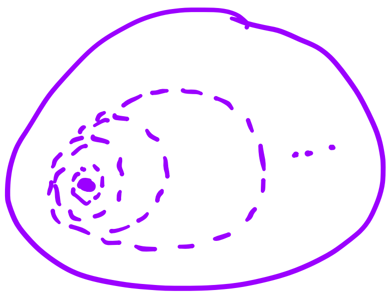
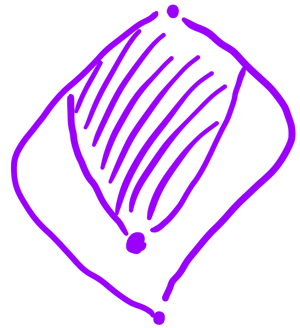
$$\begin{tikzcd} A\otimes U\otimes V \arrow[r] \arrow[d] & A\otimes V \arrow[d] \\ A\otimes U \arrow[r] & A\otimes(U\vee V) \end{tikzcd}$$

Lecture 3:

pointless topology and universal joins





topological

algebraic

A locale / frame is a complete lattice satisfying

$$u \wedge \bigvee v_i = \bigvee u \wedge v_i$$

logically:

complete Heyting algebra:

$$u \wedge v \leq w \iff u \leq v \to w$$

A frame homomorphism is a function satisfying

$$f(u \wedge v) = f(u) \wedge f(v)$$

$$f(\bigvee u_i) = \bigvee f(u_i)$$

The category **Frame** has frames as objects and frame homomorphisms.

The category **Loc** is the opposite.

If X is a topological space, its topology

$$\mathcal{O}(X) = \{U \subseteq X \text{ open}\}$$ is a frame.

If $f : X \longrightarrow Y$ is a continuous function, then

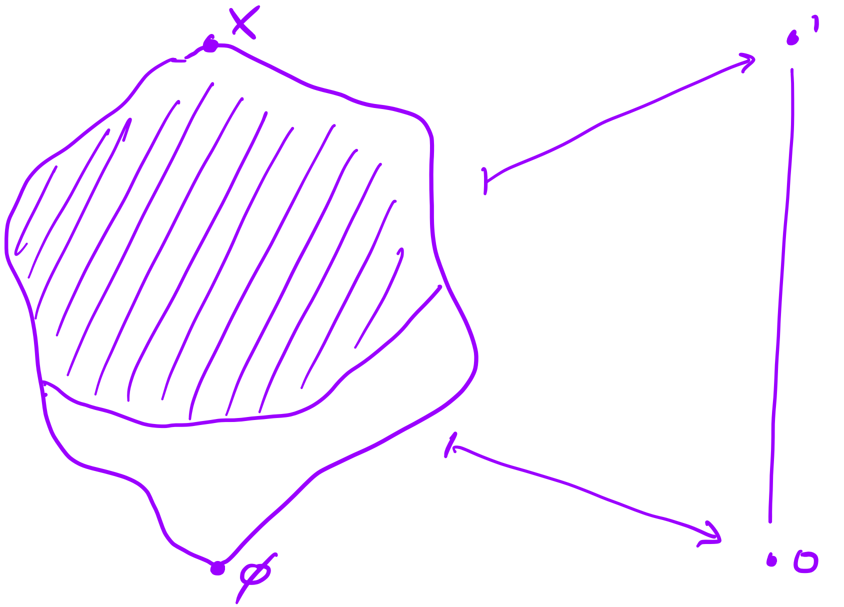
$$\begin{aligned} \mathcal{O}(Y) &\xrightarrow{\mathcal{O}(f)} \mathcal{O}(X) \\ V &\longmapsto f^{-1}(V) = \{x \in X \mid f(x) \in V\} \end{aligned}$$

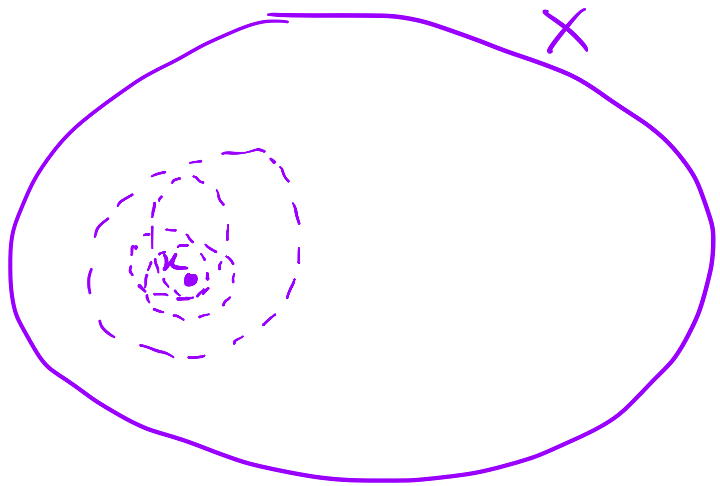
is frame homomorphism.

This gives a functor $\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Loc}$.

What's the point?

- morphism $1 \to X$ of topological spaces
- morphism $\mathcal{O}(1) \to \mathcal{O}(X)$ of locales
- morphism $\mathcal{O}(X) \to \{0,1\}$ of frames





- completely prime filter: subset $P \subseteq \mathcal{O}(X)$ that is:
  - nonempty
  - up-closed: $P \ni u \leq v \Rightarrow v \in P$
  - down-directed: $\forall u,v \in P\ \exists w \leq u \wedge v : w \in P$
  - completely prime: $\bigvee u_i \in P \Rightarrow \exists_i : u_i \in P$

Points of a locale L form a topological space with basis

$$\{P \subseteq L \text{ completely prime filter} \mid u \in P\} \qquad \text{for } u \in L.$$

Called the completely prime spectrum of L.

If $L \xrightarrow{f} M$ is locale map, then

$$\begin{array}{ccc} \mathrm{Spec}(L) & \xrightarrow{\mathrm{Spec}(f)} & \mathrm{Spec}(M) \\ P & \longmapsto & \mathcal{O}(f)^{-1}(P) \end{array}$$

is continuous function.

This gives a functor $\mathbb{L}\mathrm{oc} \xrightarrow{\mathrm{Spec}} \mathbb{T}\mathrm{op}$

(In fact an adjunction $\mathbb{T}\mathrm{op} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\underset{\perp}{\rightleftarrows}}} \mathbb{L}\mathrm{oc}$)

Same for distributive lattice L:

poset $(L, \le)$
with joins $\vee, 0$
and meets $\wedge, 1$
such that $u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w)$

Prime filters

$P \subseteq L$ up-closed
down-directed
prime: $u \vee v \in P \Rightarrow u \in P$ or $v \in P$

form topological space $\mathrm{Spec}(L)$ "prime spectrum"
with basic open sets

$$\{ P \subseteq L \text{ prime filter} \mid u \in P \}$$

for $u \in L$.

these are compact subsets of $\mathrm{Spec}(L)$.

logically:

Locale is sublocal if

$u \vee v = 1 \implies u = 1$ or $v = 1$

disjunction property:

if $\vdash u \vee v$

then $\vdash u$ or $\vdash v$

Locale is local if

$\bigvee u_i = 1 \implies \exists i : u_i = 1$

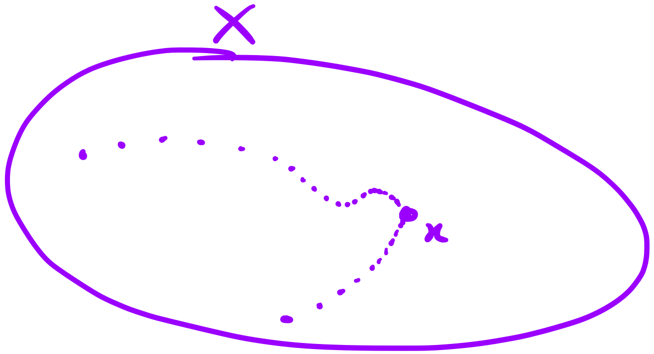
existence property:

if $\vdash \exists i : u_i$

then $\vdash u_i$ for some i

Topological space X has focal point x if

- every net converges to x
- the only open neighbourhood of x is X

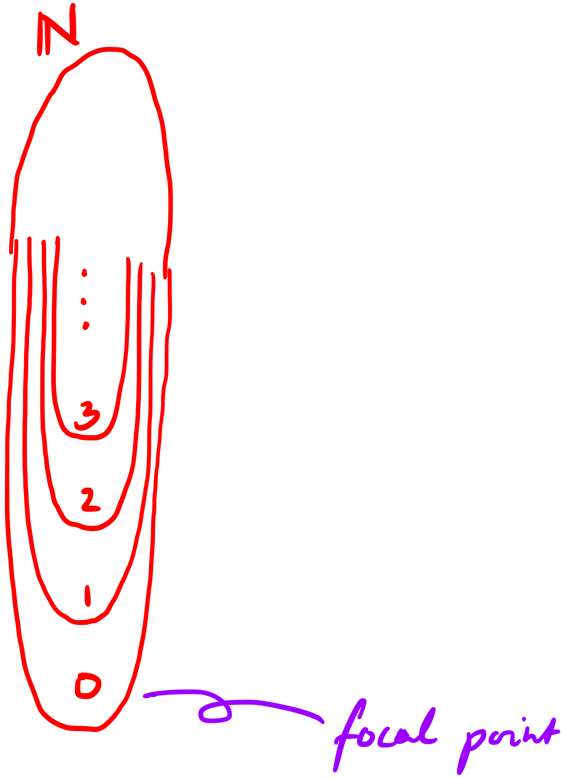


$$\boxed{\begin{array}{c} X \text{ has focal point} \\ \Longleftrightarrow \\ \mathcal{O}(X) \text{ is local} \end{array}}$$

Such spaces are "tiny" (generic)

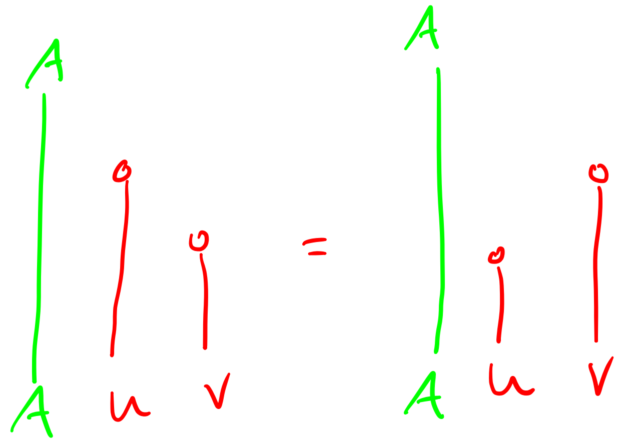
a lot like a one-point space

but don't have to be singleton:



Monoidal category $\mathbb{C}$ is <u>stiff</u> if

$$\begin{tikzcd} A \otimes U \otimes V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes V \arrow[d] \\ A \otimes U \arrow[r] & A \end{tikzcd}$$



is a pullback.

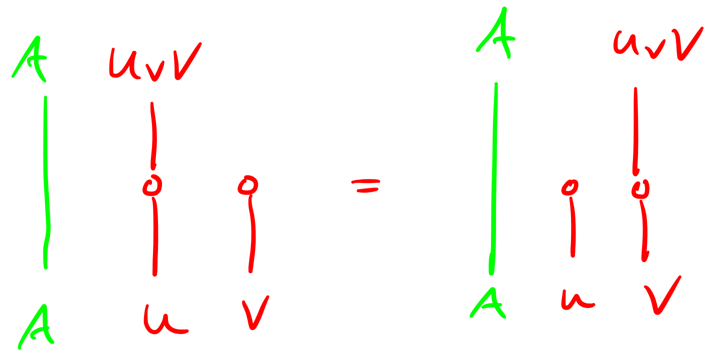
All example categories are stiff

(Don't know category that is not stiff)

Monoidal category $\mathbb{C}$ has <u>universal finite joins</u> if

- it has an initial object 0 with $A \otimes 0 \simeq 0$
- $ZI(\mathbb{C})$ has binary joins such that

$$\begin{tikzcd}
A \otimes U \otimes V \arrow[r] \arrow[d] & A \otimes V \arrow[d] \\
A \otimes U \arrow[r] & A \otimes (U \vee V)
\end{tikzcd}$$



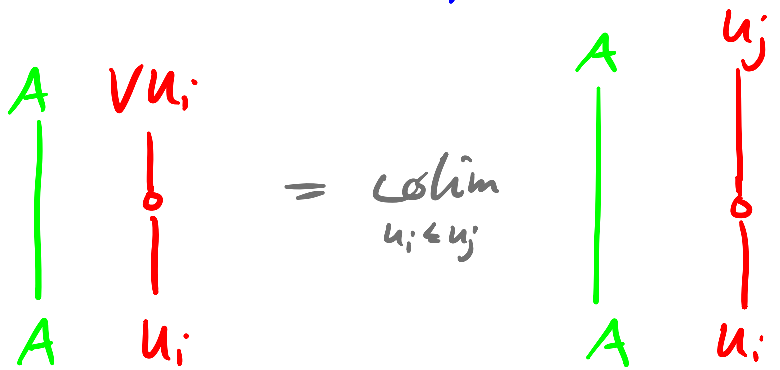
is both pullback and pushout

$$\boxed{\mathbb{C} \text{ has universal finite joins} \implies ZI(\mathbb{C}) \text{ is distributive lattice}}$$

Examples:
- semilattice has univ. fin. joins $\iff$ it is distributive lattice
- $\mathrm{Hilb}_{C_0(X)}$, $\mathbf{Set}$, $(\mathrm{Sh}(X))$ ✓
- Mod_A has univ. fin. joins $\iff$ $ZI(A)$ is distributive lattice

Monoidal category $\mathcal{C}$ has <u>universal joins</u> if

$ZI(\mathcal{C})$ has all joins, and



$$\begin{tikzcd}
A\otimes u_i \arrow[r] \arrow[drrrr, bend right] & A\otimes u_j \arrow[r] \arrow[drrr, bend right] & A\otimes u_k \arrow[r] \arrow[drr, bend right] & \cdots \arrow[r] & A\otimes \vee u_i \arrow[d, dashed] \\
 & & & & B
\end{tikzcd}$$

for any set $\{u_i\}$ of central idempotents with $\{u_i\} = \{u_i \wedge u_j\}$

$$\boxed{\mathcal{C} \text{ has universal joins } \Longrightarrow ZI(\mathcal{C}) \text{ is a frame}}$$

Examples:

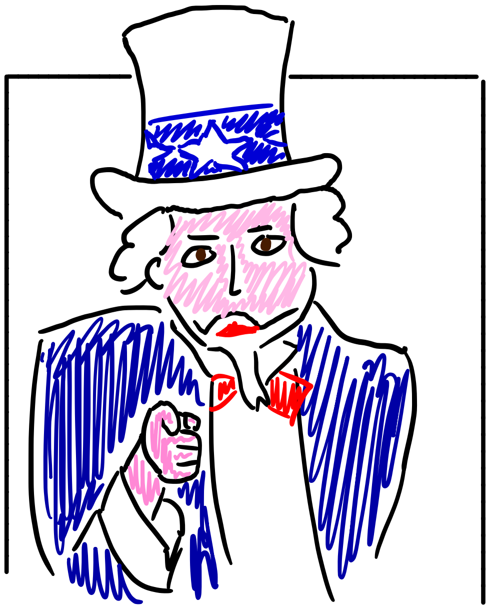
- semilattice has univ. joins $\iff$ it is a frame
- $Hilb_{C_0(X)}$, Set, $(Sh(X))$ ✓
- Mod_A has univ. joins $\iff$ $ZI(A)$ is frame

Lemma: if (finitely many) central idempotents u_i in a monoidal category $\mathbb{C}$ with universal (finite) joins satisfy $\bigvee u_i = 1$ then:

(i) $f = g : A \to B \iff \forall i : f \otimes u_i = g \otimes u_i$

(ii) $f : A \to B$ iso $\iff \forall i : f \otimes u_i$ iso

Pf: (i)



(ii) $\Leftarrow$: say $(f \otimes u_i)^{-1} = g_i : B \otimes u_i \to A \otimes u_i$

$$\begin{tikzcd} B \otimes u_i \arrow[r] \arrow[dr, "g \otimes u_i"'] & B \otimes u_j \arrow[r] \arrow[d, "g \otimes u_j"] & \cdots & B \otimes \bigvee u_i = B \arrow[dll, dashed, "g"] \\ & A & & \end{tikzcd}$$

now $g = f^{-1}$ by uniqueness

If $\mathbb{C}$ has universal joins, then

$$\begin{array}{ccc} \mathbb{C}^{\leq} & \xrightarrow{\text{supp}} & \mathcal{ZI}(\mathbb{C}) \\ f & \longmapsto & \bigvee\{u \mid f \text{ restricts to } v \Rightarrow u \leq v\} \end{array}$$

is universal support:

$$\begin{tikzcd} \mathbb{C}^{\leq} \arrow[r, "\text{supp}"] \arrow[dr, "F"'] & \mathcal{ZI}(\mathbb{C}) \arrow[d, dashed] \\ & L \in \mathbf{Loc} \end{tikzcd}$$

$$\boxed{\begin{aligned} &\text{supp}(f) = \bigwedge\{\text{supp}(A) \mid f \text{ factors through } A\} \\ &\text{supp}(g \circ f) \leq \text{supp}(f) \wedge \text{supp}(g) \\ &\text{supp}(f \otimes g) \leq \text{supp}(f) \wedge \text{supp}(g) \end{aligned}}$$

Functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ sends objects $A \in \mathcal{C}$ to objects $F(A) \in \mathcal{D}$
and $f : A \to B$ in $\mathcal{C}$ to $F(f) : F(A) \longrightarrow F(B)$ in $\mathcal{D}$
such that $F(id_A) = id_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$

$\boxed{\mathbf{Cat}: \text{categories}}$

Monoidal functor: • $\varphi_{A,B} : F(A) \otimes F(B) \longrightarrow F(A \otimes B)$

$\varphi_I : I \longrightarrow F(I)$ iso

natural + coherent:

$\boxed{\mathbf{MonCat}: \text{monoidal categories}}$

$$
\begin{tikzcd}
(F(A)\otimes F(B))\otimes F(C) \arrow[r, "\alpha"] \arrow[d, "\varphi\otimes id"'] & F(A)\otimes(F(B)\otimes F(C)) \arrow[d, "id\otimes\varphi"] \\
F(A\otimes B)\otimes F(C) \arrow[d, "\varphi"'] & F(A)\otimes F(B\otimes C) \arrow[d, "\varphi"] \\
F((A\otimes B)\otimes C) \arrow[r, "F\alpha"'] & F(A\otimes(B\otimes C))
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(A)\otimes I \arrow[r, "id\otimes\varphi"] \arrow[d, "\rho"'] & F(A)\otimes F(I) \arrow[d, "\varphi"] \\
F(A) & F(A\otimes I) \arrow[l, "F\rho"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
I\otimes F(A) \arrow[r, "\varphi\otimes id"] \arrow[d, "\lambda"'] & F(I)\otimes F(A) \arrow[d, "\varphi"] \\
F(A) & F(I\otimes A) \arrow[l, "F\lambda"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(A)\otimes F(B) \arrow[r, "\varphi"] \arrow[d, "Ff\otimes Fg"'] & F(A\otimes B) \arrow[d, "F(f\otimes g)"] \\
F(A')\otimes F(B') \arrow[r, "\varphi"] & F(A'\otimes B')
\end{tikzcd}
$$

$\boxed{\mathbb{M}\mathrm{onCat}_s : \text{stiff monoidal categories}}$

- preserves half-braiding $\sigma_A : U \otimes A \to A \otimes U$ in $\mathcal{C}$ if $F(\sigma_A) \circ \varphi_{U,A}$ extends to half-braiding in $\mathcal{D}$
- $\varphi_{A,U}$ iso

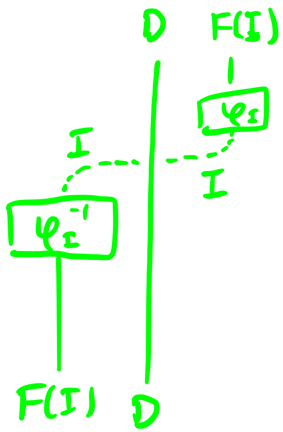
braided monoidal functors do this automatically

$\Longrightarrow$ function $ZI(\mathcal{C}) \longrightarrow ZI(\mathcal{D})$

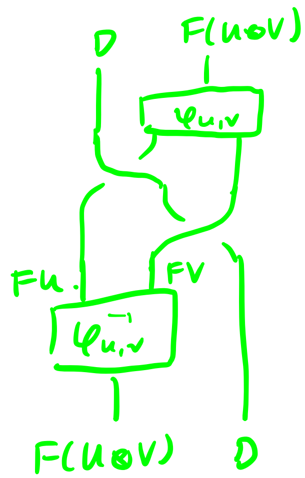
$$(U \xrightarrow{u} I) \longmapsto (F(U) \xrightarrow{F(u)} F(I) \xrightarrow{\varphi_I} I)$$

semilattice morphism if additionally:

$F(\sigma^I)_D =$



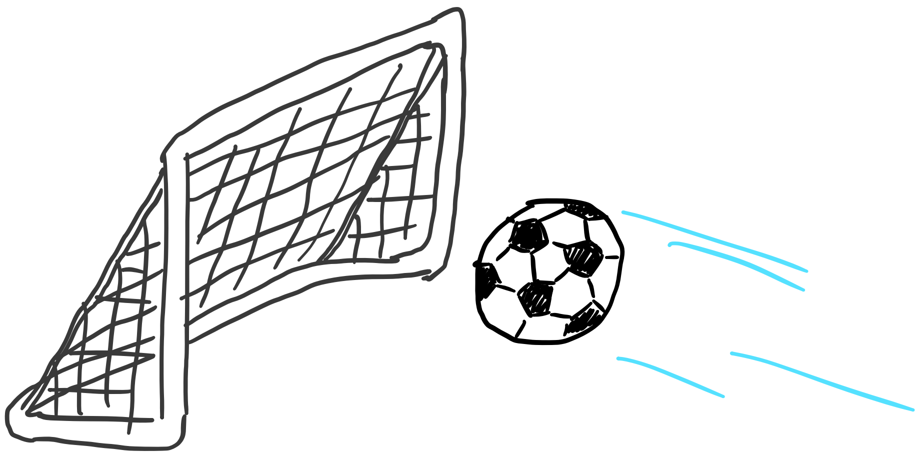
$F(\sigma^{U \otimes V})_D =$



$\boxed{\mathbb{M}\mathrm{onCat}_{(f)j} : \text{monoidal categories w. univ. (fin) joins}}$

$$F(\bigvee u_i) = \bigvee F(u_i)$$

Goal: add universal joins to given $\mathbb{C}$ for free



$$
\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[dr] & D[\mathbb{C}] \arrow[d, dashed] \\
 & \mathbb{D}
\end{tikzcd}
$$

$\mathbb{M}\mathrm{onCat}_s$

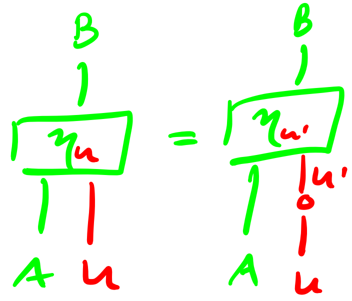
$\mathbb{M}\mathrm{onCat}(f)_s$

$D[\mathbb{C}]$:

- <u>objects</u> are pairs $\langle D, A\rangle$ with $D \subseteq ZI(\mathbb{C})$ down-closed, $A \in \mathbb{C}$

- <u>morphisms</u> $\langle D, A\rangle \longrightarrow \langle E, B\rangle$ are families $\{\eta_u : A \otimes U \to B\}_{u \in D}$ such that η_u restricts to some $v \in E$ and

$$\begin{tikzcd} A \otimes U \arrow[r, "\eta_u"] \arrow[d, dashed] & B \\ B \otimes V \arrow[r, "B \otimes v"'] & B \otimes I \arrow[u, "\lambda_B"'] \end{tikzcd}$$

$u \leq u' \implies$ 

- <u>identity</u> on $\langle D, A\rangle$ has components $A \otimes u : A \otimes U \longrightarrow A$

- <u>composition</u> of $\langle D, A\rangle \xrightarrow{\eta} \langle E, B\rangle \xrightarrow{\vartheta} \langle F, C\rangle$ is

$(\vartheta \circ \eta)_u = \vartheta_v \circ \tilde{\eta}_{u,v}$ for any $v \in E$ and

$$\begin{tikzcd} A \otimes U \arrow[r, "\eta_u"] \arrow[d, dashed, "(\vartheta\circ\eta)_u"] \arrow[dr, "\tilde{\eta}_{u,v}"] & B \\ C & B \otimes V \arrow[l, "\vartheta_v"] \arrow[u, "B \otimes v"'] \end{tikzcd}$$

$D[\mathbb{C}]$ is a well-defined category:

if $v \leq v' \in E$ and η_u restricts to v then

$$\begin{tikzcd}
 & A\otimes u \arrow[rr, "\tilde{\eta}'"] \arrow[dl, "\eta_v"'] \arrow[d, "\tilde{\eta}"] & & B\otimes v' \arrow[d, "f_{v'}"] \\
B & B\otimes v \arrow[l, "B\otimes v"] \arrow[urr] \arrow[rr, "f_v"'] & & C
\end{tikzcd} \qquad (*)$$

if η_u restricts to $v_1, v_2 \in E$, by stiffness it also restricts to $v_1 \wedge v_2$

$$\begin{tikzcd}
B\otimes V_1\otimes V_2 \arrow[rr, "B\otimes V_1\otimes v_2"] \arrow[dd, "B\otimes v_1\otimes V_2"'] & & B\otimes V_1 \arrow[dd, "B\otimes v_1"] \\
 & A\otimes u \arrow[ul] \arrow[ur] \arrow[dl] & \\
B\otimes V_2 \arrow[rr, "B\otimes v_2"'] & & B
\end{tikzcd}$$

apply (*) twice to see $f \circ \eta$ is well-defined.

(associativity and identity laws are routine.)

$D[\mathcal{C}]$ is a monoidal category:

- tensor product of objects is

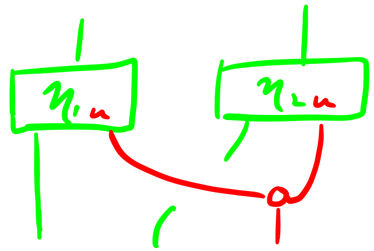
$$\langle D_1, A_1\rangle \otimes \langle D_2, A_2\rangle = \langle D_1 \cap D_2 , A_1 \otimes A_2\rangle$$

- tensor unit is

$$I = \langle Z\mathcal{I}(\mathcal{C}), I \rangle$$

- tensor product of morphisms is:

$$(\eta_1 \otimes \eta_2)_u =$$



There is a (strict) monoidal full embedding

$$\begin{aligned} \mathbb{C} &\longrightarrow D[\mathbb{C}] \\ A &\longmapsto \langle ZI(\mathbb{C}), A \rangle \\ f &\longmapsto \{ f \otimes u \}_{u \in ZI(\mathbb{C})} \end{aligned}$$

Lemma: $ZI(D[\mathcal{C}]) \simeq \{ D \subseteq ZI(\mathcal{C}) \text{ down-closed} \}$

$$\left(\langle D, I \rangle \xrightarrow[u \circ \lambda_I]{} I \right) \longleftarrow\!\shortmid \; D$$

Pf: • $\langle D, I\rangle \otimes \langle D, I \rangle = \langle D, I \otimes I \rangle$ and

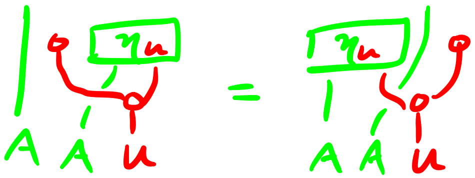
$$\begin{tikzcd} I \otimes u \arrow[r, "I \otimes u"] \arrow[d, "I \otimes u"'] & I \otimes I \arrow[d, "\lambda_I"] \\ I \otimes I \arrow[r, "\rho_I"'] & I \end{tikzcd}$$

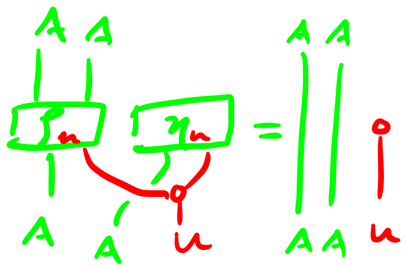
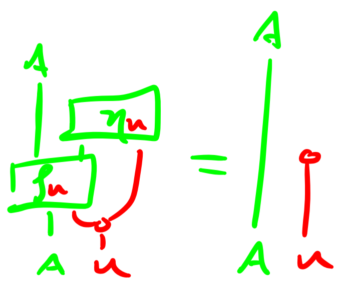
so well-defined.

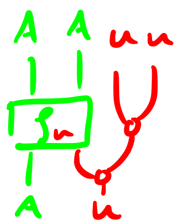
• $\langle D, I \rangle \otimes \langle E, I \rangle \simeq \langle D \cap E, I \rangle$ so semilattice morphism

• if $\langle D, I \rangle \leq \langle E, I \rangle$ then $\forall_{u \in D} \exists_{v \in E} : u \leq v$ so $D \subseteq E$ so injective

• surjective: let $\langle D, A \rangle \xrightarrow{\eta} I$ is central idempotent in $D[\mathcal{C}]$

then  so $\eta \otimes A \otimes u = A \otimes u \otimes \eta$

as $\langle D, A \rangle \otimes \eta$ has inverse ξ, also  

so  $= (A \otimes u \otimes \eta_u)^{-1}$, so $\eta_u \in ZI(\mathcal{C})$

If L is semilattice, then $\{D \subseteq L \text{ down-closed}\}$ is free frame on L

$$\bigvee D_i = \bigcup D_i$$

$$\begin{array}{ccc} L & \hookrightarrow & \{D \subseteq L \text{ down-closed}\} \\ v & \longmapsto & \{u \mid u \leq v\} \end{array}$$

$$\begin{tikzcd}
L \arrow[r, hook] \arrow[dr, "f"'] & \{D \subseteq L \text{ down-closed}\} \arrow[d, dashed] \\
& F
\end{tikzcd}
\qquad
\begin{array}{c} D \\ \downarrow \\ \bigvee\{f(u) \mid u \in D\} \end{array}$$

upgrade:

$\mathbb{C}$ is stiff monoidal

$\Downarrow$

$D[\mathbb{C}]$ has universal joins

can do same for universal finite joins by considering $D \subseteq ZI(\mathbb{C})$ that are finitely generated: $D = \bigcup_{i=1}^{n} \downarrow u_i$

Embedding theorem:

If $\mathbb{C}$ is (small) stiff monoidal category
and $\mathbb{D}$ is a monoidal category with (finite) universal joins
and $F : \mathbb{C} \to \mathbb{D}$ a morphism in $\mathbf{MonCat}_s$
then there is unique morphism $\hat{F} : D[\mathbb{C}] \to \mathbb{D}$ in $\mathbf{MonCat}_j$ ($\mathbf{MonCat}_{fj}$)
with

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook] \arrow[dr, "F"'] & D[\mathbb{C}] \arrow[d, dashed, "\hat{F}"] \\ & \mathbb{D} \end{tikzcd}$$

$$\left(\begin{tikzcd} \mathbf{MonCat}_s \arrow[r, bend left, "D"] \arrow[r, phantom, "\perp"] & \mathbf{MonCat}_{(f)j} \arrow[l, bend left] \end{tikzcd} \right)$$

Summary:

- locales can deal with topology without points
- can reconstruct points with (completely) prime spectrum
- (sub)local locales are as close to one-point space as can be
- stiff/universal (finite) join category has central idempotents forming a semilattice/distributive lattice/locale in a way that respects tensor products
- embedding theorem: can freely add universal (finite) joins to any stiff category

  can pretend stiff cat has univ joins

Next time:

- sheaf representation theorem: any cat w. univ. (fin) joins consists of global sections of cat of sheaf of (sub)local cats.

  can pretend stiff cat is local