Mini-couse on GAP – Exercises

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1 Problems

1. Use `ChineseRem` to find (if possible) the smallest solution of the following congruences:

$$\begin{cases} x \equiv 3 \bmod 10, \\ x \equiv 8 \bmod 15, \\ x \equiv 5 \bmod 84 \end{cases} \qquad \begin{cases} x \equiv 29 \bmod 52, \\ x \equiv 19 \bmod 72. \end{cases}$$

2. Compute the gcd of 42823 and 6409. Find $x, y \in \mathbb{Z}$ such that

$$\gcd(5033464705, 3138740337) = 5033464705x + 3138740337y.$$

3. Describe the following sequence: $a_1 = 3$, $a_{n+1} = 3^{a_n} \bmod 100$.

4. Use `Product` to compute $2 \cdot 4 \cdot 6 \cdots 20$.

5. Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{9999} - \frac{1}{10000} = \frac{1}{5001} + \frac{1}{5002} + \cdots + \frac{1}{10000}.$$

6. Find the last two digits of 3^{400}.

7. Find the roots of $x^2 + x + 7 \equiv 0 \bmod m$ for $m \in \{15, 189\}$.

8. Write a function that returns the binary expansion of an integer. Could you do this for other bases?

9. Use `MinimalPolynomial` to compute the minimal polynomial of $3 + \sqrt{5}$ over the rational numbers.

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10. Compute the first 100 Fibonacci numbers.

11. For $k \in \mathbb{N}$ let a_n be given by $a_1 = \cdots = a_{k+1} = 1$ and $a_n = a_{n-k} + a_{n-k-1}$ for all $n > k+1$. Write a function depending on k that constructs the sequence a_n. For more information see http://oeis.org/A103379.

12 (Somos sequence). Write a function that returns the n-th term of a_n, where $a_0 = a_1 = a_2 = a_3 = 1$ and

$$a_n = \frac{a_{n-1}a_{n-3} + a_{n-2}^2}{a_{n-4}}$$

for all $n \geq 4$. For more information see http://oeis.org/A006720.

13. Write a function that given a list `lst` of words and a letter `x`, returns a sublist of `lst` where every word starts with `x`.

14. Write a function that returns the number of prime numbers $\leq n$.

15. Use the function `Permuted` to write a function that shows all the anagrams of a given word.

16. Given a list of non-negative numbers, write a function that displays the histogram associated with this list. For example, if the argument is the list `[1,4,2]`, the function should display

```
X
XXXX
XX
```

17. Write a function that given a list of words returns the longest one.

18. Write a function that transforms a given number of seconds in days, hours, minutes and seconds.

19. Write a function that returns the average value of a given list of numbers.

20. Write a function that, given a letter, returns **`true`** if the letter is a vowel and **`false`** otherwise.

21. Use `CharacteristicPolynomial` to compute the characteristic polynomial of the matrix $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix}$. Can you compute the minimal polynomial?

22. Use the function `QuotientRemainder` to compute the quotient and the remainder of $f = 2x^4 + 3x^3 + 2x + 4$ and $g = 3x^2 + x + 2$ in the ring $\mathbb{Z}_5[x]$.

23. Compute $3x^{101} - 15x^{16} - 2x^7 - 5x^4 + 3x^3 + 2x^2 + 1 \bmod x^3 + 1$.

24. Prove that $x=2$ is the only root in $\mathbb{Z}_5$ of $x^{1000}+4x+1\in\mathbb{Z}_5[x]$.

25. Factorize x^4-1 in $\mathbb{Z}/5[x]$ and in $\mathbb{Z}/7[x]$.

26. Prove that $x^2-79x+1601$ gives a prime number for $x\in\{0,1,\dots,79\}$.

27. Write the first 50 twin primes.

28. FRACTRAN is a programming language invented by J. Conway. A FRACTRAN program is simply an ordered list of positive rationals together with an initial positive integer input n. The program is run by updating the integer n as follows:

- For the first rational f in the list for which $nf\in\mathbb{Z}$, replace n by nf.
- Repeat this rule until no rational in the list produces an integer when multiplied by n, then stop.

Write an implementation of the FRACTRAN language.

Starting with $n=2$, the program

$$\frac{17}{65},\frac{133}{34},\frac{17}{19},\frac{23}{17},\frac{2233}{69},\frac{23}{29},\frac{31}{23},\frac{74}{341},\frac{31}{37},\frac{41}{31},\frac{129}{287},\frac{41}{43},\frac{13}{41},\frac{1}{13},\frac{1}{3}$$

produces the sequence

$$2,15,825,725,1925,2275,425,390,330,290,770\dots$$

In 1987, J. Conway proved that this sequence contains the set $\{2^p : p \text{ prime}\}$. See https://oeis.org/A007542 for more information.

29. The first terms of Conway's "look and say" sequence are the following:

```
1
11
21
1211
111221
312211
```

After guessing how each term is computed, write a script to create the first terms of the sequence.

30. Write

$$\begin{pmatrix}123456\\253461\end{pmatrix},\qquad\begin{pmatrix}123456789\\234517896\end{pmatrix},\qquad\begin{pmatrix}12345\\32451\end{pmatrix},$$

as a product of disjoint cycles.

31. Write the permutations $(123)(45)(1625)(341)$ and $(12)(245)(12)$ as product of disjoint cycles.

32. Find a permutation τ such that

1. $\tau(12)(34)\tau^{-1} = (56)(13)$.
2. $\tau(123)(78)\tau^{-1} = (257)(13)$.
3. $\tau(12)(34)(567)\tau^{-1} = (18)(23)(456)$.

33. Compute $\tau\sigma\tau^{-1}$ in the following cases:

1. $\sigma = (123)$ and $\tau = (34)$.
2. $\sigma = (567)$ and $\tau = (12)(34)$.

34. Let $\sigma \in \mathbb{S}_9$ be given by $\sigma(i) = 10 - i$ for all $i \in \{1, \dots, 9\}$. Write σ as a product of disjoint cycles.

35. Find (if possible) three permutations $\alpha, \beta, \gamma \in \mathbb{S}_5$ such that $\alpha\beta = \beta\alpha$, $\beta\gamma = \gamma\beta$ and $\alpha\gamma \neq \gamma\alpha$.

36. Use the function `PermutationMat` to write the elements of $\mathbb{S}_3$ as 3×3 matrices.

37. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 2 & 3 \end{pmatrix}$ compute

$$I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4.$$

38. Write the funcion

$$(n, A) \mapsto I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{n!}A^n.$$

39. For $n \in \mathbb{N}$ the Hilbert matrix H_n is defined as

$$(H_n)_{ij} = \frac{1}{i + j - 1}, \quad i, j \in \{1, \dots, n\}.$$

Write the function $n \mapsto H_n$.

40. Use the function `KroneckerProduct` to compute

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 5 & 6 & 7 \\ 2 & 1 & 0 \\ 0 & 1 & 9 \end{pmatrix}.$$

41. Let S be the vector space (over the rationals) generated by $(0,1,0)$ and $(0,0,1)$ and T be generated by $(1,2,0)$ and $(3,1,2)$. Use `VectorSpace` to create these vector spaces and compute $\dim S$, $\dim T$, $\dim(S \cap T)$ and $\dim(S + T)$.

42. Write the coordinates of the vector $(1,0,1)$ in the basis given by $(2i,1,0)$, $(2,-i,1)$, $(0,1+i,1-i)$.

43. Walsh matrices $H(2^k)$, $k \geq 1$, are defined recursively as follows:

$$H(2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad H(2^k) = H(2) \otimes H(2^{k-1}), \quad k \geq 1,$$

Construct the function $n \mapsto H(2^n)$.

44. Use the functions `Eigenvalues` and `Eigenvalues` to compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{pmatrix} \in \mathbb{Q}^{3\times 3}.$$

The function `Eigenvectors` returns generators of the eigenspaces, where $v \neq 0$ is an eigenvector of A with eigenvalue λ if and only if $vA = \lambda v$.

45. Use the function `NullspaceMat` to compute the nullspace of the matrix A from problem 44.

The nullspace of A is defined as the set of vectors v such that $vA = 0$.

46. Compute the order of the group generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Can you recognize this group?

47. Contruct the Heisenberg group $H(\mathbb{Z}/3)$.

48. Construct the Klein group as a group of permutations.

49. Prove that all subgroups of $C_4 \times Q_8$ are normal.

50. Let G be the set of matrices of the form

$$\begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}, \quad c, d \in \mathbb{F}_4,\ c \neq 0.$$

Prove that G is a group and compute its order.

51. Use the function `IsomorphicSubgroups` to prove that $\mathbb{A}_6$ does not contain a subgroup isomorphic to $\mathbb{S}_5$ and that $\mathbb{A}_7$ contains a subgroup isomorphic to $\mathbb{S}_5$.

52. Prove that $\mathbb{A}_6$ does not contain subgroups of prime index.

53. Prove that $\mathbf{SL}_2(3)$ has a unique normal subgroup of order eight.

54. Find a subgroup of $\mathbf{SL}_2(5)$ isomorphic to $\mathbf{SL}_2(3)$.

55. Use the functions `SylowSubgroup` and `ConjugacyClassSubgroups` to construct all Sylow subgroups of $\mathbb{A}_4$ and $\mathbb{S}_4$.

56. Prove that $\mathbb{S}_5$ has a 2-Sylow subgroup isomorphic to the dihedral group of eight elements.

57. Can you recognize the structure of 2-Sylow subgroups of $\mathbb{S}_6$?

58. Use the function `Normalizer` to compute the number of conjugates of 2-Sylow subgroups of $\mathbb{A}_5$.

59. Find all Sylow subgroups of C_{27}, $\mathbf{SL}_2(5)$, $\mathbb{S}_7$, $\mathbb{S}_3 \times \mathbb{A}_4$ and $\mathbb{S}_3 \times C_{20}$.

60. Compute the conjugacy classes of subgroups of $\mathbb{S}_3 \times \mathbb{S}_3$ and find three p-Sylow subgroups, say A, B, C, such that $A \cap B = 1$ and $A \cap C \neq 1$.

61. Prove that the group

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : b, d \in \mathbb{F}_{19},\ d \neq 0 \right\}$$

is not simple.

62. Let G be the group generated by the permutations $(12)(6\,11)(8\,12)(9\,13)$, $(5\,13\,9)(6\,10\,11)(7\,8\,12)$ and $(2,4,3)(5,8,9)(6\,10\,13)(7\,11\,12)$. How many elements of G are commutators?

63. Prove that $\mathbb{A}_4 \times C_7$ does not contain subgroups of index two.

64. Prove that $\mathbb{A}_5$ does not contain subgroups of order $8, 15, 20, 24, 30, 40$.

65. It is known that an abelian subgroup of $\mathbb{S}_n$ has order $\leq 3^{\lfloor n/3 \rfloor}$. How good is this bound? For $n \in \{5, 6, 7, 8\}$ find (if possible) an abelian subgroup of $\mathbb{S}_n$ of order $3^{\lfloor n/3 \rfloor}$.

66. Prove that $\mathbf{SL}_2(5)$ does not contain subgroups isomorphic to $\mathbb{A}_5$.

67. Prove that for each d that divides 24 there exists a subgroup of $\mathbb{S}_4$ of order d.

68. Prove that $\mathbf{SL}_2(3)$ contains a unique element of order two. Prove that $\mathbf{SL}_2(3)$ does not have subgroups of order 12.

69. Prove that the derived subgroup of $\mathbf{SL}_2(3)$ is isomorphic to Q_8.

70. Can you recognize the group $\mathbf{SL}_2(3)/Z(\mathbf{SL}_2(3))$?

71. Are the groups $\mathbb{S}_5$ and $\mathbf{SL}_2(5)$ isomorphic?

72. Let $\langle r^4 = s^2 = 1, srs = r^{-1} \rangle$ be the dihedral group of eight elements. Find all subgroups containing $\langle 1, r^2 \rangle$.

73. Find all the group homomorphisms $\mathbb{S}_3 \to \mathbf{SL}_2(3)$.

74. Are there any surjective homomorphism $\mathbb{D}_{16} \to \mathbb{D}_8$? What about $\mathbb{D}_{16} \to C_2$?

75. Prove that $\mathrm{Aut}(\mathbb{A}_4) \simeq \mathbb{S}_4$.

76. Prove that $\mathrm{Aut}(\mathbb{D}_8) \simeq \mathbb{D}_8$ and that $\mathrm{Aut}(\mathbb{D}_{16}) \not\simeq \mathbb{D}_{16}$.

77. Compute the order of the group $\mathrm{Aut}(C_{11} \times C_2 \times C_3)$.

78. Prove that $\mathbb{D}_{12} \simeq \mathbb{S}_3 \times C_2$.

79. Prove that every group of order < 60 is solvable.

80. Let G be a group of order twelve such that $G \not\simeq \mathbb{A}_4$. Prove that G contains an element of order six.

81. Prove that a group of order 455 is cyclic.

82. Let G be a simple group of order 168. Compute the number of elements of order seven of G.

83. Prove that there are no simple groups of order 2540 and 9075.

84. Find a group G of order 3^6 such that $\{[x,y] : x,y \in G\} \neq [G,G]$.

85. Find a group G of order 2^7 such that $\{[x,y] : x,y \in G\} \neq [G,G]$

86. Prove that a group of order 15, 35 or 77 is cyclic.

87. Prove that a simple group of order 60 is isomorphic to $\mathbb{A}_5$.

88. Prove that the only non-abelian simple group of order < 100 is $\mathbb{A}_5$.

89. Is the following true? For any finite group G the set $\{x^2 : x \in G\}$ is a subgroup of G.

90. Prove the following theorem of Guralnick [1]. There exists a group G of order $n \leq 200$ such that $[G,G] \neq \{[x,y] : x,y \in G\}$ if and only if

$$n \in \{96, 128, 144, 162, 168, 192\}.$$

91. Prove the following extension of Guralnick's theorem (Problem 90). There exists a group G of order $n < 1024$ such that $[G,G] \neq \{[x,y] : x,y \in G\}$ if and only if n is one of the following numbers: 96, 128, 144, 162, 168, 192, 216, 240, 256, 270, 288, 312, 320, 324, 336, 360, 378, 384, 400, 432, 448, 450, 456, 480, 486, 504, 512, 528, 540, 560, 576, 594, 600, 624, 640, 648, 672, 702, 704, 720, 729, 744, 750, 756, 768, 784, 792, 800, 810, 816, 832, 840, 864, 880, 882, 888, 896, 900, 912, 918, 936, 960, 972, 1000, 1008.

92. Compute the list of normal subgroups of $\mathbf{GL}_2(3)$.

93. Compute the list of minimal subgroups of $\mathbb{A}_4$.

94. Compute the socle and the list of minimal normal subgroups of $\mathbb{A}_4$.

95. Compute the Fitting and the Frattini subgroup of $\mathbf{SL}_2(3)$.

96. Compute the list of all maximal normal subgroups of $\mathbf{SL}_2(3)$.

97. Prove that $\mathrm{PSL}_2(7)$ has a maximal subgroup of order 16.

98. Let G be a finite group and H be a subgroup. The **Chermak–Delgado** measure of H is the number $m_G(H) = |H||C_G(H)|$. Write a function to compute the Chermak–Delgado.

99. Compute $m_G(H)$ for $G \in \{\mathbb{S}_3, \mathbb{D}_8\}$ and H a subgroup of G.

100. Compute order the holomorph of $\mathbb{A}_4$. Find a permutation representation of small degree and find some minimal normal subgroup of order four. This is an exercise of [2].

101. Prove that the group $\langle (123\cdots 7), (26)(34)\rangle$ is simple, has order 168 and acts transitively on $\{1,\dots,7\}$.

102. Compute the order of the group $\langle a,b : a^2 = b^2 = (bab^{-1})^3 = 1\rangle$.

103. Prove that $\langle a,b : a^2 = aba^{-1}b = 1\rangle$ is an infinite group.

104. Compute the order of the group $\langle a,b : a^8 = b^2a^4 = ab^{-1}ab = 1\rangle$.

105. Can you recognize the group $\langle a,b : a^5 = 1, b^2 = (ab)^3, (a^3ba^4b)^2 = 1\rangle$?

106. Prove that the group $\langle a,b : a^2 = b^3 = a^{-1}b^{-1}ab = 1\rangle$ is finite and cyclic.

107. Prove that the group $\langle a,b : a^2 = b^3 = 1\rangle$ is non-abelian.

108. Compute the order of the group

$$\langle a,b,c : a^3 = b^3 = c^3 = 1,\, aba = bab,\, cbc = bcb,\, ac = ca\rangle.$$

109. Prove that the group $\langle a,b,c : bab^{-1} = a^2,\, cbc^{-1} = b,\, aca^{-1} = c^2\rangle$ is trivial. This is an exercise of Serre's book [3, §1]:

110. Prove that $B(2,3)$ is isomorphic to the Heisenberg group $H(\mathbb{Z}/3)$.

111. Find a permutation representation of the group $B(2,3)$.

112. Prove that $B(3,3)$ is a finite group of order ≤ 2187.

113. Let G be a finite group with k conjugacy classes. It is known that the probability that two elements of G commute is equal to $\mathrm{prob}(G) = k/|G|$. Compute this probability for $\mathbf{SL}_2(3)$, $\mathbb{A}_4$, $\mathbb{A}_5$, $\mathbb{S}_4$ and Q_8.

2 Some solutions

1 For the first system one obtains that $x = 173$ is the smallest solution such that $x \in \{0, 1, \ldots, 420\}$:

```
gap> Lcm(10,15,84);
420
gap> ChineseRem([10,15,84],[3,8,5]);
173
```

Since the other system does not have solutions, the function `ChineseRem` returns an error message:

```
gap> ChineseRem([52,72],[29,19]);
Error, the residues must be equal modulo 4 called from
<function "ChineseRem">( <arguments> )
 called from read-eval loop at line 149 of *stdin*
you can 'quit;' to quit to outer loop, or
you can 'return;' to continue
brk>
```

2

```
gap> Gcd(42823,6409);
17
gap> GcdRepresentation(5033464705,3137640337);
[ 107535067, -172509882 ]
```

3 The sequence is $3, 27, 87, 87, 87, \ldots$. Here is the code:

```
gap> n := 10;;
gap> s := [3];;
gap> for k in [1..n] do
> Add(s, 3^s[k] mod 100);
> od;
gap> s;
[ 3, 27, 87, 87, 87, 87, 87, 87, 87, 87, 87 ]
```

4

```
gap> Product([2,4..20]);
3715891200
```

5 It is worth noting that the exercise is a particular case of a general formula. However, here is the code to solve this particular exercise:

```
gap> n := 5000;;
gap> Sum(List([1..2*n], j->(-1)^(j+1)*1/j))=\
> Sum(List([n+1..2*n], j->1/j));
true
```

Yet another way to do this, probably less elegant, is

```
gap> s := 0;;
gap> for k in [1..5000]
> do
> s := s-(-1)^k/k;
> od;
gap> t := 0;;
gap> for k in [1..5000]
> do
> t := t+(5000+k)^-1;
> od;
gap> t=s;
true
```

6

```
gap> 3^400;
70550791086553325712464271575934796216507949612787315\
76287122320926208555158293415657929852944713415815495\
23348253559118669297930718245666941450844545352570279\
60285323760313192443283334088001
```

7 There are no solutions of $x^2+x+7\equiv 0 \bmod 15$:

```
gap> List(Filtered(Integers mod 15, x->IsZero(x^2+x+7)), Int);
[  ]
gap> List(Filtered(Integers mod 189, x->IsZero(x^2+x+7)), Int);
[ 13, 49, 76, 112, 139, 175 ]
```

12

```
a:=function(n);
> if n in [0,1,2,3] then
> return 1;
> else
> return (a(n-1)*a(n-3)+a(n-2)^2)/a(n-4);
> fi;
> end;
function( n ) ... end
```

14

```
gap> f:=function(n);
> s:=0;
> for x in [1..n] do
> if IsPrime(x)=true then
> s:=s+1;
> fi;
> od;
> return s;
> end;
function( n ) ... end
```

16

```
gap> f:=function(x)
> local n,k;
> for n in [1..Number(x)] do
> for k in [1..x[n]] do
> Print("X");
> od;
> Print("\n");
> od;
> end;
function( x ) ... end
```

21 To compute the minimal polynomial one uses `MinimalPolynomial`:

```
gap> m := [[0,-1,1],[1,2,-1],[1,1,0]];;
gap> CharacteristicPolynomial(m);
x^3-2*x^2+x
gap> MinimalPolynomial(m);
x^2-x
```

22 The quotient is $-x^2+3x+3$ and the remainder is $3x+3$:

```
gap> x := Indeterminate(GF(5));;
gap> f := 2*x^4+3*x^3+2*x+4;;
gap> g := 3*x^2+x+2;;
gap> QuotientRemainder(f,g);
[ -x_1^2+Z(5)^3*x_1+Z(5)^3, Z(5)^3*x_1+Z(5)^3 ]
```

23 The answer is $-x^2+18x-2$:

```
gap> x := Indeterminate(Rationals);;
gap> (3*x^101-15*x^16-2*x^7-5*x^4+3*x^3+2*x^2+1) mod (x^3+1);
-x^2+18*x-2
```

24

```
gap> x := Indeterminate(GF(5));;
gap> RootsOfPolynomial(x^1000+4*x+1);
[ Z(5) ]
gap> 2*Z(5)^0 = Z(5);
true
```

25

```
gap> x := Indeterminate(Integers mod 5);;
gap> Factors(x^4-1);
[ x_1+Z(5)^0, x_1+Z(5), x_1-Z(5)^0, x_1+Z(5)^3 ]
gap> x := Indeterminate(Integers mod 7);;
gap> Factors(x^4-1);
[ x_1+Z(7)^0, x_1-Z(7)^0, x_1^2+Z(7)^0 ]
```

26 Let us check that all these eighty numbers are primes.

```
gap> Filtered(List([0..79], x->x^2-79*x+1601), IsPrime);
[ 1601, 1523, 1447, 1373, 1301, 1231, 1163, 1097, 1033, 971,
  911, 853, 797, 743, 691, 641, 593, 547, 503, 461, 421, 383,
  347, 313, 281, 251, 223, 197, 173, 151, 131, 113, 97, 83,
  71, 61, 53, 47, 43, 41, 41, 43, 47, 53, 61, 71, 83, 97,
  113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421,
  461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971,
  1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601 ]
gap> Length(last);
80
```

27 Let us compute the first 50 twin primes:

```
gap> l := [];;
gap> n := 2;;
gap> repeat
> if IsPrime(n) and IsPrime(n+2) then
> Add(l, [n,n+2]);
> fi;
> n := n+1;
> until Size(l)=50;
gap> l;
[ [ 3, 5 ], [ 5, 7 ], [ 11, 13 ], [ 17, 19 ], [ 29, 31 ],
  [ 41, 43 ], [ 59, 61 ], [ 71, 73 ], [ 101, 103 ],
  [ 107, 109 ], [ 137, 139 ], [ 149, 151 ], [ 179, 181 ],
  [ 191, 193 ], [ 197, 199 ], [ 227, 229 ], [ 239, 241 ],
  [ 269, 271 ], [ 281, 283 ], [ 311, 313 ], [ 347, 349 ],
  [ 419, 421 ], [ 431, 433 ], [ 461, 463 ], [ 521, 523 ],
  [ 569, 571 ], [ 599, 601 ], [ 617, 619 ], [ 641, 643 ],
  [ 659, 661 ], [ 809, 811 ], [ 821, 823 ], [ 827, 829 ],
  [ 857, 859 ], [ 881, 883 ], [ 1019, 1021 ], [ 1031, 1033 ],
  [ 1049, 1051 ], [ 1061, 1063 ], [ 1091, 1093 ],
  [ 1151, 1153 ], [ 1229, 1231 ], [ 1277, 1279 ],
  [ 1289, 1291 ], [ 1301, 1303 ], [ 1319, 1321 ],
  [ 1427, 1429 ], [ 1451, 1453 ], [ 1481, 1483 ],
  [ 1487, 1489 ] ]
```

28

```
gap> fractran := function(n, lst, bound)
> local i, j, seq;
> seq := [n];
> for i in [1..bound] do
> for j in [1..Size(lst)] do
> if IsInt(lst[j]*n) then
> n := lst[j]*n;
> Add(seq, n);
> break;
> fi;
> od;
> od;
> return seq;
> end;
```

```
function( n, lst, bound ) ... end
gap> code := [17/91,78/85,19/51,23/38,29/33,77/29,95/23,\
77/19,1/17,11/13,13/11,15/2,1/7,55/1];;
gap> fractran(2, code, 10);
[ 2, 15, 825, 725, 1925, 2275, 425, 390, 330, 290, 770 ]
```

41 We know that $\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$. To compute $S \cap T$ we use `Intersection`. Dimensions of vector spaces are computed with `Dimension`:

```
gap> S := VectorSpace(Rationals, [[0,1,0],[0,0,1]]);;
gap> T := VectorSpace(Rationals, [[1,2,0],[3,1,2]]);;
gap> Dimension(S);
2
gap> Dimension(T);
2
gap> Dimension(Intersection(S,T));
1
```

Now $\dim(S+T) = \dim S + \dim T - \dim(S \cap T) = 2+2-1 = 3$.

42 Recall that i is `E(4)`. First we need to create the vector space `V` over the smallest field containing `E(4)`; this is done with `Field(E(4))`. With `Basis` we create the linear basis of `V` and compute the coordinates of `[1,0,1]` with the function `Coefficients`:

```
gap> lst := [[2*E(4),1,0],[2,-E(4),1],[0,1+E(4),1-E(4)]];;
gap> V := VectorSpace(Field(E(4)), lst);;
gap> Coefficients(Basis(V, lst), [1,0,1]);
[ 0, 1/2, 1/4+1/4*E(4) ]
```

51

```
gap> IsomorphicSubgroups(AlternatingGroup(6),SymmetricGroup(5));
[  ]
gap> IsomorphicSubgroups(AlternatingGroup(7),SymmetricGroup(5));
[ [ (1,2,3,4,5), (1,2) ] -> [ (3,4,5,6,7), (1,2)(3,4) ] ]
```

53

```
gap> Number(NormalSubgroups(SL(2,3)), x->Order(x)=8);
1
```

54 One can easily verify that

$$\left\langle \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix} \right\rangle \simeq \mathbf{SL}_2(3).$$

Let us see how is that we found these matrices. One possible approach requires the use of the function `IsomorphicSubgroups`:

```
gap> l := IsomorphicSubgroups(SL(2,5),SL(2,3));;
gap> gr := Image(l[1]);
```

```
Group(
[ [ [ Z(5), Z(5)^2 ], [ Z(5), Z(5) ] ],
  [ [ Z(5), 0*Z(5) ], [ Z(5)^0, Z(5)^3 ] ] ])
gap> Display(gr.1);
 . 2
 2 4
gap> Display(gr.2);
 3 3
 . 2
```

Alternatively, we could try to find all subgroups that are isomorphic to $\mathbf{SL}_2(3)$. To save tome time, we compute conjugacy classes of subgroups instead all possible subgroups. At the end, we see that there is only one conjugacy class of subgroups isomorphic to $\mathbf{SL}_2(3)$ and this conjugacy class has give elements:

```
ap> f := Filtered(ConjugacyClassesSubgroups(SL(2,5)), \\
> x->IdGroup(Representative(x))=IdGroup(SL(2,3)));
[ Group([ [ [ Z(5)^2, 0*Z(5) ], [ 0*Z(5), Z(5)^2 ] ],
      [ [ 0*Z(5), Z(5) ], [ Z(5), 0*Z(5) ] ],
      [ [ 0*Z(5), Z(5)^2 ], [ Z(5)^0, 0*Z(5) ] ],
      [ [ Z(5)^0, Z(5) ], [ Z(5)^0, Z(5)^3 ] ] ])^G ]
gap> Size(f[1]);
5
```

64 It is enough to study the orders of the representatives of conjugacy classes of subgroups.

```
gap> A5 := AlternatingGroup(5);;
gap> c := ConjugacyClassesSubgroups(A5);;
gap> Intersection([8,15,20,24,30,40], \
List(c, x->Size(Representative(x))));
[  ]
```

75

```
gap> A4 := AlternatingGroup(4);;
gap> StructureDescription(AutomorphismGroup(A4));
"S4"
```

97 The group $\mathrm{PSL}_2(7)$ has order 2448, so the following approach will work:

```
gap> 16 in List(MaximalSubgroups(PSL(2,17)), Order);
true
```

Alternatively, one can compute conjugacy classes of maximal subgroups. This approach will be better for groups of bigger order.

```
gap> List(ConjugacyClassesMaximalSubgroups(PSL(2,17)), \\
> x->Order(Representative(x)));
[ 136, 24, 24, 18, 16 ]
gap> 16 in last;
true
```

98

```
gap> ChermakDelgado := function(group, subgroup)
> return Size(subgroup)\
> *Size(Centralizer(group, subgroup));
> end;
function( group, subgroup ) ... end
```

99

```
gap> S3 := SymmetricGroup(3);;
gap> List(AllSubgroups(S3), x->ChermakDelgado(S3, x));
[ 6, 4, 4, 4, 9, 6 ]
gap> D8 := DihedralGroup(8);;
gap> List(AllSubgroups(D8), x->ChermakDelgado(D8, x));
[ 8, 16, 8, 8, 8, 8, 16, 16, 16, 16 ]
```

100 Recall that the holomorph of $\mathbb{A}_4$ is the group $\mathrm{Aut}(\mathbb{A}_4) \ltimes \mathbb{A}_4$.

```
gap> A4 := AlternatingGroup(4);;
gap> hol := SemidirectProduct(AutomorphismGroup(A4),A4);;
gap> f := IsomorphismPermGroup(hol);;
gap> g := SmallerDegreePermutationRepresentation(Image(f));;
gap> G := Image(g);;
gap> Order(G);
288
gap> MovedPoints(G);
[ 1, 2, 3, 4, 5, 6, 7, 8 ]
gap> MinimalNormalSubgroups(G);
[ Group([ (1,2)(3,4), (1,4)(2,3) ]), Group([ (5,7)
  (6,8), (5,8)(6,7) ]) ]
gap> List(last, Order);
[ 4, 4 ]
gap> GeneratorsOfGroup(G);
[ (3,4)(7,8), (2,4,3)(6,8,7), (1,2)(3,4)(5,6)(7,8),
  (1,3)(2,4)(5,7)(6,8), (6,8,7), (5,7)(6,8),
  (5,6)(7,8) ]
```

104

```
gap> G := f/[a^8,b^2*a^4,a*b^-1*a*b];;
gap> Order(G);
16
```

105

```
gap> f := FreeGroup(2);;
gap> a := f.1;;
gap> b := f.2;;
gap> G := f/[a^5, b^2*Inverse((a*b)^3), (a^3*b*a^4*b)^2];;
gap> Order(G);
60
gap> StructureDescription(G);
"A5"
```

108

```
gap> f := FreeGroup(3);;
gap> a := f.1;;
gap> b := f.2;;
gap> c := f.3;;
gap> G := f/[a*b*a*Inverse(b*a*b), \
> a*c*Inverse(c*a), b*c*b*Inverse(c*b*c), a^3, b^3, c^3];;
gap> Order(G);
648
gap> StructureDescription(G);
"(((C3 x C3) : C3) : Q8) : C3"
```

109

```
gap> f := FreeGroup(3);;/D
gap> a := f.1;;
gap> b := f.2;;
gap> c := f.3;;
gap> gr := f/[b*a*b^(-1)*a^(-2), \
> c*b*c^(-1)*b^(-2), a*c*a^(-1)*c^(-2)];;
gap> IsTrivial(gr);
true
```

112

```
gap> f := FreeGroup(3);;
gap> rels := Set(List([1..10000], x->Random(f)^3));;
gap> B33 := f/rels;;
gap> Order(B33);
2187
```

113

```
gap> probability := function(group)
> return Size(ConjugacyClasses(group))/Order(group);
> end;
function( group ) ... end
gap> probability(SL(2,3));
7/24
gap> probability(AlternatingGroup(4));
1/3
gap> probability(AlternatingGroup(5));
1/12
gap> probability(SymmetricGroup(4));
5/24
gap> probability(QuaternionGroup(8));
5/8
```

References

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3. J.-P. Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.