rr, "0_A"] \arrow[dr, dashed, "\exists!"'] & & A \\ & 0 \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Let $Z$ be the zero object in $\mathscr{A}$ and observe that since $F$ is a functor, $F$ preserves identities; as such, $F(\mathrm{id}_Z) = \mathrm{id}_{FZ}$. Similarly, since the map $F : \mathscr{A}(0,0) \to \mathscr{B}(F0, F0)$ is a morphism of groups, $F(0_Z) = 0_{FZ}$. Because functors preserve algebraic relations on morphisms, it then follows that

$$0_{FZ} = F(0_Z) = F(\mathrm{id}_Z) = \mathrm{id}_{FZ}\,.$$

Thus $FZ$ is a zero object in $\mathscr{B}$.

We now show that $F$ preserves nonempty biproducts. Consider the commuting diagram

$$\begin{tikzcd} A \arrow[dd, equal] \arrow[dr, "i_1"] & & B \arrow[dd, equal] \arrow[dl, "i_2"'] \\ & A \oplus B \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ A & & B \end{tikzcd}$$

in $\mathscr{A}$. Note that the commutative diagram above, together with the induced equational identities completely describe the biproduct (cf. Exercise C.1.17). Applying the functor $F$ to the diagram then produces the diagram

$$\begin{tikzcd} FA \arrow[dd, equal] \arrow[dr, "Fi_1"] & & FB \arrow[dd, equal] \arrow[dl, "Fi_2"'] \\ & F(A \oplus B) \arrow[dl, "Fp_1"] \arrow[dr, "Fp_2"'] & \\ FA & & FB \end{tikzcd}$$

[19] Asking functors not to preserve these is a little like asking morphisms of unital rings to not preserve the identity. For a more grounded example, it's like being quite tired and then asking for decaf coffee; either case is occasionally necessary (especially if you're a $C^*$-algebraist), but it is generally silly.

in $\mathscr{B}$. Furthermore, the fact that $F$ is a functor and that $F$ induces group homomorphisms between hom-sets then allows us to deduce the following algebraic identities hold in the above commutative diagram:

- $Fp_1 \circ Fi_1 = \mathrm{id}_{FA}$ and $Fp_2 \circ Fi_2 = \mathrm{id}_{FB}$;
- $Fp_2 \circ Fi_1 = 0$ and $Fp_1 \circ Fi_2 = 0$;
- $Fi_1 \circ Fp_1 + Fi_2 \circ Fp_2 = \mathrm{id}_{F(A \oplus B)}$.

Thus appealing to Exercise C.1.17 shows that $F(A \oplus B) \cong FA \oplus FB$ and so we are done. □

**Proposition C.4.3.** *If $\mathscr{A}$ and $\mathscr{B}$ are additive categories and if $F : \mathscr{A} \to \mathscr{B}$ preserves biproducts, then $F$ is additive.*

*Proof.* Begin by observing that, if $\Delta_A : A \to A \oplus A$ is the diagonal at $A$, the diagram

$$A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{\langle f,g \rangle} B$$

composes to

$$\langle f, g \rangle \circ \Delta_A = f + g.$$

Using that $F$ preserves biproducts shows that

$$F(f+g) = F(\langle f,g \rangle \circ \Delta_A) = F(\langle f,g \rangle) \circ F\Delta_A = \langle F(f), F(g) \rangle \circ \Delta_{FA} = F(f) + F(g).$$

However, this shows that $F_* : \mathscr{A}(A,B) \to \mathscr{B}(FA, FB)$ is a group homomorphism. □

**Corollary C.4.4.** *A functor $F : \mathscr{A} \to \mathscr{B}$ is additive if and only if it preserves all biproducts.*

**Proposition C.4.5.** *If $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ are additive, then $G \circ F : \mathscr{A} \to \mathscr{C}$ is additive.*

*Proof.* Suppose that $A, A' \in \mathscr{A}_0$. Then the map

$$\mathscr{A}(A, A') \to \mathscr{B}(FA, FA')$$

is a homomorphism of Abelian groups; similarly, so is

$$\mathscr{B}(FA, FA') \to \mathscr{C}\left(G(FA), G(FA')\right).$$

Thus it follows from the fact that **Cat** and **Grp** are categories that

$$\mathscr{A}(A, A') \to \mathscr{C}\left((G \circ F)A, (G \circ F)A'\right)$$

is a group homomorphism and so we are done. □

**Corollary C.4.6.** *If $F : \mathscr{A} \to \mathscr{B}$ is additive and $G : \mathscr{B} \to \mathscr{C}$ is a functor which is additive on the essential image of $F$, then $G \circ F$ is additive.*

**Lemma C.4.7.** *Let $\mathscr{C}$ be a category with a zero object and let $S$ be a left Ore system in $\mathscr{C}$. Then $\lambda_S(0)$ is a zero object in $S^{-1}\mathscr{C}$.*

*Proof.* Let $0$ be the zero object in $\mathscr{C}$ and note that by Theorem B.2.21, $0$ remains initial in $S^{-1}\mathscr{C}$; thus it suffices to prove that $0$ is a terminal object in $S^{-1}\mathscr{C}$ as well. Note that because $0$ is a zero object in $\mathscr{C}$, for all objects $X$ of $\mathscr{C}$ there exists a roof $X \xrightarrow{!_X} 0 \xleftarrow{\mathrm{id}_0} 0$ in $S^{-1}\mathscr{A}$, where $!_X : X \to 0$ is the unique morphism in $\mathscr{C}$. In order to see that this is unique it suffices to prove that for any roof $X \xrightarrow{f} Y \xleftarrow{s} 0$, there exists an object $Z$ and maps $\alpha, \beta$ making

$$\begin{tikzcd} & 0 \arrow[d, "\alpha"] \arrow[dr, equal] & \\ X \arrow[ur, "!_X"] \arrow[dr, "f"'] & Z & 0 \arrow[dl, "s"] \\ & Y \arrow[u, "\beta"'] & \end{tikzcd}$$

commute. To this end let $!_Y : Y \to 0$ denote the unique map in $\mathscr{C}$ and consider that the diagram

$$\begin{tikzcd} 0 \arrow[r, "s"] & Y \arrow[r, shift left, "\mathrm{id}_Y"] \arrow[r, shift right, "s \circ !_Y"'] & Y \end{tikzcd}$$

commutes in $\mathscr{C}$ with $s \in S$. Thus there exists a morphism $t \in S$ which makes the diagram

$$\begin{tikzcd} Y \arrow[r, shift left, "\mathrm{id}_Y"] \arrow[r, shift right, "s \circ !_Y"'] & Y \arrow[r, "t"] & Z \end{tikzcd}$$

commute. As such it follows that

$$t = t \circ \mathrm{id}_Y = t \circ s \circ !_Y.$$

Thus we compute that

$$t \circ s = t \circ s \circ !_Y \circ s$$

and

$$t \circ f = t \circ s \circ !_Y \circ f = t \circ s \circ !_X$$

which shows the commutativity of the diagram

$$\begin{tikzcd} & 0 \arrow[d, "t \circ s"] \arrow[dr, equal] & \\ X \arrow[ur, "!_X"] \arrow[dr, "f"'] & Z & 0 \arrow[dl, "s"] \\ & Y \arrow[u, "t"] & \end{tikzcd}$$

in $\mathscr{C}$. This proves that $(s, f) \simeq (\mathrm{id}_0, !_X)$ and so we conclude that 0 is terminal in $S^{-1}\mathscr{C}$. Because 0 is also initial, this concludes the proof. $\square$

**Corollary C.4.8.** *If $\mathscr{C}$ is an additive category and $S$ is a left Ore system in $S$, then $S^{-1}\mathscr{C}$ is an additive category and $\lambda_S : \mathscr{C} \to S^{-1}\mathscr{C}$ is an additive functor.*

*Proof.* Because $\lambda_S$ preserves finite colimits by Theorem B.2.21, it follows that $S^{-1}\mathscr{C}$ has finite coproducts; moreover from Lemma C.4.7 we also know that $S^{-1}\mathscr{C}$ has a zero object. Thus we will be done by the dual to Proposition C.1.5 if we can show that $S^{-1}\mathscr{C}$ is **Ab**-enriched.

To show that $S^{-1}\mathscr{C}$ is **Ab**-enriched[20], let us define how to add two roofs. Let $X \xrightarrow{f} A \xleftarrow{s} Y$ and $X \xrightarrow{g} B \xleftarrow{t} Y$ be two morphisms in $S^{-1}\mathscr{C}(X, Y)$. In order to define the addition of these two roofs we first

[20]This is (locally) the last place we are going to be particularly nebulous about set-theoretic conditions. We will show that the construction of addition on roofs may be done provided that morphisms in $\mathscr{C}$ may be added to begin with. We can always be careful about this as well by enriching our universe of set theory to some new Grothendieck universe $\mathscr{V}$, and then instead of asking for **Ab**-enrichment, looking for **Ab**$(\mathscr{V})$-enrichment.

note consider the span $A \xleftarrow{s} Y \xrightarrow{t} B$ and note that both $s, t \in S$. Thus we can find a cospan $A \xrightarrow{s'} C \xleftarrow{t'} B$ with $s', t' \in S$ which makes the diagram

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, "s"'] & A \arrow[d, "s'"] \\ B \arrow[r, "t'"'] & C \end{tikzcd}$$

commuute. Furthermore, we can post-compose both $f$ and $g$ by $s'$ and $t'$, respectively, to produce the roofs

$$X \xrightarrow{s' \circ f} C \xleftarrow{s' \circ s} Y$$

and:

$$X \xrightarrow{t' \circ g} C \xleftarrow{t' \circ t} Y$$

It is worth noting as well that $(s' \circ s)^{-1} \circ (s' \circ f) = s^{-1} \circ f$ and $(t' \circ t)^{-1} \circ (t' \circ g) = t^{-1} \circ g$ in $S^{-1}\mathscr{A}$. We then define the addition

$$s^{-1} \circ f + t^{-1} \circ g$$

to be represented by the roof

$$\begin{tikzcd} X \arrow[dr, "s' \circ f + t' \circ g"'] & & Y \arrow[dl, "s' \circ s"] \\ & C & \end{tikzcd}$$

in $\mathscr{A}$. This addition is well-defined and describes an **Ab**-enrichment on $S^{-1}\mathscr{A}$ (cf. Exercise C.4.10), as each of these compositions in $\mathscr{A}$ is bilinear, and the Ore equivalence classes formed in the localization $\mathscr{A} \to S^{-1}\mathscr{A}$ respect the underlying bilinearity of composition in $\mathscr{A}$. From here using the dual of Proposition C.1.5 and Lemma C.4.7 gives that $S^{-1}\mathscr{A}$ has biproducts and a zero object. This proves that $S^{-1}\mathscr{A}$ is additive and completes the proof of the corollary. $\square$

We will not spend too much time looking at additive functors in complete generality. While it is true that additive functors all preserve finite products, it is not generically true that they preserve equalizers; dually, additive functors do preserve finite coproducts but need not preserve coequalizers. We will now focus on some examples of these situations and ponder how, if at all possible, it is possible to resolve them in the next section. Before seeing these examples, we will introduce some notation with which to discuss either the ability or failure of a functor to preserve limits/colimits.

**Definition C.4.9.** A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be *left exact* if and only if it preserves all finite limits; dually, $F$ is *right exact* if and only if it preserves all finite colimits. If $F$ is both left and right exact, then we simply say that $F$ is *exact*.

**Example C.4.10.** The identity functor $\mathrm{id}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ is always exact.

**Example C.4.11.** Let $\mathscr{A} = \mathbf{QCoh}(X)$ and $\mathscr{B} = \mathbf{QCoh}(Y)$, for $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ affine schemes with a morphism $f : X \to Y$. Then the extension functor $f_* : \mathbf{QCoh}(X) \to \mathbf{QCoh}(Y)$ is exact, while the functor $f^* : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$ is left exact and the functor $f^! : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$ is right exact. This follows from the following facts: $f_*$ is both a left and right adjoint; $f^*$ is a left adjoint; $f^!$ is a right adjoint; left adjoints preserve colimits, while right adjoints preserve limits[21]; and Exercise C.4.5.

**Example C.4.12.** Let $X$ and $Y$ be topological spaces with a continuous morphism $f : X \to Y$, and let $\mathbf{Shv}(X, \mathbf{Ab})$ and $\mathbf{Shv}(Y, \mathbf{Ab})$ denote the categories of sheaves of Abelian groups on $X$ and $Y$, respectively. Then the functor $f^{-1} : \mathbf{Shv}(Y, \mathbf{Ab}) \to \mathbf{Shv}(X, \mathbf{Ab})$ is right exact while the functor $f_* : \mathbf{Shv}(X, \mathbf{Ab}) \to \mathbf{Shv}(Y, \mathbf{Ab})$ is left exact. The exactness properties once again follow from the adjunction $f^{-1} \dashv f_*$.

[21]This is a well-known categorical fact that you can find in a text covering a first course in category theory. Explicitly, this is Theorem IV.5.1 of [53] or Theorem 1.4.12 of [**?**]

**Example C.4.13.** Consider the category $R$-**Mod** of left $R$-modules over a ring with identity $R$. Then for any fixed $R$-bimodule $A$, the functor $A \otimes_R (-) : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ which sends a left $R$-module to the Abelian group $A \otimes_R M$ is always right exact but *not* left exact in general. Once again the right-exactness follows from the fact that the tensor functor is a left adjoint. To see an explicit example[22], take $R = \mathbb{Z}$, $A = \mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$, and consider the exact sequence

$$0 \longrightarrow 2\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

in $\mathbf{Ab} \cong R$-**Mod**. Then tensoring with $\mathbb{F}_2$ gives the sequence

$$0 \longrightarrow \mathbb{F}_2 \otimes_{\mathbb{Z}} 2\mathbb{Z} \longrightarrow \mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2 \longrightarrow 0$$

Now observe that $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{F}_2$ and $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2 \cong \mathbb{F}_2$; thus in order for this functor to be left exact, it must be the case that $\mathbb{F}_2 \otimes_{\mathbb{Z}} 2\mathbb{Z} \cong 0$. However, it can be checked that for $2k \in 2\mathbb{Z}$,

$$2k \otimes \overline{1} = 2 \otimes \overline{k} = \begin{cases} 2 \otimes 1 & \text{if } k \equiv 1 \pmod 2; \\ 0 & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

Thus there are at least two distinct elements of $\mathbb{F}_2 \otimes_{\mathbb{Z}} 2\mathbb{Z}$ and so $\mathbb{F}_2 \otimes_{\mathbb{Z}} (-)$ is not left exact.

**Example C.4.14.** Let $G = \mathrm{Gal}(L/K)$ be a Galois group of a Galois field extension $L/K$ and consider the category $G$-**Mod** of left $G$-modules. Define the functor $(-)^G : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$ by sending a left $G$-module $M$ to the $G$-fix group

$$M^G := \{m \in M \mid gm = m \,\forall\, g \in G\}.$$

Now let $n \in \mathbb{N}$ be nonzero, let $k$ be a field for which $n$ is a unit in $k$ and for which the map $x \mapsto x^n$ is not surjective on $k^*$, let $k^{\text{ sep}}$ be a separable closure of $k$, and define $\mu_n = \operatorname{Spec} k[x]/(x^n - 1)$. Set

$$G := \mathrm{Gal}(k^{\text{ sep}}/k)$$

and recall that there is a natural identification

$$\mu_n(k^{\text{sep}}) \cong \mathbf{Sch}_{/\operatorname{Spec} k}(\operatorname{Spec} k^{\text{sep}}, \mu_n) \cong \mathbf{Cring}\left(\frac{k[x]}{(x^n - 1)}, k^{\text{sep}}\right) \cong \{\alpha \in k^{\text{sep}} \mid \alpha^n = 1\},$$

and more generally for any commutative $k$-algebra $A$,

$$\mu_n(A) \cong \{\alpha \in A \mid \alpha^n = 1\}.$$

There is then a short exact sequence

$$0 \longrightarrow \mu_n(k^{\text{sep}}) \longrightarrow (k^{\text{sep}})^* \xrightarrow{x \mapsto x^n} (k^{\text{sep}})^* \longrightarrow 0$$

of left $G$-modules, where the action is given by $\sigma\alpha = \sigma(\alpha)$ in each case. Applying the $G$-fix functor then sends the above sequence to the sequence

$$0 \longrightarrow \mu_n(k) \longrightarrow k^* \xrightarrow{x \mapsto x^n} k^* \longrightarrow 0$$

because $k^{\text{sep}}/k$ is Galois and hence $(k^{\text{sep}})^G = k$. Then the map $x \mapsto x^n$ is not surjective on $k^*$ by assumption, so it follows that the above sequence is not exact.

**Remark C.4.15.** The above two examples show that while a formal duality argument would imply that every left exact functor which is not right exact gives rise to a right exact functor which is not left exact, both notions arise in nature and in fundamentally different ways. Moreover, the first of the two examples motivates the homology of commutative rings with identity, while the second example motivates[23] the study of Galois cohomology.

[22] We have used Exercise C.4.5 here.

[23] Or perhaps is motivated by the study of Galois cohomology. Causality is not my strong suit.

## Exercises

**Exercise C.4.1.** Let $\mathscr{A}$ be an additive category and let $\mathbf{y} : \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ be the Yoneda embedding. Is $\mathbf{y}$ additive? Is $\mathbf{y}$ exact? Hint: For the additivity, determine if $\mathbf{y}$ commutes with finite products. For the exactness, try taking the colimit of an arbitrary family of representable functors and then try to commute this with finite products.

**Exercise C.4.2.** Prove that if $\mathscr{A}$ and $\mathscr{B}$ are additive categories, then the category $\mathbf{Add}(\mathscr{A}, \mathscr{B})$ of additive functors $F : \mathscr{A} \to \mathscr{B}$ and natural transformations between them is an additive category. If $\mathscr{A}$ and $\mathscr{B}$ are Abelian, is $\mathbf{Add}(\mathscr{A}, \mathscr{B})$ Abelian? Hint: We are not worried about set-theoretic issues here[24], so potentially don't worry about whether or not $\mathbf{Add}(\mathscr{A}, \mathscr{B})$ is small.

**Exercise C.4.3.** Is there a fully faithful functor $F : \mathscr{A} \to \mathscr{B}$ between additive categories such that $F$ is not additive? Why or why not?

**Exercise C.4.4.** Prove that for a functor $F : \mathscr{A} \to \mathscr{B}$ between additive functors, the following are equivalent:

1. $F$ is additive;
2. $F$ preserves finite biproducts;
3. $F$ preserves finite products;
4. $F$ preserves finite coproducts.

**Exercise C.4.5.** Define an additive functor $F : \mathscr{A} \to \mathscr{B}$ between Abelian categories to be *categorically (left) exact* if it preserves all limits, and *homologically (left) exact* if it sends exact sequences of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

to exact sequences of the form

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC$$

in $\mathscr{B}$. Prove that $F$ is homologically left exact if and only if it is categorically left exact. Conclude via dualization the same result for right exactness and deduce that an additive functor is exact in the categorical sense if and only if it sends exact sequences to exact sequences. Harder: Does this question even make sense when $\mathscr{A}$ and $\mathscr{B}$ are additive but not Abelian? Explain why (carefully) or provide a counter example to the statement.

**Exercise C.4.6** (A classical exercise in homological algebra)**.** We say that a bimodule $A$ over $R$ is *flat* if and only if the functor $A \otimes_R (-)$ is exact. Prove that any projective object $P$ which is also a bimodule over $R$-$\mathbf{Mod}$ is flat.

**Exercise C.4.7.** Let $\mathscr{A}$ be an additive category.

1. Prove that there is a category $\mathbf{Ch}(\mathscr{A})$ of chain complexes with values in $\mathscr{A}$ with morphisms chain maps of objects. Moreover, prove that $\mathbf{Ch}(\mathscr{A})$ is additive.
2. Let $n \in \mathbb{Z}$. Prove that there are "shift" endofunctors $[n] : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{A})$ which send an object $A^\bullet$ to the complex $A[n]^\bullet$, where
$$(A[n])^k := A^{n+k}$$
for all $k \in \mathbb{Z}$ (so we have shifted the complex $A^\bullet$ to the right by $k$ positions), while the differential is given by
$$\partial_k^{A[n]} = (-1)^n \partial_{n+k}^A.$$
We assign the functor $[n]$ on chain maps similarly, i.e., if $f \in \mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$ we define $f[n]$ via
$$f[n]_k := f_{n+k}.$$

[24] Because just like wheat flour, set theory may always be enriched.

3. Show that the shift functors $[n]$ are all additive by showing that $[n] = [1]^n$ (where $[1]^n = [1] \circ \cdots \circ [1]$) if $n$ is positive and $[n] = [-1]^n$ if $n$ is negative and then using that the composition of additive functors is again additive.

4. Classify the limits and colimits that the shift functors preserve.

5. Prove that the shift functors are automorphisms of $\mathbf{Ch}(\mathscr{A})$ by showing that $[n] \circ [m] = [n+m] = [m]+[n]$ for all $n, m \in \mathbb{Z}$ and that $[0] = \mathrm{id}_{\mathbf{Ch}(\mathscr{A})}$.

**Exercise C.4.8.** Let $\mathscr{A}$ and $\mathscr{B}$ be additive categories. Prove that if $F : \mathscr{A} \to \mathscr{B}$ is additive, there is an additive functor $\widetilde{F} : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{B})$ which sends each $A^\bullet$ to $(FA)^\bullet$ in the obvious way. This exercise is important and will be used without comment in much of the rest of the text.

**Exercise C.4.9.** Prove that if $\mathscr{A}$ is an additive category and $S$ is a left Ore system in $\mathscr{A}$, then if the zero maps $!_X : X \to 0$ or $\iota_X : 0 \to X$ are in $S$, then

$$S^{-1}\mathscr{A}(X, X) = \{\mathrm{id}_X\}.$$

**Exercise C.4.10.** If $\mathscr{A}$ is an additive category and if $S$ is a left Ore system in $\mathscr{A}$, prove explicitly that the addition of roofs described in Corollary C.4.8 is well-defined. Furthermore, prove that this addition is bilinear with respect to composition of morphisms.

**Exercise C.4.11.** Let $\mathscr{A}$ be an Abelian category and let $S$ be a two-sided Ore system in $\mathscr{A}$. Prove that the category $S^{-1}\mathscr{A}$ is Abelian.

**Exercise C.4.12.** Let $\mathscr{S}$ be a Serre subcategory of an Abelian category $\mathscr{A}$ (cf. Exercise B.2.10). This exercise will prove the following theorem: There exists an Abelian category $\mathscr{A} / \mathscr{S}$ and an exact essentially surjective functor $Q : \mathscr{A} \to \mathscr{A} / \mathscr{S}$ for which there is an equivalence of categories

$$\mathscr{S} \cong \mathrm{Ker}(Q).$$

Perhaps unsurprisingly our goal is to prove that $\mathscr{S}$ essentially induces a two-sided Ore system in $\mathscr{A}$ and invoke Exercise C.4.11.

Define the class of arrows we will localize as

$$S := \{f \in \mathscr{A}_1 \mid \mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathscr{S}_0\} \subseteq \mathscr{A}_1.$$

1. Prove that $S$ is a thick subcategory, i.e., that $S$ is closed with respect to composition and identities. Prove even further that it is saturated in the sense that if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is a composable triple of morphisms in $\mathscr{A}$ for which $h \circ g, g \circ f \in S$ then $g \in S$.

2. Prove that $S$ is satisfies the left Ore square condition by showing that for a span

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \\ X & \end{tikzcd}$$

with $s \in \mathscr{S}_1$, by considering the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "i_2"] \\ X \arrow[r, "i_1"'] & X \coprod_A B \end{tikzcd}$$

where $X \coprod_A B \cong \mathrm{Coker}([s, -f] : A \to X \oplus B)$, the diagram commutes with $i_2 \in \mathscr{S}_1$. Hint: Show that the natural map $\mathrm{Ker}(s) \to \mathrm{Ker}(i_2)$ is epic and conclude that $\mathrm{Ker}(i_2) \in \mathscr{S}_0$ and show that $\mathrm{Coker}(s) \cong \mathrm{Coker}(i_2)$ to prove that $i_2 \in S$.

3. Dualize the above argument to prove that $S$ satisfies the right Ore square condition.
4. Prove that if there is a commuting diagram

$$\begin{tikzcd} A \arrow[r, "s"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd}$$

in $\mathscr{A}$ with $s \in S$ then there is a $t \in S$ making the diagram

$$\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \arrow[r, "t"] & D \end{tikzcd}$$

commute. Conclude that $S$ is a left Ore system in $\mathscr{A}$.
5. Dualize the above claim and conclude that $S$ is a two-sided Ore system in $\mathscr{A}$.
6. Define the category $\mathscr{A}\,/\,\mathscr{S} := S^{-1}\mathscr{A}$ and the quotient functor $Q : \mathscr{A} \to \mathscr{A}\,/\,\mathscr{S}$ as the localization functor $\lambda_S : \mathscr{A} \to S^{-1}\mathscr{A}$. Prove that $S^{-1}\mathscr{A}$ is Abelian and that $\lambda_S$ is exact. Hint: Exercise C.4.11.
7. Prove that $\mathrm{Ker}(\lambda_S) \cong \mathscr{S}$. Can this isomorphism be taken to be a strict equality?
8. Show that $\lambda_S$ is essentially surjective and exact.
9. What is the universal property that $\mathscr{A}\,/\,\mathscr{S}$ satisfies?

**Exercise C.4.13.** In this exercise we'll be using Exercises B.2.10 and C.4.12 to give a description of the module category $S^{-1}R$-**Mod**. Let $R$ be a unital ring and let $S$ be a two-sided Ore set in $R$. Prove using Exercise B.2.10 Part 4 and Exercise C.4.12 that there is an isomorphism of categories

$$S^{-1}R\text{-}\mathbf{Mod} \cong R\text{-}\mathbf{Mod}\,/(R\text{-}\mathbf{Mod})_S.$$

**Exercise C.4.14.** Let $\mathscr{A}$ be an Abelian category and let $S$ be a left Ore system in $\mathscr{A}$ such that for any object $X \in \mathscr{A}_0$, the zero map $!_X : X \to 0$ is in $S$. Prove that for any $X, Y \in \mathscr{A}$, $S^{-1}\mathscr{A}(X, Y) \cong 0$. Conclude that every object $X \in S^{-1}\mathscr{A}$ is isomorphic to the zero object 0.

# Appendix D

# Cohomology Objects, Derived Functors, and Derived Functor Cohomology

Let us begin this section by considering a motivating question that we will use to try and fix the potential nonexactness[1] of additive functors: Using an arbitrary Abelian category $\mathscr{A}$, is it possible to measure when a sequence in $\mathscr{A}$ is exact, and if so, how can we record this?

To dissect this question, consider the sequence $A^\bullet$

$$\cdots \longrightarrow A^{k-1} \xrightarrow[\partial_{k-1}]{} A^k \xrightarrow[\partial_k]{} A^{k+1} \longrightarrow \cdots$$

in $\mathbf{Ch}(\mathscr{A})$. Recall that from the fact that $\partial_k \circ \partial_{k-1} = 0$, we have a canonical monomorphism

$$\gamma : \mathrm{Ker}(\mathrm{coker}\,\partial_{k-1}) \to \mathrm{Ker}\,\partial_k$$

which is an isomorphism if and only if the sequence is exact at $A^k$. However, this map $\gamma$ is an isomorphism if and only if $\gamma$ is epic. Thus, by taking the cokernel of $\gamma$, we can check if $\gamma$ is an isomorphism by checking if $\mathrm{Coker}\,\gamma \overset{?}{\cong} 0$. This gives us the first proposition below:

**Proposition D.0.1.** *Let $A^\bullet$ be a sequence in $\mathbf{Ch}(\mathscr{A})$. Then if $\gamma : \mathrm{Ker}(\mathrm{coker}\,\partial_{k-1}) \to \mathrm{Ker}\,\partial_k$ is the canonical monomorphism, the sequence $A^\bullet$ is exact at $A^k$ if and only if* $\mathrm{Coker}\,\gamma \cong 0$.

*Proof.* First note that if $A^\bullet$ is exact at $A^k$, then $\mathrm{Ker}(\mathrm{coker}\,\partial_{k-1}) \cong \mathrm{Ker}\,\partial_k$ through the map $\gamma$. However, in this case $\gamma$ is epic and so $\mathrm{Coker}\,\gamma \cong 0$.

On the other hand, note that if $\mathrm{Coker}\,\gamma \cong 0$, $\gamma$ is epic. In this case, since $\gamma$ is always monic, $\gamma$ is an isomorphism from $\mathrm{Ker}(\mathrm{coker}\,\partial_{k-1}) \to \mathrm{Ker}\,\partial_k$. This shows $A^\bullet$ is exact at $A^k$. □

**Definition D.0.2.** Let $A^\bullet$ be an object in $\mathbf{Ch}(\mathscr{A})$. The *i-th cohomology object of $A^\bullet$* is the object

$$H^i(A^\bullet) := \mathrm{Coker}(\gamma_k)$$

where $\gamma_k : \mathrm{Ker}(\mathrm{coker}\,\partial_{k-1}) \to \mathrm{Ker}\,\partial_k$ is the canonical map.

**Remark D.0.3.** We should think of these cohomology objects as a proto-cohomology/proto-invariants of a functor. While these objects do not mention or have anything to do with any functors anywhere, we will use them to construct the actual cohomolgy of a functor later on

[1] Because we are all mathematicians here, the use the hyphen after the prefix "non" is to be shunned and ignored. When asking if you should use it, think of the following French phrase: Non.

**Remark D.0.4.** A tool we will use in this section for computing cohomology is that of the homology of the opposite complex. This arises as follows: If we consider a complex $A^\bullet$ in $\mathbf{Ch}(\mathscr{A})$, define the complex $A^\bullet_{\mathrm{op}}$ by reversing all the arrows of $A^\bullet$, taking $A^n_{\mathrm{op}} := A^{-n}$, and adapting morphism indexes as necessary. In this way we define the *n-th homology object* of $A^\bullet_{\mathrm{op}}$ to be the cokernel of the new $\widetilde{\gamma}_n : \mathrm{Ker}(\mathrm{coker}\,\widetilde{\partial}_{n-1}) \to \mathrm{Ker}\,\widetilde{\partial}_n$. Note that $H_{-n}(A^\bullet_{\mathrm{op}}) = H^n(A^\bullet)$.

Exercise D.0.1 shows the connection between the definition of a cohomology object given above and how it is likely seen in a first course on modules[2]. In our strategy and goal to show how to fix the nonexactness of an additive functor, we will show how we can take an exact sequence of chain complexes

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

and produce a sequence in $\mathbf{Ch}(\mathscr{A})$

$$\cdots \longrightarrow H^n(A^\bullet) \longrightarrow H^n(B^\bullet) \longrightarrow H^n(C^\bullet) \xrightarrow{\delta^n} H^{n+1}(A^\bullet) \longrightarrow \cdots$$

which is exact at every $n \in \mathbb{Z}$. This will be a fundamental tool in constructing right derived functors, as we will see below; as such, we will give this construction in order to provide the technical tools as to why right derived functors, well, "right derive."

**Lemma D.0.5.** *Let $A^\bullet \in \mathbf{Ch}(\mathscr{A})_0$ be a sequence. For all $k \in \mathbb{Z}$, the map $\partial_k : A^k \to A^{k+1}$ induces a morphism*

$$\zeta_k : \mathrm{Coker}\,\partial_{k-1} \to \mathrm{Ker}\,\partial_{k+1}$$

*such that* $\mathrm{Ker}\,\zeta_k = H^k(A^\bullet)$ *and* $\mathrm{Coker}(\zeta_k) = H^{k+1}(A^\bullet)$.

*Proof.* We first prove how to construct $\zeta_k$. Begin by recalling that since $A^\bullet$ is an object of $\mathbf{Ch}(\mathscr{A})$, $\partial_n \circ \partial_{n-1} = 0$ for all $n \in \mathbb{Z}$. Furthermore, from this and the image factorization, we (again) consider the canonical map

$$\gamma_k : \mathrm{Ker}(\mathrm{coker}\,\partial_{k-1}) \to \mathrm{Ker}\,\partial_k.$$

Now observe that if we give the coimage factorization of $\partial_n$, we get a monic $m : \mathrm{Coker}(\ker \partial_n) \to A^{n+1}$ which makes the diagram

$$\begin{tikzcd} A^n \arrow[rr, "\partial_n"] \arrow[dr, "\mathrm{coker}(\ker \partial_n)"'] & & A^{n+1} \\ & \mathrm{Coker}(\ker \partial_n) \arrow[ur, "m"'] & \end{tikzcd}$$

commute. Furthermore, from this it follows that

$$0 = \partial_n \circ \partial_{n-1} = m \circ \mathrm{coker}(\ker \partial_n) \circ \partial_{n-1},$$

and hence, because $m$ is monic, it follows that

$$0 = \mathrm{coker}(\ker \partial_n) \circ \partial_{n-1} = 0.$$

From this and the universal property of the cokernel, we can find a unique morphism $\rho_n : \mathrm{Coker}\,\partial_{n-1} \to \mathrm{Coker}(\ker \partial_n)$ making the diagram

$$\begin{tikzcd} A^{n-1} \arrow[r, shift left, "\partial_{n-1}"] \arrow[r, shift right, "0"'] & A^n \arrow[r, "\mathrm{coker}\,\partial_{n-1}"] \arrow[dr, "\mathrm{coker}(\ker \partial_n)"'] & \mathrm{Coker}\,\partial_{n-1} \arrow[d, dashed, "\exists! \rho_n"] \\ & & \mathrm{Coker}(\ker \partial_n) \end{tikzcd}$$

[2] Or often how it is used in practice in such nasty things as singular cohomology. The trick is to take specific chain complexes and do things that way.

Now invoke Corollary C.2.11 and observe that

$$\operatorname{Coker}(\ker \partial_n) \cong \operatorname{Ker}(\operatorname{coker} \partial_n);$$

call this isomorphism $\theta$. Following this by the canonical map $\gamma_{n+1} : \operatorname{Ker}(\operatorname{coker} \partial_n) \to \operatorname{Ker} \partial_{n+1}$, we define the map $\zeta_n : \operatorname{Coker} \partial_{n-1} \to \operatorname{Ker} \partial_{n+1}$ to be the composite:

$$\begin{tikzcd}
\operatorname{Coker} \partial_{n-1} \arrow[r, "\rho_n"] \arrow[d, "\zeta_n"'] & \operatorname{Coker}(\ker \partial_n) \arrow[d, "\cong"] \\
\operatorname{Ker} \partial_{n+1} & \operatorname{Ker}(\operatorname{coker} \partial_n) \arrow[l, "\gamma_{n+1}"]
\end{tikzcd}$$

We now prove that $\operatorname{Coker}(\zeta_n) \cong H^{n+1}(A^\bullet)$. To see this, note that in the factorization given by $\rho_n$, we have that

$$\operatorname{coker}(\ker \partial_n) = \rho_n \circ \operatorname{coker}(\partial_{n-1})$$

where both cokernel morphisms are epic. Thus by the dual of Exercise C.1.14 it follows that

$$0 \cong \operatorname{Coker}(\operatorname{coker}(\ker \partial_n)) = \operatorname{Coker}(\rho_n \circ \operatorname{coker}(\ker \partial_{n-1})) = \operatorname{Coker}(\rho_n),$$

so $\rho_n$ is epic by Proposition C.1.29. Thus we compute that, by using the dual to Exercise C.1.14 and the fact that $\theta$ and $\rho_n$ are epic,

$$\operatorname{Coker}(\zeta_n) = \operatorname{Coker}(\gamma_{n+1} \circ \theta \circ \rho_n) = \operatorname{Coker} \gamma_{n+1} = H^{n+1}(A^\bullet).$$

Finally, we are left with showing that $\operatorname{Ker}(\zeta_n) \cong H^n(A^\bullet)$; however this is dual to the proof that $\operatorname{Coker}(\zeta_n) = H^{n+1}(A^\bullet)$ and follows from the following: $\rho_n^{\mathrm{op}}$ is the canonical map $\widetilde{\gamma}_{-n} : \operatorname{Ker}(\operatorname{coker} \partial_n^{\mathrm{op}}) \to \operatorname{Ker} \partial_{n-1}^{\mathrm{op}}$; $\operatorname{Coker} \widetilde{\gamma}_n$ is object-isomorphic to $\operatorname{Ker} \zeta_n$; and finally that $\operatorname{Coker} \widetilde{\gamma}_n$ is object isomorphic to $h_{-n}(A^\bullet_{\mathrm{op}})$ and $h_{-n}(A^\bullet_{\mathrm{op}}) \cong H^n(A^\bullet)$. □

**Theorem D.0.6.** *Let*

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

*be a short exact sequence in* $\mathbf{Ch}(A^\bullet)$. *Then there is a (long) exact sequence of cohomology objects:*

$$\cdots \longrightarrow H^k(A^\bullet) \longrightarrow H^k(B^\bullet) \longrightarrow H^k(C^\bullet) \xrightarrow{\delta_k} H^{k+1}(A^\bullet) \longrightarrow \cdots$$

*Proof.* We prove this by applying the Snake Lemma twice. First note that since the short exact sequence of complexes implies that for all $n \in \mathbb{Z}$, we get the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & A^n \arrow[r, "f_n"] \arrow[d, "\partial_n^A"'] & B^n \arrow[r, "g_n"] \arrow[d, "\partial_n^B"'] & C^n \arrow[r] \arrow[d, "\partial_n^C"] & 0 \\
0 \arrow[r] & A^{n+1} \arrow[r, "f_{n+1}"'] & B^{n+1} \arrow[r, "g_{n+1}"'] & C^{n+1} \arrow[r] & 0
\end{tikzcd}$$

in $\mathscr{A}$ in which both rows are exact. Applying Lemma C.2.22 and Exercise C.2.12 then give rise to an exact sequence

$$0 \longrightarrow \operatorname{Ker} \partial_n^A \longrightarrow \operatorname{Ker} \partial_n^B \longrightarrow \operatorname{Ker} \partial_n^C \longrightarrow \operatorname{Coker} \partial_n^A \longrightarrow \operatorname{Coker} \partial_n^B \longrightarrow \operatorname{Coker} \partial_n^C \longrightarrow 0$$

in $\mathscr{A}$ for all $n \in \mathbb{Z}$.

Fix now some $k \in \mathbb{Z}$ and consider the diagram

$$\begin{tikzcd}
 & \operatorname{Coker} \partial^A_{k-1} \arrow[r] \arrow[d, "\zeta^A_k"'] & \operatorname{Coker} \partial^B_{k-1} \arrow[r] \arrow[d, "\zeta^B_k"'] & \operatorname{Coker} \partial^C_{k-1} \arrow[r] \arrow[d, "\zeta^C_k"] & 0 \\
0 \arrow[r] & \operatorname{Ker} \partial^A_{k+1} \arrow[r] & \operatorname{Ker} \partial^B_{k+1} \arrow[r] & \operatorname{Ker} \partial^C_{k+1} &
\end{tikzcd}$$

in $\mathscr{A}$ where the $\zeta_k$ maps are those constructed in Lemma D.0.5; verifying that this diagram commutes is routine, while the exactness of the rows is shown above. Applying the Snake Lemma again gives the exact sequence

$$\operatorname{Ker} \zeta^A_k \longrightarrow \operatorname{Ker} \zeta^B_k \longrightarrow \operatorname{Ker} \zeta^C_k \longrightarrow \operatorname{Coker} \zeta^A_k \longrightarrow \operatorname{Coker} \zeta^B_k \longrightarrow \operatorname{Coker} \zeta^C_k$$

in $\mathscr{A}$. However, from Lemma C.2.22 we get that the exact sequence above takes the form

$$H^k(A^\bullet) \longrightarrow H^k(B^\bullet) \longrightarrow H^k(C^\bullet) \longrightarrow H^{k+1}(A^\bullet) \longrightarrow H^{k+1}(B^\bullet) \longrightarrow H^{k+1}(C^\bullet)$$

instead. Running this argument now for every $k \in \mathbb{Z}$ verifies the long exact sequence and completes the proof of the theorem. $\square$

**Proposition D.0.7.** *The cohomology object long exact sequence is natural in the sense that if the diagram*

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d, "\varphi"'] & B^\bullet \arrow[r] \arrow[d, "\psi"'] & C^\bullet \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & X^\bullet \arrow[r] & Y^\bullet \arrow[r] & Z^\bullet \arrow[r] & 0
\end{tikzcd}$$

*is a commuting diagram in* $\mathbf{Ch}(\mathscr{A})$ *with exact rows, for* $\mathscr{A}$ *an Abelian category, then the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(A^\bullet) \arrow[r] \arrow[d] & H^n(B^\bullet) \arrow[r] \arrow[d] & H^n(C^\bullet) \arrow[r] \arrow[d] & H^{n+1}(A^\bullet) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^n(X^\bullet) \arrow[r] & H^n(Y^\bullet) \arrow[r] & H^n(Z^\bullet) \arrow[r] & H^{n+1}(X^\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

*commutes for all* $n \in \mathbb{Z}$.

*Proof.* We begin by mimicking the proof of Theorem D.0.6: We use the exactness of each row of the diagram to produce, for all $n \in \mathbb{Z}$, the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \partial^A_n \arrow[r] \arrow[d, "\overline{\varphi}_n"'] & \operatorname{Ker} \partial^B_n \arrow[r] \arrow[d, "\overline{\psi}_n"'] & \operatorname{Ker} \partial^C_n \arrow[r] \arrow[d, "\overline{\rho}_n"'] & \operatorname{Coker} \partial^A_n \arrow[r] \arrow[d, "\widetilde{\varphi}_n"] & \operatorname{Coker} \partial^B_n \arrow[r] \arrow[d, "\widetilde{\psi}_n"] & \operatorname{Coker} \partial^C_n \arrow[r] \arrow[d, "\widetilde{\rho}_n"] & 0 \\
0 \arrow[r] & \operatorname{Ker} \partial^X_n \arrow[r] & \operatorname{Ker} \partial^Y_n \arrow[r] & \operatorname{Ker} \partial^Z_n \arrow[r] & \operatorname{Coker} \partial^X_n \arrow[r] & \operatorname{Coker} \partial^Y_n \arrow[r] & \operatorname{Coker} \partial^Z_n \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact by the Snake Lemma (cf. Lemma C.2.22) and Exercise C.2.12. Proceeding as in the proof of Theorem D.0.6, with the $\zeta_n$ morphisms the morphisms constructed in Lemma D.0.5, we construct the commuting diagram

$$
\begin{tikzcd}
 & \operatorname{Coker} \partial_n^A \arrow[rr] \arrow[dr, "\widetilde{\varphi}_n"] \arrow[dd, "\zeta_n^A"'] & & \operatorname{Coker} \partial_n^B \arrow[rr] \arrow[dr, "\widetilde{\psi}_n"] \arrow[dd, "\zeta_n^B"'] & & \operatorname{Coker} \partial_n^C \arrow[r] \arrow[dr, "\widetilde{\rho}_n"] \arrow[dd, "\zeta_n^C"] & 0 \\
 & & \operatorname{Coker} \partial_n^X \arrow[rr] \arrow[dd, "\zeta_n^X"'] & & \operatorname{Coker} \partial_n^Y \arrow[rr] \arrow[dd, "\zeta_n^Y"'] & & \operatorname{Coker} \partial_n^Z \arrow[u, bend right] \arrow[dd, "\zeta_n^Z"] \\
0 \arrow[r] & \operatorname{Ker} \partial_{n+1}^A \arrow[rr] \arrow[dr, "\overline{\varphi}_{n+1}"'] & & \operatorname{Ker} \partial_{n+1}^B \arrow[rr] \arrow[dr, "\overline{\psi}_{n+1}"'] & & \operatorname{Ker} \partial_{n+1}^C \arrow[dr, "\overline{\rho}_{n+1}"] & \\
 & 0 \arrow[r] & \operatorname{Ker} \partial_{n+1}^X \arrow[rr] & & \operatorname{Ker} \partial_{n+1}^Y \arrow[rr] & & \operatorname{Ker} \partial_{n+1}^Z
\end{tikzcd}
$$

in which every row is exact. Using the Snake Lemma again (together with Lemma D.0.5 and Exercise C.2.16) while varying $n$ takes the above diagram to the commuting diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & H^n(A^\bullet) \arrow[r] \arrow[d] & H^n(B^\bullet) \arrow[r] \arrow[d] & H^n(C^\bullet) \arrow[r] \arrow[d] & H^{n+1}(A^\bullet) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^n(X^\bullet) \arrow[r] & H^n(Y^\bullet) \arrow[r] & H^n(Z^\bullet) \arrow[r] & H^{n+1}(X^\bullet) \arrow[r] & \cdots
\end{tikzcd}
$$

with long exact rows. □

**Remark D.0.8.** The above theorem, together with the naturality of the Snake Lemma (cf. Exercise C.2.16) shows that the cohomology long exact sequence is natural in the sense that chain maps between exact sequences gives rise to maps between cohomology sequences. This is explored explicitly in an exercise below.

Before moving to show how this can be used to fix a lack of exactness for additive functors, let us show how to define cohomology functors $H^k : \mathbf{Ch}(\mathscr{A}) \to \mathscr{A}$: On objects, this is not surprising[3] as we define $H^k(A^\bullet)$ by, well, $H^k(A^\bullet)$;[4] however, the definition of $H^k$ for morphisms is a little more involved. Let us now see that we can well-define $H^k$ on morphisms. To do this, however, we will need a lemma to construct some maps explicitly (and uniquely).

**Lemma D.0.9.** *Let $\mathscr{A}$ be an Abelian category and let $f : A^\bullet \to B^\bullet$ be a morphism in $\mathbf{Ch}(\mathscr{A})$. Then for all $k \in \mathbb{Z}$, the following hold:*

1. *There exists a unique morphism $\alpha_{k-1} : \operatorname{Ker}(\operatorname{coker} \partial_{k-1}^A) \to \operatorname{Ker}(\operatorname{coker} \partial_{k-1}^B)$ making the diagram*

$$
\begin{tikzcd}
 & \operatorname{Ker}(\operatorname{coker} \partial_{k-1}^A) \arrow[dr, "\ker(\operatorname{coker} \partial_{k-1}^A)"] \arrow[ddd, dashed, "\exists! \alpha_{k-1}"] & \\
A^{k-1} \arrow[ur, "\varepsilon_{k-1}^A"] \arrow[rr, "\partial_{k-1}^A"] \arrow[d, "f_{k-1}"'] & & A^k \arrow[d, "f_k"] \\
B^{k-1} \arrow[rr, "\partial_{k-1}^B"'] \arrow[dr, "\varepsilon_{k-1}^B"'] & & B^k \\
 & \operatorname{Ker}(\operatorname{coker} \partial_{k-1}^B) \arrow[ur, "\ker(\operatorname{coker} \partial_{k-1}^B)"'] &
\end{tikzcd}
$$

*commute in $\mathscr{A}$, where the $\varepsilon_{k-1}^-$ are the epic maps in the image factorization of $\partial_{k-1}^-$;*

[3]Unless you are like me and you are routinely surprised by everyday things. Every morning I rediscover espresso and am utterly amazed.

[4]Or, for those members of TGoPwHTbtHT (The Group of People who Hate Tautologies because they Hate Tautologies), we take $H^k(A^\bullet) := \operatorname{Coker} \gamma_k$, as in Definition D.0.2.

*2. The map $\alpha_{k-1}$ makes the diagram*

$$\begin{tikzcd}
\operatorname{Ker}(\operatorname{coker}\partial_{k-1}^A) \arrow[r, "\alpha_{k-1}"] \arrow[d, "\gamma_k^A"'] & \operatorname{Ker}(\operatorname{coker}\partial_{k-1}^B) \arrow[d, "\gamma_k^B"] \\
\operatorname{Ker}\partial_k^A \arrow[r, "\overline{f}_k"'] \arrow[d, "\ker\partial_k^A"'] & \operatorname{Ker}\partial_k^B \arrow[d, "\ker\partial_k^B"] \\
A^k \arrow[r, "f_k"'] & B^k
\end{tikzcd}$$

*commute in $\mathscr{A}$;*

*3. There is a unique morphism $\sigma_k : \operatorname{Coker}\gamma_k^A \to \operatorname{Coker}\gamma_k^B$ making the diagram*

$$\begin{tikzcd}
\operatorname{Ker}\partial_k^A \arrow[r, "\overline{f}_k"] \arrow[d, "\operatorname{coker}\gamma_k^A"'] & \operatorname{Ker}\partial_k^B \arrow[d, "\operatorname{coker}\gamma_k^B"] \\
\operatorname{Coker}\gamma_k^A \arrow[r, dashed, "\exists!\sigma_k"'] & \operatorname{Coker}\gamma_k^B
\end{tikzcd}$$

*commute in $\mathscr{A}$.*

*Proof.* We prove the lemma in three parts. For the first part, we observe that since $f : A^\bullet \to B^\bullet$ is a chain map, the diagram

$$\begin{tikzcd}
A^{k-1} \arrow[r, "\partial_{k-1}^A"] \arrow[d, "f_{k-1}"'] & A^k \arrow[d, "f_k"] \\
B^{k-1} \arrow[r, "\partial_{k-1}^B"'] & B^k
\end{tikzcd}$$

commutes. We now post compose the diagram by $\operatorname{coker}\partial_{k-1}^B$ to get that

$$\operatorname{coker}(\partial_{k-1}^B) \circ f_k \circ \ker(\operatorname{coker}\partial_{k-1}^A) \circ \varepsilon_{k-1}^A = \operatorname{coker}(\partial_{k-1}^B) \circ f_k \circ \partial_{k-1}^A = \operatorname{coker}\partial_{k-1}^B \circ \partial_{k-1}^B \circ f_{k-1} = 0;$$

since $\varepsilon_{k-1}^A$ is epic, we find that $\operatorname{coker}\partial_{k-1}^B \circ f_k \circ \ker(\operatorname{coker}\partial_{k-1}^A) = 0$. This implies that there is a unique map $\alpha_{k-1}$ making the diagram

$$\begin{tikzcd}
\operatorname{Ker}(\operatorname{coker}\partial_{k-1}^B) \arrow[r, "\ker(\operatorname{coker}\partial_{k-1}^B)"] & B^k \arrow[r, shift left, "\operatorname{coker}\partial_{k-1}^B"] \arrow[r, shift right, "0"'] & \operatorname{Coker}\partial_{k-1}^B \\
\operatorname{Ker}(\operatorname{coker}\partial_{k-1}^A) \arrow[u, dashed, "\exists!\alpha_{k-1}"] \arrow[ur, "f_k \circ \ker(\operatorname{coker}\partial_{k-1}^A)"'] & &
\end{tikzcd}$$

commute; however, rewriting the diagram as

$$\begin{tikzcd}
\operatorname{Ker}(\operatorname{coker}\partial_{k-1}^A) \arrow[r, "\alpha_{k-1}"] \arrow[d, "\ker(\operatorname{coker}\partial_{k-1}^A)"'] & \operatorname{Ker}(\operatorname{coker}\partial_{k-1}^B) \arrow[d, "\ker(\operatorname{coker}\partial_{k-1}^B)"] \\
A^k \arrow[r, "f_k"'] & B^k
\end{tikzcd}$$

gives the right side of Condition (1); the other side of the diagram follows dually and hence is omitted. This verifies the first statement of the lemma.

For the second statement of the lemma, we need only verify that $\gamma_k^B \circ \alpha_{k-1} = \overline{f}_k \circ \alpha_{k-1}$ in the diagram:

$$\begin{tikzcd}
\operatorname{Ker}(\operatorname{coker} \partial_{k-1}^A) \arrow[r, "\alpha_{k-1}"] \arrow[d, "\gamma_k^A"'] & \operatorname{Ker}(\operatorname{coker} \partial_{k-1}^B) \arrow[d, "\gamma_k^B"] \\
\operatorname{Ker} \partial_k^A \arrow[r, "\overline{f}_k"'] \arrow[d, "\ker \partial_k^A"'] & \operatorname{Ker} \partial_k^B \arrow[d, "\ker \partial_k^B"] \\
A^k \arrow[r, "f_k"'] & B^k
\end{tikzcd}$$

To see this, recall that $\ker \partial_k^A \circ \gamma_k^A = \ker(\operatorname{coker} \partial_{k-1}^A)$ and that $\ker \partial_k^B \circ \gamma_k^B = \ker(\operatorname{coker} \partial_{k-1}^B)$. We then calculate that

$$f_k \circ \ker(\operatorname{coker} \partial_{k-1}^A) = f_k \circ \ker \partial_k^A \circ \gamma_k^A = \ker \partial_k^B \circ \overline{f}_k \circ \gamma_k^A$$

on one hand, while on the other we have that

$$f_k \circ \ker(\operatorname{coker} \alpha_{k-1}^A) = \ker(\operatorname{coker} \partial_{k-1}^B) \circ \alpha_{k-1} = \ker \partial_k^B \circ \gamma_k^B \circ \alpha_{k-1}.$$

Using that $\ker \partial_k^B$ is monic then implies that $\gamma_k^B \circ \alpha_{k-1} = \overline{f}_k \circ \gamma_k^A$ and verifies Condition (2).

Finally, we verify the last condition of the Lemma. However, this is immediate from the commuting square

$$\begin{tikzcd}
\operatorname{Ker}(\operatorname{coker} \partial_{k-1}^A) \arrow[r, "\alpha_{k-1}"] \arrow[d, "\gamma_k^A"'] & \operatorname{Ker}(\operatorname{coker} \partial_{k-1}^B) \arrow[d, "\gamma_k^B"] \\
\operatorname{Ker} \partial_k^A \arrow[r, "\overline{f}_k"'] & \operatorname{Ker} \partial_k^B
\end{tikzcd}$$

as this allows us to verify that

$$\operatorname{coker} \gamma_k^B \circ \overline{f}_k \circ \gamma_k^A = \operatorname{coker} \gamma_k^B \circ \gamma_k^B \circ \alpha_{k-1} = 0.$$

Thus there exists a unique morphism $\sigma_k$ making the diagram

$$\begin{tikzcd}
\operatorname{Ker}(\operatorname{coker} \partial_{k-1}^A) \arrow[r, shift left, "\gamma_k^A"] \arrow[r, shift right, "0"'] & \operatorname{Ker} \partial_k^A \arrow[r, "\operatorname{coker} \gamma_k^A"] \arrow[dr, "\operatorname{coker} \gamma_k^B \circ \overline{f}_k"'] & \operatorname{Coker} \gamma_k^A \arrow[d, dashed, "\exists ! \sigma_k"] \\
 & & \operatorname{Coker} \gamma_k^B
\end{tikzcd}$$

commute by the universal property of the cokernel. Rewriting the diagram as

$$\begin{tikzcd}
\operatorname{Ker} \partial_k^A \arrow[r, "\overline{f}_k"] \arrow[d, "\operatorname{coker} \gamma_k^A"'] & \operatorname{Ker} \partial_k^B \arrow[d, "\operatorname{coker} \gamma_k^B"] \\
\operatorname{Coker} \gamma_k^A \arrow[r, dashed, "\exists ! \sigma_k"'] & \operatorname{Coker} \gamma_k^B
\end{tikzcd}$$

then completes the proof of the lemma. $\square$

This allows us to define the cohomology functors, as the lemma above tells us that the map $\sigma_k$ is unique which will give us the actual functoriality of the assignment.

**Proposition D.0.10.** *For all $k \in \mathbb{Z}$, there are cohomology functors $H^k : \mathbf{Ch}(\mathscr{A}) \to \mathscr{A}$ which are given on objects by*

$$H^k(A^\bullet) = \operatorname{Coker} \gamma_k^A$$

*and on morphisms $f : A^\bullet \to B^\bullet$ by*

$$H^k(f) = \sigma_k,$$

*where $\sigma_k$ is the map constructed in Lemma D.0.9.*

*Proof.* Both assignments are well-defined[5], so it suffices to verify that the assignments are functorial, i.e., that $H^k(g \circ f) = H^k(g) \circ H^k(f)$ and that $H^k(\mathrm{id}_{A^\bullet}) = \mathrm{id}_{H^k(A^\bullet)}$.

We first show that $H^k$ preserves the identity morphism. Begin observing that the diagram

$$\begin{tikzcd}
\operatorname{Ker} \partial_k \arrow[r, "\mathrm{id}_{\operatorname{Ker}\partial_k}"] \arrow[d, "\operatorname{coker}\gamma_k"'] & \operatorname{Ker} \partial_k \arrow[d, "\operatorname{coker}\gamma_k"] \\
\operatorname{Coker} \gamma_k \arrow[r, "\mathrm{id}_{A^k}"'] & \operatorname{Coker} \gamma_k
\end{tikzcd}$$

commutes. Considering the proof of Lemma D.0.9, we find that the map $\sigma_k$ is the unique map making the above diagram commute, so it must be the case that $\sigma_k = \mathrm{id}_{\operatorname{Coker}\gamma_k}$. Thus it follows that

$$H^k(\mathrm{id}_{A^\bullet}) = \mathrm{id}_{\operatorname{Coker}\gamma_k} = \mathrm{id}_{H^k(A^\bullet)},$$

which shows that $H^k$ preserves identities.

It now remains to show that $H^k$ preserves composition. Suppose that $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to C^\bullet$ are morphisms in $\mathbf{Ch}(\mathscr{A})$ and let $\sigma_k$ and $\tau_k$ be the unique maps making the diagrams

$$\begin{tikzcd}
\operatorname{Ker} \partial_k^A \arrow[r, "\overline{f}_k"] \arrow[d, "\operatorname{coker}\gamma_k^A"'] & \operatorname{Ker} \partial_k^B \arrow[d, "\operatorname{coker}\gamma_k^B"] \\
\operatorname{Coker} \gamma_k^A \arrow[r, dashed, "\sigma_k"'] & \operatorname{Coker} \gamma_k^B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\operatorname{Ker} \partial_k^B \arrow[r, "\overline{g}_k"] \arrow[d, "\operatorname{coker}\gamma_k^B"'] & \operatorname{Ker} \partial_k^C \arrow[d, "\operatorname{coker}\gamma_k^C"] \\
\operatorname{Coker} \gamma_k^B \arrow[r, dashed, "\tau_k"'] & \operatorname{Coker} \gamma_k^C
\end{tikzcd}$$

commute in $\mathscr{A}$, respectively. Furthermore, let $\rho_k$ be the unique morphism making the diagram

$$\begin{tikzcd}
\operatorname{Ker} \partial_k^A \arrow[r, "\overline{(g_k \circ f_k)}"] \arrow[d, "\operatorname{coker}\gamma_k^A"'] & \operatorname{Ker} \partial_k^C \arrow[d, "\operatorname{coker}\gamma_k^C"] \\
\operatorname{Coker} \gamma_k^A \arrow[r, dashed, "\rho_k"'] & \operatorname{Coker} \gamma_k^C
\end{tikzcd}$$

commute. A routine check using the universal properties that define the maps $\overline{h}_k : \operatorname{Ker} \partial_k^A \to \partial_k^B$, for $h \in \mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$, gives that

$$\overline{(g_k \circ f_k)} = \overline{g}_k \circ \overline{f}_k.$$

[5] By the convention that when we have an object with a universal property we fix one such object with said property for all time.

Using this and stacking the diagrams defining $\sigma_k$ and $\tau_k$, we get that

$$\begin{tikzcd} \operatorname{Ker}\partial_k^A \arrow[r, "\overline{g}_k\circ\overline{f}_k"] \arrow[d, "\operatorname{coker}\gamma_k^A"'] & \operatorname{Ker}\partial_k^C \arrow[d, "\operatorname{coker}\gamma_k^C"] \\ \operatorname{Coker}\gamma_k^A \arrow[r, "\tau_k\circ\sigma_k"'] & \operatorname{Coker}\gamma_k^C \end{tikzcd}$$

commutes. From this it follows from the universal property that $\rho_k = \tau_k \circ \sigma_k$ and so we derive that

$$H^k(g \circ f) = \rho_k = \tau_k \circ \sigma_k = H^k(g) \circ H^k(f),$$

which proves the proposition. ☐

**Proposition D.0.11.** *The cohomology functors $H^k$ are all additive.*

*Proof.* A careful reading of the proof of Lemma D.0.9 shows that all of the universal properties that go into defining the cohomology functors on morphisms preserve the addition of morphisms. Consequently, it follows that

$$H^k(\varphi + \psi) = H^k(\varphi) + H^k(\psi)$$

for all $k \in \mathbb{Z}$, and we are done. ☐

**Remark D.0.12.** When we wish to refer nebulously to cohomology objects of a sequence without specifying degrees, we will write $H^*(A^\bullet)$. In particular, we say that

$$H^*(A^\bullet) \cong H^*(B^\bullet)$$

if and only if there are isomorphisms

$$H^n(A^\bullet) \cong H^n(B^\bullet)$$

for all $n \in \mathbb{Z}$.

We have now seen how to define cohomology object functors, which will play an essential role in determining the cohomology of a functor (as this is where we can talk about higher invariants of "holes" in a functor [6]), as at the moment we can only talk about cohomology intrinsically in the category $\mathbf{Ch}(\mathscr{A})$.

In order to get around the issue of having cohomology only at the level of a single category at a time (and hence not having a theory that can move between categories), we will need to study when it is that two maps have the same cohomology, and when this can be "witnessed" by some other map (maybe in an outside category). This leads us to the notion of what it means for two maps to be chain homotopic in $\mathbf{Ch}(\mathscr{A})$.

**Remark D.0.13.** Before we define chain homotopies, a caveat is warranted: We need to know that two maps give induce the same cohomology, but to detect this we will use graded morphisms. We warn the reader here because of one main reason: *Graded morphisms are not chain morphisms!* In particular, graded morphisms will not preserve the degrees of morphisms, and so cannot be morphisms in $\mathbf{Ch}(\mathscr{A})$.

[6]When I say this, I should be clear that I am speaking **only** of a formal analogy. Cohomology can be thought of as seeing derived (higher) invariants of some left exact functor around some sort of hole or object in spaces. For instance, Galois cohomology (and group cohomology) study the right-derived functors of the fix functor $(-)^G : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$ which sends a group module it its submodule of fixed points; in this way, group cohomology can be seen as looking for higher geometric invariants of the group action. I have done the perverse (but not faiseuax pervers) thing and used this intuition in how I see cohomology

**Definition D.0.14.** A *graded morphism of objects $A^\bullet, B^\bullet$ in $\mathbf{Ch}(\mathscr{A})$ of degree $k \in \mathbb{Z}$* is a collection of morphisms $f_n : A^n \to B^{n+k}$ in $\mathscr{A}$ such that the diagram

$$\begin{tikzcd} A^n \arrow[r, "\partial_n^A"] \arrow[d, "f_n"'] & A^{n+1} \arrow[d, "f_{n+1}"] \\ B^{n+k} \arrow[r, "\partial_{n+k}^B"'] & B^{n+k+1} \end{tikzcd}$$

commutes in $\mathscr{A}$.

**Remark D.0.15.** A morphism $f \in \mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$ is precisely a degree zero graded morphism of complexes.

**Definition D.0.16.** Let $\varphi, \psi : A^\bullet \to B^\bullet$ be morphisms in $\mathbf{Ch}(\mathscr{A})$. We say that $\varphi$ and $\psi$ are *chain homotopic* if there exists a degree $-1$ graded morphism $h : A^\bullet \to B^\bullet$,

$$\begin{tikzcd} \cdots \arrow[r] & A^n \arrow[r, "\partial_n^A"] \arrow[d, "h_n"'] & A^{n+1} \arrow[r] \arrow[d, "h_{n+1}"] & \cdots \\ \cdots \arrow[r] & B^{n-1} \arrow[r, "\partial_{n-1}^B"'] & B^n \arrow[r] & \cdots \end{tikzcd}$$

such that for all $n \in \mathbb{Z}$,

$$\varphi_n - \psi_n = \partial_{n-1}^B \circ h_n + h_{n+1} \circ \partial_n^A.$$

In this case, we write $\varphi \simeq \psi$.[7]

**Remark D.0.17.** The idea behind this definition is that the homotopy $h$ intertwines its way around the diagram in such a way that under cohomology, $H^k$ sees the same thing at $\varphi$ and $\psi$, and hence their difference under homotopy should be zero. This is why we ask that the composite on the right is equal to $\varphi_{n+1} - \psi_{n+1}$,; in fact, we will prove that two chain homotopic maps give the same cohomology, which should clarify the seemingly ad hoc nature of the definition.

**Remark D.0.18.** For the topologists in the crowd[8], the definition of chain homotopy says that two maps $f^\bullet, g^\bullet$ are homotopic if they differ by a boundry. The idea is that when we kill boundries to make homology and cohomology classes, $H^n(\varphi) = H^n(\psi)$ for all $n \in \mathbb{Z}$ so the information encoded by the maps $\varphi$ and $\psi$ are the same up to a smooth deformation of your space. That being said, this intuition is mired in the tyranny of real topology[9] and should be used only to help build a formal analogy.

**Definition D.0.19.** Two complexes $A^\bullet, B^\bullet \in \mathbf{Ch}(\mathscr{A})_0$ are called *homotopic* if there exist chain maps $\varphi : A^\bullet \to B^\bullet$ and $\psi : B^\bullet \to A^\bullet$ such that $\psi \circ \varphi \simeq \mathrm{id}_{A^\bullet}$ and $\varphi \circ \psi \simeq \mathrm{id}_{B^\bullet}$. Such maps $\varphi$ and $\psi$ are called *homotopy equivalences.*

**Proposition D.0.20.** *Let $\varphi, \psi : A^\bullet \to B^\bullet$ be morphisms in $\mathbf{Ch}(\mathscr{A})$ such that there is a chain homotopy $h : \varphi \to \psi$. Then $H^*(\varphi) = H^*(\psi)$.*

[7] Note that we have just defined a relation on what is generically a proper class. This should not bother you, as we can always enrich our universe of set theory, but those sensitive to foundational issues should note this. There are also local smallness reasons that make this okay.

[8] I'm sorry.

[9] For instance, a smooth deformation of $\mathbb{Q}_p$ is a lot harder to talk about (and also meaningless in the standard model structure on **Top**).

*Proof.* Begin by supposing that $n \in \mathbb{Z}$ and observe that the diagram

$$\begin{tikzcd}
A^n \arrow[r, "\partial_n^A"] \arrow[d, "h_n"'] & A^n \arrow[d, "h_{n+1}"] \\
B^{n-1} \arrow[r, "\partial_{n-1}^B"'] & B^n
\end{tikzcd}$$

commutes for all $n$. Mimicking the proof of Part (1) of Lemma D.0.9 gives a commuting diagram

$$\begin{tikzcd}
& \operatorname{Ker}(\operatorname{coker}\partial_n^A) \arrow[ddd, dashed, "\exists!\alpha_{n-1}^n"] \arrow[dr, "\ker(\operatorname{coker}\partial_n^A)"] & \\
A^n \arrow[ur, "\varepsilon_n^A"] \arrow[rr, "\partial_n^A"] \arrow[d, "h_n"'] & & A^{n+1} \arrow[d, "h_{n+1}"] \\
B^{n-1} \arrow[rr, "\partial_{n-1}^B"'] \arrow[dr, "\varepsilon_{n-1}^B"'] & & B^n \\
& \operatorname{Ker}(\operatorname{coker}\partial_{n-1}^B) \arrow[ur, "\ker(\operatorname{coker}\partial_{n-1}^B)"'] &
\end{tikzcd}$$

with the map $\alpha_{n-1}^n$ uniquely determined. This allows us to compute that

$$\begin{aligned}
h_{n+1} \circ \partial_n^A + \partial_{n-1}^B \circ h_n &= h_{n+1} \circ \ker(\operatorname{coker}\partial_n^A) \circ \varepsilon_n^A + \circ \ker(\operatorname{coker}\partial_{n-1}^B) \circ \varepsilon_{n-1}^B \circ h_n \\
&= \ker(\operatorname{coker}\partial_{n-1}^B) \circ \alpha_{n-1}^n \circ \varepsilon_n^A + \ker(\operatorname{coker}\partial_{n-1}^B) \circ \alpha_{n-1}^n \circ \varepsilon_n^A;
\end{aligned}$$

furthermore, a routine computation shows that, since $\ker(\operatorname{coker}\partial_{n-1}^B)$ is killed by $\operatorname{coker}\gamma_n^B$, we get that

$$H^n(h_{n+1} \circ \partial_n^A + \partial_{n-1}^B \circ h_n) = 0.$$

Since $h$ was a homotopy between $\varphi$ and $\psi$, we have that

$$\varphi_n - \psi_n = h_{n+1} \circ \partial_n^A + \partial_{n-1}^B \circ h_n;$$

thus, using Proposition D.0.11,

$$H^n(\varphi) - H^n(\psi) = H^n(\varphi - \psi) = H^n(\varphi_n - \psi_n) = H^n(h_{n+1} \circ \partial_n^A + \partial_{n-1}^B \circ h_n) = 0$$

from whence it follows that

$$H^n(\varphi) = H^n(\psi).$$

Because of the fact that $n \in \mathbb{Z}$ was arbitrary, it follows that $H^k(\varphi) = H^k(\psi)$ for all $k \in \mathbb{Z}$. $\square$

It is our general strategy from here to show that this notion of chain homotopy is strong enough that if we only care about the cohomology of a complex and the cohomology of an additive functor, it suffices to work with the localization of chain homotopies on $\mathbf{Ch}(\mathscr{A})$. This will in turn lead us to derived categories and quasi-isomorphisms; however, we need to develop some of the theory of chain homotopies up to this point. In particular, we need to show that additive functors preserve chain homotopies and preserve whenever maps give equivalent cohomology. After this we will show that being chain homotopic is an equivalence relation on morphisms, so in the localization category it makes sense to have exactly one representative morphism for every class of chain homotopic maps.

While we will not explore this last idea of localizing at homotopy in this section, after developing some basics about homotopies, we will move on to discuss resolutions (which will lead us to discuss quasi-isomorphisms) of a sequence, and in turn discuss right derived functors. This will lead us to a general definition and description of cohomology of a functor, and how to fix a lack of exactness.

**Proposition D.0.21.** *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor and let $\varphi, \psi : A^\bullet \to B^\bullet$ be morphisms in $\mathbf{Ch}(\mathscr{A})$ with $\varphi \simeq \psi$ via the homotopy $h : \varphi \to \psi$. Then $F\varphi \simeq F\psi$.*

*Proof.* Since $F$ is additive, it follows that

$$\begin{aligned} F\varphi - F\psi = F(\varphi - \psi) &= F(h_{n+1} \circ \partial_n^A + \partial_{n-1}^B \circ h_n) = F(h_{n+1} \circ \partial_n^A) + F(\partial_{n-1}^B \circ h_n) \\ &= F(h_{n+1}) \circ F(\partial_n^A) + F(\partial_{n-1}^B) \circ F(h_n) \\ &= Fh_{n+1} \circ \partial_n^{FA} + \partial_{n-1}^{FB} \circ Fh_n, \end{aligned}$$

where $Fh$ is a degree $-1$ morphism on $\mathbf{Ch}(\mathscr{B})$ and $F\partial_k^A = \partial_k^{FA}$ gives the differentials on the sequence $(FA)^\bullet$ (similarly for $(FB)^\bullet$). This proves that $F\varphi \simeq F\psi$. □

**Corollary D.0.22.** *If $\varphi \simeq \psi$ for morphisms $\varphi, \psi \in \mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$, and if $F : \mathscr{A} \to \mathscr{B}$ is an additive functor, then $H^*(F\varphi) = H^*(F\psi)$.*

**Proposition D.0.23.** *The relation $\simeq$ is an equivalence relation on $\mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$ for all $A^\bullet$, and $B^\bullet$ in $\mathbf{Ch}(\mathscr{A})_0$.*

*Proof.* We first show that the relation is reflexive. Consider the degree $-1$ zero map, i.e., $0 : A^\bullet \to B^\bullet$ given by taking $0_n : A^n \to B^{n-1}$ to be the zero map. Then for any $f \in \mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$,

$$f_n - f_n = 0 = 0 + 0 = \partial_{n-1}^B \circ 0_n + 0_{n+1} \circ \partial_n^A.$$

Thus $f \simeq f$ via the zero homotopy.

We now show that the relation is symmetric. Assume that $f \simeq g$, for $f, g \in \mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$, and let $h : f \to g$ be the witnessing homotopy. Then we observe that for all $n \in \mathbb{Z}$

$$g_n - f_n = -(f_n - g_n) = -(\partial_{n-1}^B \circ h_n + h_{n+1} \circ \partial_n^A) = \partial_{n-1}^B \circ (-h_n) + (-h_{n+1}) \circ \partial_n^A.$$

Because $-h$ is also a degree $-1$ morphism of complexes, it follows that $g \simeq f$.

Finally assume that $\varphi, \psi, \rho \in \mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$ with $\varphi \simeq \psi$ via the homotopy $h$ and $\psi \simeq \rho$ via the homotopy $k$. It then follows that

$$\begin{aligned} \rho_n - \varphi_n = \rho_n - \psi_n + \psi_n - \varphi_n &= \partial_{n-1}^B \circ k_n + k_{n+1} \circ \partial_n^A + \partial_{n-1}^B \circ h_n + h_n \circ \partial_{n-1}^A \\ &= \partial_{n-1}^B \circ (k_n + h_n) + (k_{n+1} + h_{n+1}) \circ \partial_n^A, \end{aligned}$$

and hence $h + k$ is a homotopy witnessing $\varphi \simeq \rho$. □

**Lemma D.0.24.** *If $\varphi, \psi \in \mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$ are chain homotopic through the degree 1 map $h$, and if $\rho : Z^\bullet \to A^\bullet$ and $\tau : B^\bullet \to C^\bullet$ are complex morphisms, then:*

1. *$\tau \circ \varphi \simeq \tau \circ \psi$;*
2. *$\varphi \circ \rho \simeq \psi \circ \rho$.*

*Proof.* Begin by observing that (2) follows from (1) mutatis mutandis, so we need only prove (1). In order to do this, however, assume that $h : \varphi \to \psi$ is a homotopy witnessing $\varphi \simeq \psi$ and fix $k \in \mathbb{Z}$. Now consider the commuting diagram

$$\begin{tikzcd} A^k \arrow[r, "\partial_k^A"] \arrow[d, "h_k"'] & A^{k+1} \arrow[d, "h_{k+1}"] \\ B^{k-1} \arrow[r, "\partial_{k-1}^B"'] \arrow[d, "\tau_{k-1}"'] & B^k \arrow[d, "\tau_k"] \\ C^{k-1} \arrow[r, "\partial_{k-1}^C"'] & C^k \end{tikzcd}$$

in $\mathscr{A}$ given from the fact that $h$ is a degree $-11$ morphism and $\tau$ is a morphism of complexes. From this we derive that

$$\begin{aligned}\tau_k \circ \varphi_k - \tau_k \circ \psi_k = \tau_k \circ (\varphi_k - \psi_k) &= \tau_k \circ (\partial^B_{k-1} \circ h_k + h_{k+1} \circ \partial^A_k) \\ &= \tau_k \circ \partial^B_{k-1} \circ h_k + \tau_k \circ h_{k+1} \circ \partial^A_k = \partial^C_{k-1} \circ \tau_{k-1} \circ h_k + \tau_k \circ h_{k+1} \circ \partial^A_k\end{aligned}$$

which shows that $\tau \circ \varphi \simeq \tau \circ \psi$ via the homotopy $k$ given degreewise by

$$k_n := \tau_{n-1} \circ h_n,$$

which concludes the proof. $\square$

**Proposition D.0.25.** *If $\varphi \simeq \psi$ and if $\rho \simeq \tau$, with $\rho \circ \varphi, \rho \circ \psi, \tau \circ \varphi$, and $\tau \circ \psi$ all defined, then prove that*

$$\rho \circ \varphi \simeq \tau \circ \psi.$$

*Proof.* From Lemma D.0.24 and the transitivity of the chain homotopy relation, we get that

$$\rho \circ \varphi \simeq \rho \circ \psi \simeq \tau \circ \psi.$$

$\square$

**Corollary D.0.26.** *The relation $\simeq$ of being chain homotopic is a composition-respecting equivalence relation.*

**Proposition D.0.27.** *The quotient category $K(\mathscr{A}) := \mathbf{Ch}(\mathscr{A})_{/\simeq}$ exists and is additive. Moreover, every additive functor $F : \mathbf{Ch}(\mathscr{A}) \to \mathscr{B}$ which kills homotopy (in the sense that if $\varphi \simeq \psi$, $F\varphi = F\psi$) uniquely factors through $K(\mathscr{A})$.*

*Proof.* By Corollary D.0.26, the chain homotopy relations $\simeq$ are equivalence relations that preserve composition. Thus by Lemma B.1.3 the quotient category

$$K(\mathscr{A}) := \mathbf{Ch}(\mathscr{A})_{/\simeq}$$

exists, and it is routine to verify that this quotient is bilinear with respect to composition in each of $X$ and $Y$. Finally, by Corollary D.0.22 and Lemma B.1.5 it follows that there exist unique functors,[10] for all $k \in \mathbb{Z}$, $F : K(\mathscr{A}) \to \mathscr{A}$ which make the diagram

$$\begin{tikzcd} \mathbf{Ch}(\mathscr{A}) \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathscr{B} \\ & K(\mathscr{A}) \arrow[ur, dashed, "\exists ! F"'] & \end{tikzcd}$$

commute. $\square$

**Definition D.0.28.** For any Abelian category $\mathscr{A}$, we call the category $K(\mathscr{A}) := \mathbf{Ch}(\mathscr{A})_{/\simeq}$ the *naïve homotopy category.*[11]

**Corollary D.0.29.** *Any cohomology object functor $H^k : \mathbf{Ch}(\mathscr{A}) \to \mathscr{A}$ factors uniquely through the naïve homotopy category $K(\mathscr{A})$.*

[10] Which will perversely name $F$ as well. Our notation here is bad, but eventually we will simply identify cohomology functors with their derived versions, and this brings us one step towards that. This is the actual perverse part of perverse sheaves (which are themselves neither perverse nor sheaves).

[11] The reason for this is from a naïve application of some Quillen model theory that we will likely not discuss in this document, which itself is a naïve choice on my part.

Let us now move to discuss injective resolutions of complexes in an Abelian category. The basic idea of an injective complex is to approximate an object $A$ by a sequence of injective objects $I^\bullet$ in such a way that $H^0(I^\bullet) \cong A$. From this we will be able to show how to take the cohomology of a functor and hence show how to fix[12] the issue of functors not being exact, at least in some cases.

**Definition D.0.30.** A nonnegative complex $I^\bullet$ of the form

$$\cdots \longrightarrow 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

is said to be *injective* if each $I^k$ is injective and *acyclic* if $H^n(I^\bullet) = 0$ for all $n \geq 1$.

**Remark D.0.31.** An injective complex $I^\bullet$ is acyclic if and only if the sequence

$$\cdots \longrightarrow 0 \longrightarrow H^0(I^\bullet) \longrightarrow I^0 \longrightarrow I^2 \longrightarrow \cdots$$

is exact. Note that the existence of the map $H^0(I^\bullet) \to I^0$ is left as Exercise D.0.7.

**Definition D.0.32.** Let $\mathscr{A}$ be an Abelian category and let $X \in \mathscr{A}_0$. An *injective resolution of* $X$ is an injective and acyclic complex $I^\bullet$ of the form

$$\cdots \longrightarrow 0 \longrightarrow I^0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \cdots$$

together with a monomorphism $m : X \to I^0$ such that

$$H^0(I^\bullet) \cong X.$$

An alternative way to define injective resolutions is through what is called a quasi-isomorphism. We will define this here, but leave questions about quasi-isomorphisms themselves until the next section. The idea for a quasi-isomorphism is simply that it is a morphism which induces isomorphisms on each degree of cohomology.

**Definition D.0.33.** A *quasi-isomorphism* in an Abelian category $\mathscr{A}$ is a chain morphism $q \in \mathbf{Ch}(\mathscr{A})(X^\bullet, Y^\bullet)$ such that for all $k \in \mathbb{Z}$,

$$H^k(q) : H^k(X^\bullet) \to H^k(Y^\bullet)$$

is an isomorphism in $\mathscr{A}$.

**Example D.0.34.** Any isomorphism is a quasi-isomorphism.

**Proposition D.0.35.** *An injective and acyclic complex $I^\bullet$ is an injective resolution of $X$ if and only if there is a quasi-isomorphism*

$$q : X^\bullet \to I^\bullet,$$

*where $X^\bullet$ is a sequence with $X^k = 0$ for all $k \neq 0$ and $X^0 = X$.*

*Proof.* Recall that since $I^\bullet$ is an injective resolution, the sequence

$$0 \longrightarrow H^0(I^\bullet) \longrightarrow I^0 \xrightarrow{\partial_0} I^1 \longrightarrow \cdots$$

is exact. Now, by writing the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "q_0"] & 0 \arrow[r] \arrow[d, "q_1"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

[12]After far too much work, and not in the way you would likely want. It's a fix in the same way the Taylor series of a function "fixes" the lack of analyticity of the function.

we note that

$$H^k(X^\bullet) = \begin{cases} 0 & \text{if}\, k \neq 0; \\ X & \text{if}\, k = 0 \end{cases}$$

and

$$H^k(I^\bullet) = \begin{cases} 0 & \text{if}\, k \neq 0; \\ H^0(I^\bullet) & \text{if}\, k = 0. \end{cases}$$

It then follows that if there is an embedding $X \to I_0$ with $X \cong H^0(I^\bullet)$, this embedding may be used to define the chain map $q$, which is immediately a quasi-isomorphism. On the other hand, if there is such a quasi-isomorphism, then $H^0(q)$ gives an isomorphism

$$X = H^0(X^\bullet) \cong H^0(I^\bullet).$$

Thus post-composing $H^0(q)$ with the embedding $H^0(I^\bullet) \to I^0$ gives the desired monomorphism. $\square$

We will now proceed to give a short study of injective resolutions[13] in order to have some idea of why we use them to define cohomology. The first results will be to justify injective resolutions as places used for collecting cohomological data. After this we will show that any two injective resolutions are of the same homotopy type.

**Proposition D.0.36.** *Let $\mathscr{A}$ be an Abelian category and let $A^\bullet$ and $I^\bullet$ be sequences of the form*

$$\cdots \longrightarrow 0 \longrightarrow A^0 \xrightarrow{\partial_0^A} A^1 \xrightarrow{\partial_1^A} \cdots$$

*and*

$$\cdots \longrightarrow 0 \longrightarrow I^0 \xrightarrow{\partial_0^I} I^1 \xrightarrow{\partial_1^I} \cdots$$

*where $A^\bullet$ is exact at $A^n$ for all $n > 0$ and $I^\bullet$ is injective. Then for every morphism $\varphi \in \mathscr{A}(H^0(A^\bullet), H^0(I^\bullet))$ there exists a morphism $\alpha \in \mathbf{Ch}(\mathscr{A})(A^\bullet, I^\bullet)$ for which $H^0(\alpha) = \varphi$. Furthermore, if $\alpha, \beta \in \mathbf{Ch}(\mathscr{A})(A^\bullet, I^\bullet)$ have the property that*

$$H^0(\alpha) = \varphi = H^0(\beta),$$

*then $\alpha \simeq \beta$.*

*Proof.* We begin by observing that for all $k < 0$, $\alpha_k = \mathrm{id}_0$; as such we need only define the $\alpha_n$ for $n \geq 0$, which we will do via (strong) induction on $n$.

For the base case of $n = 0$, we consider that since $A^\bullet$ is exact for all $n > 0$, the diagram

$$0 \longrightarrow H^0(A^\bullet) \xrightarrow{\mu} A^0$$

is an exact sequence in $\mathscr{A}$. Thus $\mu$ is monic. We then consider the diagram:

$$\begin{tikzcd} H^0(A^\bullet) \arrow[r, "\mu"] \arrow[d, "\varphi"'] & A^0 \\ H^0(I^\bullet) \arrow[r] & I^0 \end{tikzcd}$$

Because $I^0$ is injective, there exists a morphism $\alpha_0 : A^0 \to I^0$ making the resulting square

$$\begin{tikzcd} H^0(A^\bullet) \arrow[r, "\mu"] \arrow[d, "\varphi"'] & A^0 \arrow[d, dashed, "\exists\alpha_0"] \\ H^0(I^\bullet) \arrow[r] & I^0 \end{tikzcd}$$

[13] Short relative to the rest of this chapter, perhaps. Whether it is short exact is something only you can verify, however.

commute in $\mathscr{A}$. This establishes the base case.

We proceed now via induction. Assume that there exists an $m \in \mathbb{N}$ with $m \geq 1$ such that for all $0 \leq k \leq m-1$, the map $\alpha_k$ has been constructed. We now consider the diagram, if $m = 1$,

$$\begin{tikzcd}
H^0(A^\bullet) \arrow[r] \arrow[d, "\varphi"'] & A^0 \arrow[r, "\partial_0^A"] \arrow[d, "\alpha_0"] & A^1 \\
H^0(I^\bullet) \arrow[r] & I^0 \arrow[r, "\partial_0^I"'] & I^!
\end{tikzcd}$$

and the diagram

$$\begin{tikzcd}
A^{m-2} \arrow[r, "\partial_{m-2}^A"] \arrow[d, "\alpha_{m-2}"'] & A^{m-1} \arrow[r, "\partial_{m-1}^A"] \arrow[d, "\alpha_{m-1}"] & A^m \\
I^{m-2} \arrow[r, "\partial_{m-2}^I"'] & I^{m-1} \arrow[r, "\partial_{m-1}^I"'] & I^m
\end{tikzcd}$$

if $m \geq 2$; establishing the existence of $\alpha_1$ if $m = 1$ follows mutatis mutandis from the second diagram, so we will work only in that case. We now observe that

$$\partial_{m-1}^I \circ \alpha_{m-1} \circ \partial_{m-2}^A = \partial_{m-1}^I \circ \partial_{m-2}^I \circ \alpha_{m-2} = 0,$$

which implies that $\partial_{m-1}^I \circ \alpha_{m-1}$ factors through the cokernel of $\partial_{m-2}^A$. Explicitly, there exists a unique morphism $\gamma$ making the diagram

$$\begin{tikzcd}
A^{m-1} \arrow[r, "\operatorname{coker} \partial_{m-2}^A"] \arrow[dr, "\partial_{m-1}^I \circ \alpha_{m-1}"'] & \operatorname{Coker} \partial_{m-2}^A \arrow[d, dashed, "\exists! \gamma"] \\
 & I^m
\end{tikzcd}$$

commute in $\mathscr{A}$. However, taking the image factorization of $\partial_{n-1} \circ \alpha_{n-1}$, as in the diagram

$$\begin{tikzcd}
A^{m-1} \arrow[rr, "\partial_{m-1}^I \circ \alpha_{m-1}"] \arrow[dr, "\varepsilon"'] & & I^m \\
 & \operatorname{Ker}(\operatorname{coker}(\partial_{m-1}^I \circ \alpha_{m-1})) \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

gives the commuting diagram

$$\begin{tikzcd}
A^{m-1} \arrow[r, "\varepsilon"] \arrow[d, "\operatorname{coker} \partial_{m-2}^A"'] & \operatorname{Ker}(\operatorname{coker}(\partial_{m-1}^I \circ \alpha_{m-1})) \arrow[d, "\sigma"] \\
\operatorname{Coker} \partial_{m-2}^A \arrow[r, "\gamma"'] & I^m
\end{tikzcd}$$

Using properties of orthogonal factorization systems together with the fact that $\sigma$ is monic and $\operatorname{coker} \partial_{m-2}^A$ is epic gives the existence of a unique map $\rho : \operatorname{Coker} \partial_{m-2} \to \operatorname{Ker}(\operatorname{coker}(\partial_{m-1}^I \circ \alpha_{m-1}))$ which makes the diagram

$$\begin{tikzcd}
A^{m-1} \arrow[r, "\varepsilon"] \arrow[d, "\operatorname{coker} \partial_{m-2}^A"'] & \operatorname{Ker}(\operatorname{coker}(\partial_{m-1}^I \circ \alpha_{m-1})) \arrow[d, "\sigma"] \\
\operatorname{Coker} \partial_{m-2}^A \arrow[r, "\gamma"'] \arrow[ur, dashed, "\exists! \rho"] & I^m
\end{tikzcd}$$

commute. Furthermore, it is routine to check that $\rho$ is epic. Thus there exists an epimorphism

$$\rho : \mathrm{Coker}(\partial^A_{m-2}) \to \mathrm{Ker}(\mathrm{coker}(\partial^I_{m-1} \circ \alpha_{m-1})).$$

Using the exactness of $A^\bullet$ at all $k > 0$ gives the isomorphism

$$\mathrm{Coker}(\partial^A_{m-2}) \cong \mathrm{Coim}(\partial^A_{m-1}) = \mathrm{Coker}(\ker(\partial^A_{m-1}));$$

pre-composing $\rho$ with the above isomorphism gives an epimorphism

$$\mathrm{Coker}(\ker(\partial^A_{m-1})) \to \mathrm{Ker}(\mathrm{coker}(\partial^I_{m-1} \circ \alpha_{m-1})).$$

From here using the injectivity of $I^m$ gives rise to a map $\alpha_m$ which makes the diagram

$$\begin{tikzcd}
A^{m-1} \arrow[r, "\partial^A_{m-1}"] \arrow[d, "\partial^I_{m-1}\circ\alpha_{m-1}"'] & A^m \arrow[dl, dashed, "\exists\alpha_m"] \\
I^m &
\end{tikzcd}$$

commute in $\mathbf{Ch}(\mathscr{A})$. This establishes the inductive and hence proves the existence of the morphism $\alpha$ by the Principle of Mathematical Induction.

Assume now that $\alpha, \beta \in \mathbf{Ch}(\mathscr{A})(A^\bullet, I^\bullet)$ with the property that $H^0(\alpha) = \varphi = H^0(\beta)$. We now must construct a homotopy $k : \alpha \to \beta$; since $A^m = 0 = I^m$ for all $m \leq 0$, we define

$$k_m : A^m \to I^{m-1}$$

by $k_m = 0$. Thus we must only define $k_m$ for $m > 0$. Observe that if $\alpha \simeq \beta$ through $k$, it must be the case that we can write

$$\alpha_0 - \beta_0 = \partial^B_{-1} \circ k_0 + k_1 \circ \partial^A_0 = k_1 \circ \partial^A_0.$$

To show that it is possible to construct such a $k_1$, first observe that

$$H^0(A^\bullet) \cong \mathrm{Ker}\,\partial^A_0$$

and

$$H^0(I^\bullet) \cong \mathrm{Ker}\,\partial^I_0.$$

Thus we have that $H^0(\alpha)$ is the unique morphisms making the diagram

$$\begin{tikzcd}
A^0 \arrow[r, "\alpha_0"] & I^0 \\
\mathrm{Ker}\,\partial^A_0 \arrow[u, "\ker\partial^A_0"] \arrow[r, dashed, "\exists!"'] & \mathrm{Ker}\,\partial^I_0 \arrow[u, "\ker\partial^I_0"']
\end{tikzcd}$$

commute, and similarly for $H^0(\beta)$. Thus we compute that

$$(\alpha_0 - \beta_0) \circ \ker \partial^A_0 = \ker(\partial^I_0) \circ H^0(\alpha_0 - \beta_0) = \ker(\partial^I_0) \circ (H^0(\alpha) - H^0(\beta)) = \ker(\partial^I_0) \circ 0 = 0.$$

This implies that $\alpha_0 - \beta_0$ factors uniquely as

$$\begin{tikzcd}
A^0 \arrow[r, "\mathrm{coker}(\ker\partial^A_0)"] \arrow[dr, "\alpha_0-\beta_0"'] & \mathrm{Coker}(\ker\partial^A_0) \arrow[d, dashed, "\exists!\gamma"] \\
& I^0
\end{tikzcd}$$

Now write $\mathrm{Coker}(\ker \partial_0^A) = \mathrm{Coim}\, \partial_0^A$ and let $\theta : \mathrm{Coim}\, \partial_0^A \xrightarrow{\cong} \mathrm{Im}\, \partial_0^A$ be the isomorphism of Corollary C.2.11. Then, writing $\mathrm{Im}\, \partial_0^A = \mathrm{Ker}(\mathrm{coker}\, \partial_0^A)$, there is a monomorphism which arises as the composite:

$$\mathrm{Coker}(\ker \partial_0^A) \xrightarrow{\theta} \mathrm{Ker}(\mathrm{coker}\, \partial_0^A) \xrightarrow{\ker(\mathrm{coker}\, \partial_0^A)} A^1$$

It then follows, because $I^0$ is injective, that there exists a morphism $k_1$ which makes the diagram

$$\begin{tikzcd}
\mathrm{Coker}(\ker \partial_0^A) \arrow[r, "\ker(\mathrm{coker}\, \partial_0^A)"] \arrow[d, "\gamma"'] & A^1 \arrow[dl, dashed, "\exists k_1"] \\
I^0 &
\end{tikzcd}$$

commute in $\mathscr{A}$. We then calculate that, because $\ker(\mathrm{coker}\, \partial_0^A) \circ \theta \circ \mathrm{coker}(\ker \partial_0^A) = \partial_0^A$ (cf. Theorem C.2.12),

$$k_1 \circ \partial_0^A = k_1 \circ \ker(\mathrm{coker}\, \partial_0^A) \circ \theta \circ \mathrm{coker}(\ker \partial_0^A) = \gamma \circ \mathrm{coker}(\ker \partial_0^A) = \alpha_0 - \beta_0.$$

This proves that $k_1$ exists and establishes the base case of our induction.

We now proceed inductively. Assume that there exists an $m \in \mathbb{N}$ with $m \geq 1$ such that for all $1 \leq \ell \leq m$, $k_\ell$ has been constructed with the property that the diagrams

$$\begin{tikzcd}
A^{\ell-1} \arrow[r, "\partial_{\ell-1}^A"] \arrow[d, "k_{\ell-1}"'] & A^\ell \arrow[d, "k_\ell"] \\
B^{\ell-2} \arrow[r, "\partial_{\ell-2}^B"'] & B^{\ell-1}
\end{tikzcd}$$

commute for $1 \leq \ell \leq m$, and whenever $0 \leq \ell \leq m-1$, the identity

$$\alpha_\ell - \beta_\ell = k_{\ell+1} \circ \partial_\ell^A + \partial_{\ell-1}^B \circ k_\ell$$

holds. Now, in order to define the map $k_{m+1}$, define the morphism $\rho_m : A^m \to B^m$ via

$$\rho_m := \alpha_m - \beta_m - \partial_{m-1}^B \circ k_m.$$

Note that if we can show that there exists a homotopy fibre $k_{m+1} : A^{m+1} \to B^m$ such that $\rho_m = k_{m+1} \circ \partial_m^A$ we will be done, as:

$$\frac{\dfrac{\rho_m = k_{m+1} \circ \partial_m^A}{\alpha_m - \beta_m - \partial_{m-1}^B \circ k_m = k_{m+1} \circ \partial_m^A}}{\alpha_m - \beta_m = \partial_{m-1}^B \circ k_m + k_{m+1} \circ \partial_m^A}$$

We now calculate that

$$\begin{aligned}
\rho_m \circ \partial_{m-1}^A &= (\alpha_m - \beta_m - \partial_{m-1}^B \circ k_m) \circ \partial_{m-1}^A = (\alpha_m - \beta_m) \circ \partial_{m-1}^A - \partial_{m-1}^B \circ k_m \circ \partial_{m-1}^A \\
&= \partial_{m-1}^B \circ (\alpha_{m-1} \circ \beta_{m-1}) - \partial_{m-1}^B \circ k_m \circ \partial_{m-1}^A \\
&= \partial_{m-1}^B \circ (k_m \circ \partial_{m-1}^A + \partial_{m-2}^B \circ k_{m-1}) - \partial_{m-1} \circ k_m \circ \partial_{m-1}^A \\
&= \partial_{m-1}^B \circ k_m \circ \partial_{m-1}^A + \partial_{m-1}^B \circ k_m \circ \partial_{m-1}^A = \partial_{m-1}^B \circ \partial_{m-2}^B \circ k_{m-1} + \partial_{m-1}^B \circ \partial_{m-2}^B \circ k_{m-1} \\
&= 0.
\end{aligned}$$

Thus it follows that there exists a unique morphism $\psi_m : \operatorname{Coker} \partial^A_{m-1} \to I^m$ making the diagram

$$\begin{tikzcd} A^m \arrow[r, "\operatorname{coker} \partial^A_{m-1}"] \arrow[dr, "\rho_m"'] & \operatorname{Coker} \partial^A_{m-1} \arrow[d, dashed, "\exists! \psi_m"] \\ & I^m \end{tikzcd}$$

commute. Use the exactness of $A^\bullet$ to give the isomorphism $\operatorname{Coker} \partial^A_{m-1} \cong \operatorname{Coim} \partial^A_m$; post-composing this isomorphism with the isomorphism

$$\operatorname{Coim} \partial^A_m \cong \operatorname{Im} \partial^A_m = \operatorname{Ker}(\operatorname{coker} \partial^A_m)$$

and then use the natural monomorphism $\operatorname{Ker}(\operatorname{coker} \partial^A_m) \to A^{m+1}$ to produce a monomorphism

$$\sigma_m : \operatorname{Coker} \partial^A_{m-1} \to A^{m+1}.$$

It is then straightforward to verify using Theorem C.2.12 to show that the pair $(\operatorname{coker} \partial^A_{m-1}, \sigma_m)$ gives an epic/monic factorization of $\partial^A_m$, i.e.,

$$\partial^A_m = \sigma_m \circ \operatorname{coker} \partial^A_{m-1}.$$

Because $I^m$ is injective and $\sigma_m$ is monic, there exists a morphism $k_{m+1}$ making the diagram

$$\begin{tikzcd} \operatorname{Coker} \partial^A_{m-1} \arrow[r, "\sigma_m"] \arrow[d, "\psi_m"'] & A^{m+1} \arrow[dl, dashed, "\exists k_{m+1}"] \\ I^m \end{tikzcd}$$

commute. We then compute that

$$k_{m+1} \circ \partial^A_m = k_{m+1} \circ \sigma_m \circ \operatorname{coker} \partial^A_{m-1} = \psi_m \circ \operatorname{coker} \partial^A_{m-1} = \rho_m,$$

which completes the inductive step. As such it follows that the homotopy $k : \alpha \to \beta$ exists by the Principle of Mathematical induction, completing the proof of the proposition. $\square$

Applying the proposition[14] above allows us to prove that any two injective resolutions of an object are homotopy equivalent. This will allow us to in turn prove that derived functors, in the case $\mathscr{A}$ has enough injectives, can be defined in terms of injective resolutions.

**Corollary D.0.37.** *Let $A$ be an object of an Abelian category $\mathscr{A}$ and let $I^\bullet$ and $J^\bullet$ be injective resolutions of $\mathscr{A}$. Then $I^\bullet$ and $J^\bullet$ are of the same homotopy type.*

*Proof.* By Proposition D.0.36, we can find maps $\varphi : I^\bullet \to J^\bullet$ and $\psi : J^\bullet \to I^\bullet$ whose compositions induce the identity on $H^0(I^\bullet) = A = H^0(J^\bullet)$. However, since the identity morphisms $\mathrm{id}_{I^\bullet}$ and $\mathrm{id}_{J^\bullet}$ also induce the identity on $H^0(I^\bullet)$ and $H^0(J^\bullet)$, so it follows that $\varphi \circ \psi \simeq \mathrm{id}_{J^\bullet}$ and $\psi \circ \varphi \simeq \mathrm{id}_{I^\bullet}$. This proves the corollary. $\square$

We are now in a place to introduce derived functors and show how they fix a lack of exactness that an arbitrary additive functor may have. We will develop the theory of derived functors in the presence of categories that have enough injectives[15] and focus on the theory of *right* derived functors of left exact functors, as this will allow us to define cohomology in many of the various settings we see in algebraic geometry (as the

[14] For the shear effort and length of the proof, this should probably be a theorem. However, sometimes life is cruel, so this it is demoted to the rank of proposition.

[15] If one instead has enough projectives or is interested in the perverse notion of homology (as opposed to cohomology, which is totally different), all you need to do is take th formal duals of what we do here.

Abelian categories $\mathbf{QCoh}(X)$ and $\mathcal{O}_X$-$\mathbf{Mod}$ both have enough injectives when $X$ is a scheme, and the global sections functor in both cases is left exact — this strategy gives us the sheaf cohomology of the scheme in the second case). We will not, however, start to assume our additive functors are left exact until it becomes necessary.

We would like to use the cohomology object functors we have defined[16] in order to define our derived functors, and hence our cohomology functors. Let us explore this in more detail: Assume that $\mathscr{A}$ is an Abelian category with enough injectives and assume that $F : \mathscr{A} \to \mathscr{B}$ is an additive functor between Abelian categories. We would like to define the $n$-th right derived functor of $F$, for $n \in \mathbb{N}$, to be approximated by the functorial image of some injective resolution of each object and morphism.

Explicitly, let $A$ be an object of $\mathscr{A}$ and let $A \to I^\bullet$ be an injective resolution of $A$. We then define the $n$-th right derived functor, denoted by $R^nF$, on objects by

$$R^nF(A) := H^n(FI^\bullet).$$

For morphisms we work a little harder. If $\varphi \in \mathscr{A}(A, B)$, and if $I^\bullet$ and $J^\bullet$ are injective resolutions of $A$ and $B$, respectively, by Proposition D.0.36 we can find a chain morphism $\alpha \in \mathbf{Ch}(\mathscr{A})(I^\bullet, J^\bullet)$ with $H^0(\alpha) = \beta$. Thus we define $R^nF$ at $\alpha$ via

$$R^nF(\alpha) = H^n(F\alpha).$$

These definitions are fine, but unfortunately there is an issue: We don't know that they are even remotely well-defined! Our next few lemmas will be to study in what sense these definitions are well-defined[17], and how functorial these definitions are.

**Lemma D.0.38.** *Let $\varphi \in \mathscr{A}(A, B)$, let $I^\bullet$ and $J^\bullet$ be injective resolutions of $A$ and $B$, respectively, and let $\alpha, \beta \in \mathbf{Ch}(\mathscr{A})(I^\bullet, J^\bullet)$ with $H^0(\alpha) = \varphi = H^0(\beta)$. Then $H^n(F\alpha) = H^n(F\beta)$. In particular, $R^nF$ is well-defined on morphisms.*

*Proof.* By Proposition D.0.36, $\alpha \simeq \beta$. Thus, since $F$ is the additive prolongation of $F : \mathscr{A} \to \mathscr{B}$ to $\mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{B})$, it follows from Corollary D.0.22 that

$$H^n(F\alpha) = H^n(F\beta).$$

This shows that $R^nF$ is well-defined on morphisms, as any two chain maps which induce $\varphi$ in degree-zero cohomology are homotopic. $\square$

**Lemma D.0.39.** *If $\varphi \in \mathscr{A}(A, B)$ and if $\psi \in \mathscr{A}(B, C)$ then*

$$R^nF(\psi \circ \varphi) = R^nF(\psi) \circ R^nF(\varphi).$$

*Furthermore,*

$$R^nF(\mathrm{id}_A) = \mathrm{id}_{H^n(FA)}.$$

*In particular, $R^nF$ is functorial on morphisms.*

*Proof.* We first prove the rigidity of $R^nF$ on the identity; as such assume that $I^\bullet$ is an injective acyclic complex. Since the identity map on $I^\bullet$ induces the identity in $H^0$, so any map $\alpha \in \mathbf{Ch}(\mathscr{A})(I^\bullet, I^\bullet)$ over $\mathrm{id}_{H^0(I^\bullet)}$ satisfies $\alpha \simeq \mathrm{id}_{I^\bullet}$. Thus it follows that

$$R^nF(\mathrm{id}_A) = H^n(F\alpha) = H^n(F\,\mathrm{id}_{I^\bullet}) = \mathrm{id}_{H^n(FI^\bullet)} = \mathrm{id}_{R^nF(A)},$$

which shows that $R^nF$ preserves the identity.

[16] Perhaps this should say "laboriously defined," but let's not pat our backs too much here.
[17] Hopefully in the literal sense.

We now verify that $R^nF$ preserves composition. Let $I^\bullet, J^\bullet$, and $K^\bullet$ be injective resolutions of $A, B$, and $C$, respectively. Now use Proposition D.0.36 to find chain maps $\alpha \in \mathbf{Ch}(\mathscr{A})(A,B)$, $\beta \in \mathbf{Ch}(\mathscr{A})$, and $\gamma \in \mathbf{Ch}(\mathscr{A})(A,C)$ such that $H^0(\alpha) = \varphi$, $H^0(\beta) = \psi$, and $H^0(\gamma) = \psi \circ \varphi$. Then it follows that

$$H^0(\beta \circ \alpha) = H^0(\beta) \circ H^0(\alpha) = \psi \circ \varphi = H^0(\gamma)$$

and so it follows that $\beta \circ \alpha \simeq \gamma$. Using Corollary D.0.22 then gives that

$$H^n(F\beta) \circ H^n(F\alpha) = H^n(F\gamma),$$

and hence it follows that

$$RF^n(\beta \circ \alpha) = R^nF(\beta) \circ R^nF(\alpha).$$

Thus $R^nF$ is functorial in its morphism assignment. $\square$

**Lemma D.0.40.** *Let $A$ be an object of $\mathscr{A}$. Then $R^nF(A)$ is well-defined uniquely up to chain homotopy.*

*Proof.* We first show that any two injective resolutions of $A$, $I^\bullet$ and $J^\bullet$, give uniquely isomorphic objects in $\mathscr{B}$. Begin by Corollary D.0.37 to show that $I^\bullet$ and $J^\bullet$ are of the same homotopy type in $\mathbf{Ch}(\mathscr{A})$; as such, find chain maps $\rho : I^\bullet \to J^\bullet$ and $\sigma : J^\bullet \to I^\bullet$ such that $\sigma \circ \rho \simeq \mathrm{id}_{I^\bullet}$ and $\rho \circ \sigma \simeq \mathrm{id}_{J^\bullet}$. As such it follows that $H^n(F\sigma) \circ H^n(F\rho) = \mathrm{id}_{H^n(FI^\bullet)}$ and $H^n(F\rho) \circ H^n(F\sigma) = \mathrm{id}_{H^n(FJ^\bullet)}$, showing that

$$H^n(FI^\bullet) \cong H^n(FJ^\bullet).$$

The fact that this isomorphism is unique up to homotopy follows from the fact that the homotopy type of two chain objects is uniquely determined.

We now verify that these isomorphisms are natural in the relevant sense. To see this, assume that $A$ and $B$ are objects of $\mathscr{A}$ and that $I^\bullet$ and $J^\bullet$ are injective resolutions of $A$, while $K^\bullet$ and $L^\bullet$ are injective resolutions of $B$. Find mutual chain homotopy equivalences $\rho_1 : I^\bullet \to J^\bullet, \sigma_1 : J^\bullet \to I^\bullet$, $\rho_2 : K^\bullet \to L^\bullet$, and $\sigma_2 : L^\bullet \to K^\bullet$. Furthermore, assume that $\varphi : A \to B$ is a morphism in $\mathscr{A}$ and let $\alpha : I^\bullet \to K^\bullet$ and $\beta : J^\bullet \to L^\bullet$ be chain morphisms lifting $\varphi$, i.e.,

$$H^0(\alpha) = \varphi = H^0(\beta).$$

Write $\eta_{I^\bullet,J^\bullet} = H^n(F\rho_1)$ and $\eta_{K^\bullet,L^\bullet} = H^n(F\rho_2)$ as the canonical isomorphisms from $H^n(FI^\bullet) = H^n(FJ^\bullet)$ and from $H^n(FK^\bullet)$ to $H^n(FL^\bullet)$, respectively. We calculate that

$$\eta_{K^\bullet,L^\bullet} \circ H^n(F\alpha) = H^n(F\rho_1) \circ H^n(F\alpha) = H^n(F(\rho_1 \circ \alpha))$$

and similarly that

$$H^n(F\beta) \circ \eta_{K^\bullet,L^\bullet} = H^n(F\beta) \circ H^n(F\rho_1) = H^n(F(\rho_1 \circ \beta)).$$

It then follows that

$$\begin{aligned}\beta \circ \rho_1 = \mathrm{id}_{L^\bullet} \circ \beta \circ \rho_1 &\simeq \rho_2 \circ \sigma_2 \circ \beta \circ \rho_1 \simeq \rho_2 \circ \alpha \circ \sigma_1 \circ \rho_1 \\ &\simeq \rho_2 \circ \alpha \circ \mathrm{id}_{I^\bullet} = \rho_2 \circ \alpha.\end{aligned}$$

Thus we have that

$$H^n(F\beta) \circ \eta_{I^\bullet,J^\bullet} = H^n(F(\beta \circ \rho_1)) = H^n(F(\rho_2 \circ \alpha)) = \eta_{K^\bullet,L^\bullet} \circ H^n(F\alpha),$$

which shows that the diagram

$$\begin{tikzcd} H^n(FI^\bullet) \arrow[r, "H^n(F\alpha)"] \arrow[d, "\eta_{I^\bullet,J^\bullet}"'] & H^n(FK^\bullet) \arrow[d, "\eta_{K^\bullet,L^\bullet}"] \\ H^n(FJ^\bullet) \arrow[r, "H^n(F\beta)"'] & H^n(FL^\bullet) \end{tikzcd}$$

commutes. It thus follows that any two choices for the object $R^nF(A)$ are uniquely naturally isomorphic. $\square$

**Remark D.0.41.** In what proceeds, we define $R^nF(A)$ by fixing a *specific* injective resolution for all objects $A$ of $\mathscr{A}$ once and for all and making sure that our morphisms keep track of these choices. While the above lemma suggest taking a psuedo-functorial approach to the theory so as to keep track of choices more explicitly, we will not take this perspective in these notes. Note that Lemma D.0.40 shows that after making this choice, no significant obstructions arise in the well-definedness of the object function of $R^nF(A)$.

**Definition D.0.42.** If $\mathscr{A}$ is an Abelian category with enough injectives and if $F : \mathscr{A} \to \mathscr{B}$ is an additive functor between Abelian categories, then the *$n$-th right derived functor of $F$*, for all $n \in \mathbb{N}$, is defined by

$$R^nF(A) = H^n(FI^\bullet),$$

for an injective resolution $I^\bullet$ of $A$ chosen as in Remark D.0.41, on objects $A \in \mathscr{A}_0$, and by

$$RF^n(\varphi) = H^n(F\alpha)$$

for some lift of $\varphi$ to $\alpha$ on morphisms $\varphi \in \mathscr{A}_1$.

It is immediate from Remark D.0.41 and Lemmas D.0.38 and D.0.39 that $R^nF$ is well-defined and is a functor. We will see later that these do give rise to long exact sequences when $F$ is left exact, and that these right derived functors $R^nF$ "approximate" the functor $F$ as in a Taylor expansion in a relevant sense. In the mean time we will present two topos-theoretic results and then move to discuss some basic properties of these functors before studying the right derived functors of left exact functors.

We move now to calculate the right derived functors of injective objects. Afterwards, we will prove some basic results about right derived functors and then move to prove the long exact sequence of such functors.

**Proposition D.0.43.** *If $I$ is an injective object in $\mathscr{A}$, then for any additive functor $F : \mathscr{A} \to \mathscr{B}$,*

$$R^nF(I) = \begin{cases} FI & if n = 0; \\ 0 & if n \geq 1. \end{cases}$$

*Proof.* Since $I$ is injective, the sequence

$$\cdots \longrightarrow 0 \longrightarrow I \longrightarrow 0 \longrightarrow \cdots$$

is an injective resolution of $I$; call this resolution $I[0]$ to denote that $I$ is concentrated in degree 0. Applying the functor $F$ sends $I$ to the sequence

$$\cdots \longrightarrow 0 \longrightarrow FI \longrightarrow 0 \longrightarrow \cdots$$

in $\mathbf{Ch}(\mathscr{B})$. Applying $H^n$ for each $n$ gives that

$$R^nF(I) = H^n(I[0]) = \begin{cases} FI & \text{if } n = 0; \\ 0 & \text{if } n \geq 1. \end{cases}$$

This proves the proposition. □

**Proposition D.0.44.** *If $\mathscr{A}$ has enough injectives and if the functor $F : \mathscr{A} \to \mathscr{B}$ is left exact, then the functors $R^0F$ and $F$ are naturally isomorphic.*

*Proof.* Begin by letting $A$ be an object of $\mathscr{A}$ and find an injective resolution $I^\bullet$ of $A$. By definition, the sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1$$

is exact; applying $F$ and using that $F$ is additive and left exact shows that the sequence

$$0 \longrightarrow FA \longrightarrow FI^0 \longrightarrow FI^1$$

is also exact. However, it is routine to check that

$$R^0F(A) = H^0(FI^\bullet) \cong FA;$$

the naturality of this isomorphism follows from construction of the cohomology object functors and the functoriality of $F$. $\square$

**Proposition D.0.45.** *The functors $R^nF$ are additive for all $n \in \mathbb{N}$.*

*Proof.* Since the functor $F$ (and its prolongment) are both additive, and since the functors $H^n$ are additive for all $n \in \mathbb{N}$, we are done by the fact that the composition of additive functors remains additive if we know that the process of taking injective resolutions preserves addition as well. However, it is a consequence of Proposition D.0.36 that this holds up to chain homotopy, and so we conclude that $R^nF$ is additive for all $n \in \mathbb{N}$. $\square$

We are finally in a position to prove the long exact sequence of right derived functors[18]. This will use the Snake Lemma and many of the Abelian categorical techniques we have introduced and used throughout this text. After proving this sequence, we will show how to produce a long exact sequence of right derived functors in the presence of a natural transformation; this will lead to a change of cohomology type result, which is useful in the study of, for example, $\ell$-adic étale cohomology. In the meantime, let us state the long exact theorem, and in doing so, restate the many[19] assumptions we are making so as to make the theorem more explicit.

**Theorem D.0.46.** *Let $\mathscr{A}$ and $\mathscr{B}$ be Abelian categories, assume that $\mathscr{A}$ has enough injectives, and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Then if the sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*is exact in $\mathscr{A}$, there exist connecting morphisms $\delta_n : R^nF(A) \to R^{n+1}F(A)$ which induce a long exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & R^0F(A) \arrow[r] & R^0F(B) \arrow[r] & R^0F(C) \arrow[r, "\delta_0"] & R^1F(A) \arrow[r] & \cdots \arrow[d] \\
\cdots & R^{n+1}F(A) \arrow[l] & R^nF(C) \arrow[l, "\delta_n"'] & R^nF(B) \arrow[l] & R^nF(A) \arrow[l] & \cdots \arrow[l, "\delta_{n-1}"']
\end{tikzcd}$$

*in $\mathscr{B}$.*

*Proof.* We will show that we can inductively construct injective resolutions $I^\bullet, J^\bullet$, and $K^\bullet$ of $A, B$, and $C$ which themselves fit into an exact sequence in $\mathbf{Ch}(\mathscr{A})$. Begin by finding injective objects $I^0$ and $K^0$ for which there are monomorphisms $i : A \to I^0$ and $k : C \to K^0$. We now use Exercise C.3.12 together with the suggested hint to give a monomorphism $j : B \to I^0 \oplus K^0$ which makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "i"'] & B \arrow[r] \arrow[d, "j"'] & C \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & I^0 \arrow[r] & I^0 \oplus K^0 \arrow[r] & K^0 \arrow[r] & 0
\end{tikzcd}$$

commute in $\mathscr{A}$ with exact rows. Applying the Snake Lemma (cf. Lemma C.2.22) gives the exact sequence

$$0 \longrightarrow \mathrm{Ker}(i) \longrightarrow \mathrm{Ker}(j) \longrightarrow \mathrm{Ker}(k) \longrightarrow \mathrm{Coker}(i) \longrightarrow \mathrm{Coker}(j) \longrightarrow \mathrm{Coker}(k) \longrightarrow 0$$

[18]Perhaps better known as the long exact cohomology sequence, but this is a choice of terminology at this point. What matters, however, is that we're finally there! Pat yourself on the back, you universal homological algebraist, you!

[19]Way too many. Things are so complex at the moment that we may as well be algebraically closed and Euclidean.

which, because each of the morphisms $i, j$, and $k$ are monic, simplifies to the exact sequence

$$0 \longrightarrow \mathrm{Coker}(i) \longrightarrow \mathrm{Coker}(j) \longrightarrow \mathrm{Coker}(k) \longrightarrow 0$$

in $\mathscr{A}$. We now find injective objects $I^1$ and $K^1$ together with monomorphisms

$$i_1 : \mathrm{Coker}(i) \to I^1$$

and

$$k_1 : \mathrm{Coker}(k) \to K^1;$$

applying Exercise C.3.12 again gives rise to a monomorphism $j_1 : \mathrm{Coker}(j) \to I^1 \oplus K^1$ and the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Coker}(i) \arrow[r] \arrow[d, "i_1"'] & \mathrm{Coker}(j) \arrow[r] \arrow[d, "j_1"'] & \mathrm{Coker}(k) \arrow[r] \arrow[d, "k_1"] & 0 \\
0 \arrow[r] & I^1 \arrow[r] & I^1 \oplus K^1 \arrow[r] & K^1 \arrow[r] & 0
\end{tikzcd}$$

with exact rows. This allows us to produce the morphism, together with epic-monic factorization

$$\begin{tikzcd}
I^0 \arrow[rr, "\partial_0^I"] \arrow[dr, "\mathrm{coker}(i)"'] & & I^1 \\
& \mathrm{Coker}(i) \arrow[ur, "i_1"'] &
\end{tikzcd}$$

and similarly for $\partial_0^{I \oplus K} : I^0 \oplus K^0 \to I^1 \oplus K^1$ and $\partial_0^K : K^0 \to K^1$. Proceed via induction on $n$ to produce chain maps $\delta_n^I : I^n \to I^{n+1}$, and similarly for $I^n \oplus K^n$ and $K^n$, in order to generate the injective sequences $I^\bullet, K^\bullet$, and $I^\bullet \oplus K^\bullet$. A routine calculation gives that

$$H^0(I^\bullet) = \mathrm{Ker}\, \partial_0^I = \mathrm{Ker}(i_1 \circ \mathrm{coker}(i)) = \mathrm{Ker}(\mathrm{coker}(i)) \cong A,$$

and similarly for $K^\bullet$ and $I^\bullet \oplus K^\bullet$. Finally, it is a routine check to show that for $k \geq 1$,

$$H^k(I^\bullet) = 0;$$

note that this follows because the canonical maps $\gamma_{k-1} : \mathrm{Im}\, \partial_{k-1}^I \to \mathrm{Ker}\, \partial_k^I$ correspond (via Theorem C.2.12) to maps $\rho_{k-1} : \mathrm{Coker}(i_{k-1}) \to \mathrm{Ker}\, \partial_k^I$, where the $i_k$ morphisms embed the $\mathrm{Coker}(i_{k-1})$ into an injective object and $i_0 := i$. From here a routine analysis of the canonical maps $\rho_{k-1}$ shows that for $k \geq 1$, $\rho_{k-1}$ is an isomorphism, from whence it follows that $H^k(I^\bullet) = 0$. The results for $K^\bullet$ and $I^\bullet \oplus K^\bullet$ follow mutatis mutandis. This shows that there are injective resolutions $I^\bullet, K^\bullet$, and $I^\bullet \oplus K^\bullet$ which fit into the exact sequence

$$0 \longrightarrow I^\bullet \longrightarrow I^\bullet \oplus K^\bullet \longrightarrow K^\bullet \longrightarrow 0$$

in $\mathbf{Ch}(\mathscr{A})$. Applying the functor $F$, using that $F$ and its prolongment are additive, and then using that the sequence above is exact exactly because the morphisms are built from the biproduct maps, we get that the sequence

$$0 \longrightarrow FI^\bullet \longrightarrow FI^\bullet \oplus FK^\bullet \longrightarrow FK^\bullet \longrightarrow 0$$

is exact in $\mathbf{Ch}(\mathscr{B})$. Applying now the $H^k$ functors for all $k \in \mathbb{Z}$ gives rise to the exact sequence, by Theorem D.0.6 and the fact that each sequence is identically zero in negative degrees,

$$\begin{tikzcd}
0 \arrow[r] & H^0(FI^\bullet) \arrow[r] & H^0(FI^\bullet \oplus FK^\bullet) \arrow[r] & H^0(FK^\bullet) \arrow[r, "\delta_0"] & H^1(FI^\bullet) \arrow[r] & \cdots \arrow[d] \\
\cdots & H^{n+1}(FI^\bullet) \arrow[l] & H^n(FK^\bullet) \arrow[l, "\delta_n"] & H^n(FI^\bullet \oplus FK^\bullet) \arrow[l] & H^n(FI^\bullet) \arrow[l] & \cdots \arrow[l, "\delta_{n-1}"]
\end{tikzcd}$$

in $\mathscr{B}$. However, this is exactly the sequence to be constructed in the statement of the theorem once we identify $H^n(FI^\bullet) =: R^nF(A)$, and similarly for $B$ and $C$, for all $n \in \mathbb{N}$. Thus this concludes the theorem. $\square$

Some of the benefits of defining right derived functors in this way is that it allows a quick introduction to change-of-functor type results. In particular, we can use this to define, later on, change of cohomology functors, and in turn use this to indirectly build new cohmology theories. For this, however, we need a basic structural lemma that we will use to prove that we can not only right derive natural transformations, but also show that the long exact sequence of Theorem D.0.46 is natural in both functors and objects.

**Lemma D.0.47.** *Let $\mathscr{A}$ and $\mathscr{B}$ be additive categories and let $F, G : \mathscr{A} \to \mathscr{B}$ be additive functors. Then if $\alpha : F \to G$ is a natural transformation, there exists a natural transformation $\alpha : F \to G : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{B})$, where $F$ and $G$ are the prolongments of $F$ and $G$ to the chain categories.*

*Proof.* Begin by observing that $\alpha_{A^\bullet} : FA^\bullet \to GA^\bullet$ is defined by taking $\alpha_{A^\bullet}$ to be the chain map induced by $\alpha_{A^n} : FA_n \to GA_n$ for every $n \in \mathbb{Z}$. This implies that the diagrams, for all $n \in \mathbb{Z}$,

$$\begin{tikzcd}
FA^{n-1} \arrow[r, "F\partial^A_{n-1}"] \arrow[d, "\alpha_{A^{n-1}}"'] & FA^n \arrow[d, "\alpha_{A^n}"] \\
GA^{n-1} \arrow[r, "G\partial^A_{n-1}"'] & GA^n
\end{tikzcd}$$

commute by the naturality of $\alpha$. However, if $\varphi \in \mathbf{Ch}(\mathscr{A})(A^\bullet, B^\bullet)$, the squares

$$\begin{tikzcd}
FA^n \arrow[r, "\alpha_{A^n}"] \arrow[d, "F\varphi_n"'] & GA^n \arrow[d, "G\varphi_n"] \\
FB^n \arrow[r, "\alpha_{B^n}"'] & GB^n
\end{tikzcd}$$

commute for all $n \in \mathbb{Z}$ as well. Thus it follows that the cube

$$\begin{tikzcd}
 & FB^{n-1} \arrow[rr, "F\partial^B_{n-1}"] \arrow[dd, "\alpha_{B^{n-1}}" near start] & & FB^n \arrow[dd, "\alpha_{B^n}"] \\
FA^{n-1} \arrow[ur, "F\varphi_{n-1}"] \arrow[rr, "F\partial^A_{n-1}" near start] \arrow[dd, "\alpha_{A^{n-1}}"'] & & FA^n \arrow[ur, "F\varphi_n"'] \arrow[dd, "\alpha_{A^n}" near start] & \\
 & GB^{n-1} \arrow[rr, "G\partial^B_{n-1}"' near start] & & GB^n \\
GA^{n-1} \arrow[ur, "G\varphi_{n-1}"] \arrow[rr, "G\partial^A_{n-1}"'] & & GA^n \arrow[ur, "G\varphi_n"'] &
\end{tikzcd}$$

commutes as well. This shows that the square

$$\begin{tikzcd}
FA^\bullet \arrow[r, "\alpha_{A^\bullet}"] \arrow[d, "F\varphi"'] & GA^\bullet \arrow[d, "G\varphi"] \\
FB^\bullet \arrow[r, "\alpha_{G^\bullet}"'] & GB^\bullet
\end{tikzcd}$$

commutes in $\mathbf{Ch}(\mathscr{B})$, which proves the lemma. $\square$

**Proposition D.0.48.** *Assume that $F, G : \mathscr{A} \to \mathscr{B}$ are additive functors between Abelian categories and assume that $\mathscr{A}$ has enough injectives. Let $\alpha : F \to G$ be a natural transformation and let*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & B \arrow[r, "\beta"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & X \arrow[r, "\gamma"'] & Y \arrow[r, "\delta"'] & Z \arrow[r] & 0
\end{tikzcd}$$

*be a diagram with exact rows in $\mathscr{A}$. Then:*

1. *For all $n \in \mathbb{N}$, there are natural transformations $R^n\alpha : R^nF \to R^nG$;*
2. *For all $n \in \mathbb{N}$, the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & R^nF(A) \arrow[r] \arrow[d] & R^nF(B) \arrow[r] \arrow[d] & R^nF(C) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & R^nF(X) \arrow[r] & R^nF(Y) \arrow[r] & R^nF(Z) \arrow[r] & \cdots
\end{tikzcd}$$

   *is commutative;*
3. *For all $n \in \mathbb{N}$, the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & R^nF(A) \arrow[r] \arrow[d] & R^nF(B) \arrow[r] \arrow[d] & R^nF(C) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & R^nG(A) \arrow[r] & R^nG(B) \arrow[r] & R^nG(C) \arrow[r] & \cdots
\end{tikzcd}$$

   *is commutative.*

*Proof.* For (1), we observe the following: First, that by Lemma D.0.47 we can lift the natural transformation $\alpha$ to a natural transformation:

$$\begin{tikzcd}
\mathbf{Ch}(\mathscr{A}) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Ch}(\mathscr{B})
\end{tikzcd}$$

We then define $R^n\alpha : R^nF \to R^nG$ by taking $R^n\alpha$ to be the whiskering $H^n * \alpha$, i.e., we define $R^n\alpha$ via the horizontal composite

$$\begin{tikzcd}
\mathbf{Ch}(\mathscr{A}) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Ch}(\mathscr{B}) \arrow[r, "H^n"] & \mathscr{B}
\end{tikzcd}$$

in **Cat** after taking appropriate injective resolutions. This establishes (1).

For (2) and (3), we use the same basic set-up. We find injective resolutions $I_1^\bullet$ of $A$, $I_2^\bullet$ of $X$, $K_1^\bullet$ of $C$, and $K_2^\bullet$ of $Z$; moreover, we can find morphisms $\varphi' : I_1^\bullet \to K_1^\bullet$ and $\rho' : K_1^\bullet \to K_2^\bullet$ such that

$$H^0(\varphi') = \varphi$$

and

$$H^0(\rho') = \rho.$$

Now use Exercise C.3.12 on each row to give injective resolutions $J_1^\bullet$ of $B$ and $J_2^\bullet$ of $Y$ with $J_1^\bullet \cong I_1^\bullet \oplus K_1^\bullet$ and $J_2^\bullet \cong I_2^\bullet \oplus K_2^\bullet$. Note since we can replace $J_1^\bullet$ and $J_2^\bullet$ with the corresponding direct sums and then conjugate by the appropriate isomorphisms, it suffices to show (2) and (3) in the case where $J_i^\bullet = I_i^\bullet \oplus K_i^\bullet$ for $i = 1$ and $i = 2$.

We proceed now to show (2). From the above construction, there is a map $\psi' : I_1^\bullet \oplus K_1^\bullet \to I_2^\bullet \oplus K_2^\bullet$ for which $h_0(\psi') = \psi$ and for which the diagram

$$\begin{tikzcd}
0 \arrow[r] & I_1^\bullet \arrow[r] \arrow[d, "\varphi'"'] & I_1^\bullet \oplus K_1^\bullet \arrow[r] \arrow[d, "\psi'"'] & K_1^\bullet \arrow[r] \arrow[d, "\rho'"] & 0 \\
0 \arrow[r] & I_2^\bullet \arrow[r] & I_2^\bullet \oplus K_2^\bullet \arrow[r] & K_2^\bullet \arrow[r] & 0
\end{tikzcd}$$

commutes. Now, because $F$ is additive, applying $F$ to the diagram above gives the diagram

$$\begin{tikzcd}
0 \arrow[r] & FI_1^\bullet \arrow[r] \arrow[d, "F\varphi'"'] & FI_1^\bullet \oplus FK_1^\bullet \arrow[r] \arrow[d, "F\psi'"'] & FK_1^\bullet \arrow[r] \arrow[d, "F\rho'"] & 0 \\
0 \arrow[r] & FI_2^\bullet \arrow[r] & FI_2^\bullet \oplus FK_2^\bullet \arrow[r] & FK_2^\bullet \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact because $F$ preserves biproducts. Using Proposition D.0.7, we derive the existence of the commuting diagram, for $n \in \mathbb{N}$,

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(FI_1^\bullet) \arrow[r] \arrow[d] & H^n(FI_1^\bullet \oplus FK_1^\bullet) \arrow[r] \arrow[d] & H^n(FK_1^\bullet) \arrow[r] \arrow[d] & H^{n+1}(FI_1^\bullet) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^n(FI_2^\bullet) \arrow[r] & H^n(FI_2^\bullet \oplus FK_2^\bullet) \arrow[r] & H^n(K_2^\bullet) \arrow[r] & H^{n+1}(FK_2^\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

in which both rows are (long) exact. However, identifying each of the objects above with the $R^nF(-)$ establishes (2).

For (3), we consider the exact sequence

$$0 \longrightarrow I_1^\bullet \longrightarrow I_1^\bullet \oplus K_1^\bullet \longrightarrow K_1^\bullet \longrightarrow 0$$

of injective resolutions, as we did in (2). Applying $F$ and $G$ to the diagram above and then using the natural transformation $\alpha$ constructed as in Lemma D.0.47 gives rise to the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & FI_1^\bullet \arrow[r] \arrow[d, "\alpha_{I_1^\bullet}"'] & F(I_1^\bullet \oplus K_1^\bullet) \arrow[r] \arrow[d, "\alpha_{I_1^\bullet \oplus K_1^\bullet}"'] & FK_1 \arrow[r] \arrow[d, "\alpha_{K_1^\bullet}"] & 0 \\
0 \arrow[r] & GI_1^\bullet \arrow[r] & G(I_1^\bullet \oplus K_1^\bullet) \arrow[r] & GK_1^\bullet \arrow[r] & 0
\end{tikzcd}$$

in $\mathbf{Ch}(\mathscr{B})$. Furthermore, because both $F$ and $G$ are additive functors, both rows are exact. Proceeding mutatis mutandis as in (2) then establishes (3).

□

**Remark D.0.49.** In this proof we should potentially be doing this in a stacky way and working with the pseudofunctorial nature of our definition of derived functors. While we have not done this here, it is not a difficult adaptation from what we have developed in this article.[20]

**Remark D.0.50.** We will not explicitly discuss left derived functors, save for in Exercise D.0.10 and in this comment,[21] as left derived functors are formally dual to right derived functors. In particular, we can define left derived functors as right derived functors on the opposite category[22]. Alternatively, and more explicitly, left derived functors may be defined by taking an Abelian category with enough projectives, working with projective resolutions, and then proceeding mutatis mutandis.

[20]So of course it is annoyingly complicated.
[21]Exploring these are *left* as an exercise.
[22]Which, while not particularly helpful, is technically correct and reduces the writing I have to do.

With all this work, we can now proceed to define and work with cohomology functors associated to an additive functor! These will fix the issues of additive functors not preserving exact sequences, and instead turn them into long exact sequences. We will also use this in our discussion of geometric invariants, but for now we can begin to get to know cohomology in the most basic cases.

**Definition D.0.51.** If $F : \mathscr{A} \to \mathscr{B}$ is a left exact additive functor and if $\mathscr{A}$ has enough injectives, then for all $n \in \mathbb{N}$, the $n$-th right derived cohomology of $F$ with coefficients in $A$ is the object

$$H^n(F; A) := R^n F(A).$$

In particular, the $n$-th right derived cohomology functor $H^n(F; -)$ is defined to be the $n$-th right derived functor of $F$.

**Remark D.0.52.** In some cases, we denote cohomology differently. For instance, group cohomology with coefficients in a $G$-module $A$ is denoted by $H^n(G; A)$, the $n$-th simplicial cohomology of a topological space $X$ with $A$-valued coefficients is denoted by $H^n_{\text{simp}}(X; A)$, the $n$-th sheaf cohomology of a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules is denoted $H^n(\mathscr{F})$, and other examples are given when necessary.

**Example D.0.53.** Let $G$-**Mod** be the Abelian category of left $G$-modules and let $(-)^G : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$ be the $G$-fix functor, i.e., $A \mapsto A^G = \{a \in A \mid \forall\, g \in G, ga = a\}$. Then the group cohomology functors of $n$ are the right derived functors of $(-)^G$, i.e.,

$$H^n(G; A) = H^n((-)^G; A) = R^n\big((-)^G\big)(A).$$

**Example D.0.54.** If $\mathscr{A} = \mathcal{O}_X\text{-}\mathbf{Mod}$ or if $\mathscr{A} = \mathbf{QCoh}(X)$ for a scheme $X$, then the global sections functor $\Gamma : \mathscr{A} \to \mathbf{Ab}$ given by $\mathscr{F} \mapsto \Gamma(\mathscr{F}, |X|)$ is left exact. The $n$-th sheaf cohomology group of an $\mathscr{A}$-sheaf $\mathscr{F}$ is then

$$H^n(\mathscr{F}) = H^n(\Gamma; \mathscr{F}) = R^n\Gamma(\mathscr{F}).$$

## Exercises

**Exercise D.0.1.** Show that if $\mathscr{A} = R\text{-}\mathbf{Mod}$, then $H^i(A^\bullet) \cong \operatorname{Ker}\partial_i / \operatorname{Im}\partial_{i-1}$.

**Exercise D.0.2.** Complete the proof of Lemma D.0.5 by showing that $\rho_n^{\text{op}}$ is the canonical map $\delta_n : \operatorname{Ker}(\operatorname{coker}\partial_n^{\text{op}}) \to \operatorname{Ker}\partial_{n-1}^{\text{op}}$ in $\mathscr{A}^{\text{op}}$ and then using the homology/cohomolgy calculus from the opposite category to the main category.

**Exercise D.0.3.** Prove that the long exact cohomology sequence is natural in the following sense: If we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X^\bullet \arrow[r] & Y^\bullet \arrow[r] & Z^\bullet \arrow[r] & 0 \end{tikzcd}$$

in $\mathbf{Ch}(\mathscr{A})$ in which both rows are exact, then there are morphisms making the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^k(A^\bullet) \arrow[r] \arrow[d] & H^k(B^\bullet) \arrow[r] \arrow[d] & H^k(C^\bullet) \arrow[r] \arrow[d] & H^{k+1}(A^\bullet) \arrow[r] \arrow[d] & H^{k+1}(B^\bullet) \arrow[r] \arrow[d] & H^{k+1}(C^\bullet) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^k(X^\bullet) \arrow[r] & H^k(Y^\bullet) \arrow[r] & H^k(Z^\bullet) \arrow[r] & H^{k+1}(X^\bullet) \arrow[r] & H^{k+1}(Y^\bullet) \arrow[r] & H^{k+1}(Z^\bullet) \arrow[r] & \cdots \end{tikzcd}$$

commute in $\mathscr{A}$.

**Exercise D.0.4.** Let $\mathscr{A}$ be an Abelian category. Is there a category $\mathscr{C}$ whose objects are complexes in $\mathscr{A}$ and whose morphisms are graded morphisms of nonfixed degree? Is this category, if it exists, Abelian? Prove or disprove. Hint: Prove that if $f$ is of degree $k$ and if $g$ is of degree $\ell$, then $g \circ f$ is of degree $\ell + k$.

**Exercise D.0.5.** Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor and let $h$ be a degree $k$ morphism on $\mathbf{Ch}(\mathscr{A})$. Prove that $Fh$, defined in the obvious way, gives a degree $k$ morphism on $\mathbf{Ch}(\mathscr{B})$.

**Exercise D.0.6.** Prove that in an Abelian category $\mathscr{A}$ that if there is a commuting square of the form

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d] & X_2 \arrow[r, "g"] \arrow[d] & X_3 \arrow[d] \\ Y_1 \arrow[r, "h"'] & Y_2 \arrow[r, "k"'] & Y_3 \end{tikzcd}$$

where both horizontal rows compose to the zero map, then there is a commuting diagram of the form

$$\begin{tikzcd} & \operatorname{Ker}(\operatorname{coker} f) \arrow[dr, "\ker(\operatorname{coker} f)"] \arrow[ddd, dashed, "\exists!\alpha"] & \\ X_1 \arrow[ur, "\varepsilon"] \arrow[rr, "f"] \arrow[d] & & x_2 \arrow[d] \\ Y_1 \arrow[rr, "h"'] \arrow[dr, "\varepsilon'"'] & & Y_2 \\ & \operatorname{Ker}(\operatorname{coker} h) \arrow[ur, "\ker(\operatorname{coker} h)"'] & \end{tikzcd}$$

in $\mathscr{A}$.

**Exercise D.0.7.** Let $I$ be an injective object and assume that $I$ is the degree zero object in a sequence $A^\bullet$ where $A^{-n} = 0$ for all integers $n \geq 1$. Prove that there exists a monomorphism $H^0(A^\bullet)$ into $I$. More generally, prove that if $A^\bullet$ is a sequence such that $A^k = 0$ for all $k \leq 0$, then there is a map $H^0(A^\bullet) \to A^0$ which is monic.

**Exercise D.0.8.** Let $\mathscr{A}$ be an Abelian category. Prove that $\mathscr{A}$ has enough injectives if and only if every object has an injective resolution.

**Exercise D.0.9.** Prove that the following are equivalent for an Abelian category $\mathscr{A}$:

1. $\mathscr{A}$ has enough injectives;

2. If $A^\bullet$ is a complex for which there exists an integer $n$ such that for all $k \leq n$, $H^k(A^\bullet) = 0$, then there is an injective resolution $q : A^\bullet \to I^\bullet$ in the following sense: The map $q$ is a quasi-isomorphism, $I^k = 0$ for all $k \leq \ell$, for some $\ell \in \mathbb{Z}$, and $I^m$ is injective for all $m \in \mathbb{Z}$;

3. If $A^\bullet$ is a complex in $\mathbf{Ch}(\mathscr{A})$ such that there exists an $m \in \mathbb{Z}$ for which $A^k = 0$ if $k \leq m$, then there exists an injective resolution $q : A^\bullet \to I^\bullet$ where $I^k = 0$ for all $k \leq m$ and for all $n \in \mathbb{Z}$, $q_n : A^n \to I^n$ is a monomorphism.

**Exercise D.0.10.** Define the left derived functors of an additive functor $F : \mathscr{A} \to \mathscr{B}$ when $\mathscr{A}$ has enough projectives. Prove explicitly that your definitions are well-defined up to unique isomorphism and that they indeed give functors. Moreover, prove that there is a long exact sequence of left derived functors, natural transformations of additive functors lift to natural transformations of left derived functors, and that this long exact sequence is natural in both morphisms of short exact sequences and in transition of functors.

Hint: Dualize the construction of right derived functors by talking about right derived functors on the opposite category.

**Exercise D.0.11.** Let $\mathscr{A}$ be an Abelian category and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence in $\mathscr{A}$. Prove that there exists an exact sequence of complexes

$$0 \longrightarrow I^\bullet \xrightarrow{\alpha} J^\bullet \xrightarrow{\beta} K^\bullet \longrightarrow 0$$

in $\mathbf{Ch}(\mathscr{A})$, where $I^\bullet, J^\bullet$, and $K^\bullet$ are injective resolutions of $A, B$, and $C$, while $\alpha$ and $\beta$ are maps for which $H^0(\alpha) = f$ and $H^0(\beta) = g$.

**Exercise D.0.12.** Show that the category $K(\mathscr{A})$ is more pathological than you expect in that nonzero objects $A^\bullet$ in $\mathbf{Ch}(\mathscr{A})$ can become zero in $K(\mathscr{A})$. This exercise seeks to show that: Find an Abelian category $\mathscr{A}$ and an object $A^\bullet$ of $\mathbf{Ch}(\mathscr{A})$ for which $A^\bullet \not\cong 0$ but the maps $\mathrm{id}_{A^\bullet} \simeq 0 : A^\bullet \to A^\bullet$, where 0 is the map factoring as

$$\begin{tikzcd} A^\bullet \arrow[rr, "0"] \arrow[dr, dashed, "\exists!"'] & & A^\bullet \\ & 0 \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Prove that $0 \cong A^\bullet$ in $K(\mathscr{A})$. Such an object is called *contractible*.

**Exercise D.0.13.** Recall the shift functors $[n] : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{A})$ of Exercise C.4.7.

1. Prove that for any $n, k \in \mathbb{Z}$ there are isomorphisms

$$k_n A[k] : \operatorname{Ker} \partial_{n+k}^A \to \operatorname{Ker} \partial_n^{A[k]}$$

for all complexes $A^\bullet \in \mathbf{Ch}(\mathscr{A})_0$.

2. Prove that if $n, k \in \mathbb{Z}$,

$$H^k(A[n]^\bullet) \cong H^{n+k}(A^\bullet)$$

for all complexes $A^\bullet \in \mathbf{Ch}(\mathscr{A})_0$. Call these isomorphisms $s_n A[k] : H^{n+k}(A^\bullet) \to H^n(A[k]^\bullet)$.

3. Prove that the isomorphisms $k_n(-)[k]$ and $s_n(-)[k]$ are natural in the sense that if $A^\bullet$ is any complex in $\mathbf{Ch}(\mathscr{A})_0$ then the diagram

$$\begin{tikzcd} \operatorname{Ker} \partial_{n+k}^A \arrow[r, "k_n A[k]"] \arrow[d] & \operatorname{Ker} \partial_n^{A[k]} \arrow[d] \\ H^{n+k}(A^\bullet) \arrow[r, "s_n A[k]"'] & H^n(A[k]^\bullet) \end{tikzcd}$$

commutes in $\mathscr{A}$.

4. Conclude that the shift functors induce isomorphisms

$$R^k F(A[n]^\bullet) \cong R^{k+n} F(A^\bullet)$$

for any left exact functor $F$ (assuming that $\mathscr{A}$ has enough injectives).

5. As a special case, prove that if $k \equiv 0 \mod 2$ then $\operatorname{Ker} \partial_{n+k}^A = \operatorname{Ker} \partial_n^{A[k]}$ and $H^n(A[k]^\bullet) = H^{n+k}(A^\bullet)$ for all $n \in \mathbb{Z}$. Prove also that if $\mathscr{A} = R$-$\mathbf{Mod}$, where $R$ is a ring of characteristic 2, then $\operatorname{Ker} \partial_{n+k}^A = \operatorname{Ker} \partial_n^{A[k]}$ for all $k, n \in \mathbb{Z}$.

**Exercise D.0.14.** Prove that there is an equivalence of categories

$$K(\mathscr{A})^{\mathrm{op}} \simeq K(\mathscr{A}^{\mathrm{op}})$$

and that to give an additive functor $K(\mathscr{A}^{\mathrm{op}}) \to \mathscr{B}^{\mathrm{op}}$ is equivalent to giving an additive functor $K(\mathscr{A}) \to \mathscr{B}$.

**Exercise D.0.15.** For those of you who know some topos theory and/or are comfortable arguing with adjunction calculus, this will help us determine things like saying that the topos-valued cohomology is the right derived functors of internal global sections. If this is meaningless to you, or you want to read this explained explicitly[23], see Appendix. Ref

In what follows, let $\mathcal{E}$ be a Grothendieck topos and $\mathscr{C}$ a Cartesian closed Category.

1. Let $A$ be an internal Abelian group in $\mathscr{C}$. Prove that for any object $X \in \mathscr{C}_0$, the internal hom object[24] $[X, A]$ of $\mathscr{C}$ is an internal Abelian group in $\mathscr{C}$. Hint: Internalize the proofs above and use the adjunction calculus of $(-) \times X \dashv [X, -] : \mathscr{C} \to \mathscr{C}$ to help you find the zero map.

2. Let $\mathscr{A} := \mathbf{Ab}(\mathscr{C})$ be the category of internal Abelian groups in $\mathscr{C}$ and let $X \in \mathscr{C}_0$.

   - Show that the restriction of the functor $[X, -]$ to the category $\mathscr{A}$ takes values in $\mathscr{A}$, i.e., $[X, -]|_{\mathscr{A}}$ factors as:
   $$\mathscr{A} \to \mathscr{A}$$
   - Prove the functor $[X, -]|_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ is left exact.

3. Let $\mathscr{A} := \mathbf{Ab}(\mathcal{E})$. Prove that $\mathscr{A}$ is an Abelian category and conclude that for any $X \in \mathcal{E}_0$, the functor $[X, -]|_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ and its prolongment $[X, -] : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{A})$ are left exact.

4. Let $\top$ be the terminal object in $\mathcal{E}$. The object $[\top, X]$ is called the $\mathcal{E}$-valued global sections of $X$.

   - Prove that the right derived functors $R^q[\top, -]$ of $[\top, -]|_{\mathscr{A}}$ exist for any $q \in \mathbb{Z}$. Hint: You may take for granted that $\mathscr{A}$ has enough injectives (this is a clean but routine argument involving some adjunction pushing and topos-theoretic results).
   - We define the $\mathcal{E}$ cohomology of an internal Abelian group to be $H^q_{\mathcal{E}}(A) := R^q[\top, A]$. Describe how to capture étale cohomology in this formulation and show how this generalizes the usual right derived functor cohomology of $\mathcal{E}(\top, -) : \mathscr{A} \to \mathbf{Ab}$.

5. **The next few exercises are tricky!** Let $\mathcal{O}_X$ be a commutative ring object in $\mathcal{E}$.

   - Assuming that $\mathscr{A}$ is symmetric monoidal closed and has enough injectives (and in particular that the internal Abelian group $\mathbb{Q}/\mathbb{Z}$ defined as the cokernel of the unique map of internal rings $\mathbb{Z} \to \mathbb{Q}$ internal to $\mathcal{E}$ (note that $\mathbb{Z}$ is constructed as the internal Grothendieck group of the natural numbers object $\mathbb{N}$ of $\mathcal{E}$) is injective (if you don't like this, use the constant sheaf on $\mathbb{Q}/\mathbb{Z}$) in $\mathscr{A}$), prove that the internal hom object $[\mathcal{O}_X, \mathbb{Q}/\mathbb{Z}]$ of $\mathscr{A}$ is an injective $\mathcal{O}_X$-module (Hint: Show that $[\mathcal{O}_X, \mathbb{Q}/\mathbb{Z}]$ is injective by internalizing the usual proof that $\mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ is injective in $R$-$\mathbf{Mod}$).
   - Show that for any $\mathcal{O}_X$-module $A$ in $\mathcal{E}$, the functor $[\top, -]$ restricts to a functor from $\mathcal{O}_X$-$\mathbf{Mod}$ to $\mathscr{A}$.
   - Prove that the right derived functors $R^k[\top, -] : \mathcal{O}_X\text{-}\mathbf{Mod} \to \mathscr{A}$ exist for any $k \in \mathbb{Z}$.

**Exercise D.0.16.** Let $\mathscr{A}$ be an Abelian category with enough injectives and fix an object $A^\bullet \in K(\mathscr{A})_0$.

[23] I guess this book contains at least one fully solved exercise after all!

[24] I vote we call these "homjects," analogously to how we call the graded hom complex (which we'll meet later) a "homplex."

- Prove that the functor

$$K(\mathscr{A})(A^\bullet, -) : K(\mathscr{A}) \to \mathbf{Ab}, \quad X^\bullet \mapsto K(\mathscr{A})(A^\bullet, X^\bullet)$$

  is left exact.

- Consider a sequence of complexes

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A^\bullet[1]$$

  in $K(\mathscr{A})$ chain homotopic to a mapping cone sequence, i.e., there is a morphism $\varphi \in \mathbf{Ch}(\mathscr{A})(X^\bullet, Y^\bullet)$ for which there is a commuting diagram

$$\begin{tikzcd}
A^\bullet \arrow[r, "f"] \arrow[d, "\cong"'] & B^\bullet \arrow[r, "g"] \arrow[d, "\cong"'] & C^\bullet \arrow[r, "h"] \arrow[d, "\cong"] & A[1] \arrow[d, "\cong"] \\
X^\bullet \arrow[r, "\varphi"'] & Y^\bullet \arrow[r, "\alpha_\varphi"'] & \mathrm{Cone}(\varphi) \arrow[r, "\rho_\varphi"'] & X[1]
\end{tikzcd}$$

  in $K(\mathscr{A})$. Prove that for any $Z^\bullet \in K(\mathscr{A})_0$, the sequence of maps

$$K(\mathscr{A})(Z^\bullet, A^\bullet) \xrightarrow{f_*} K(\mathscr{A})(Z^\bullet, B^\bullet) \xrightarrow{g_*} K(\mathscr{A})(Z^\bullet, C^\bullet)$$

  is exact.

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