Lecture 2:

Central idempotents and support

If $\mathbb{C}$ is monoidal, then so is $\mathbb{C}/I$:

<u>objects</u>: $U \xrightarrow{u} I$

<u>arrows</u>:
$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "u"'] & & V \arrow[dl, "v"] \\ & I & \end{tikzcd}$$

<u>tensor</u>: $U \otimes V \xrightarrow{u \otimes v} I \otimes I \xrightarrow{\rho_I} I$

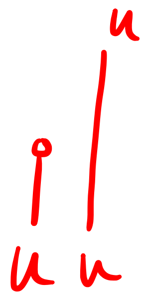
Object $U \xrightarrow{u} I$ may be <u>idempotent</u> in the sense $U \simeq U \otimes U$.

Interested in objects that are idempotent in <u>canonical</u> way.

A morphism $\mathcal{U} \xrightarrow{u} I$ is:

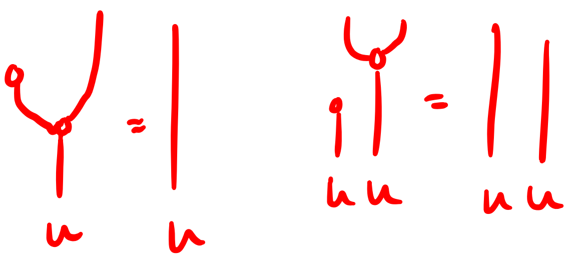
<u>left idempotent</u>

if



$\mathcal{U}\otimes\mathcal{U} \xrightarrow{u\otimes\mathcal{U}} I\otimes\mathcal{U} \xrightarrow{\lambda_{\mathcal{U}}} \mathcal{U}$

is invertible

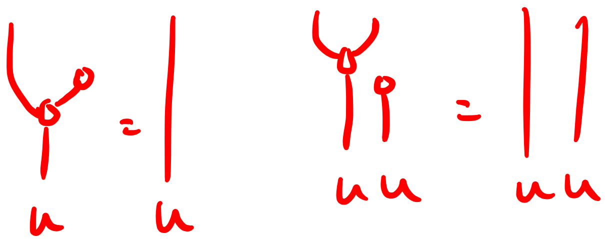


<u>right idempotent</u>

if

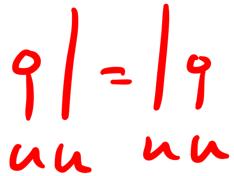
$\mathcal{U}\otimes\mathcal{U} \xrightarrow{\mathcal{U}\otimes u} \mathcal{U}\otimes I \xrightarrow{\rho_{\mathcal{U}}} \mathcal{U}$

is invertible

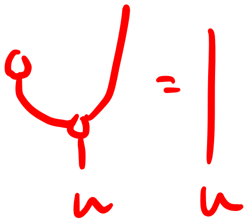


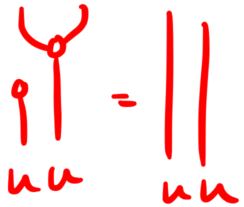
Morphism $U \xrightarrow{u} I$ is a central idempotent if

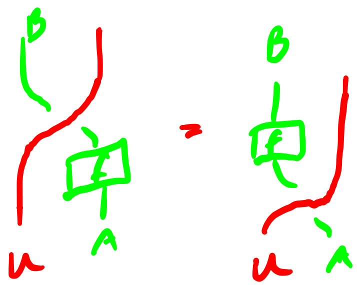
left idempotent in the centre, such that

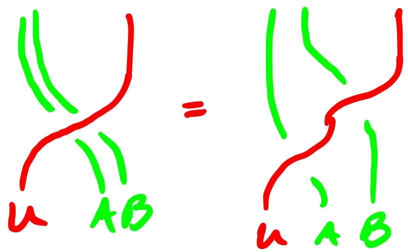


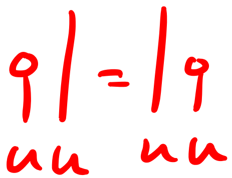
$$\begin{tikzcd}
u \otimes u \arrow[r, "u \otimes u"] \arrow[d, "u \otimes u"'] & I \otimes u \arrow[d, "\lambda_u"] \\
u \otimes I \arrow[r, "\rho_u"'] & u
\end{tikzcd}$$

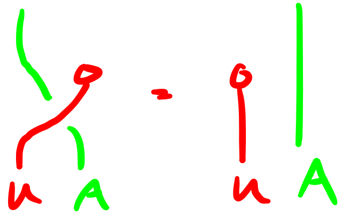












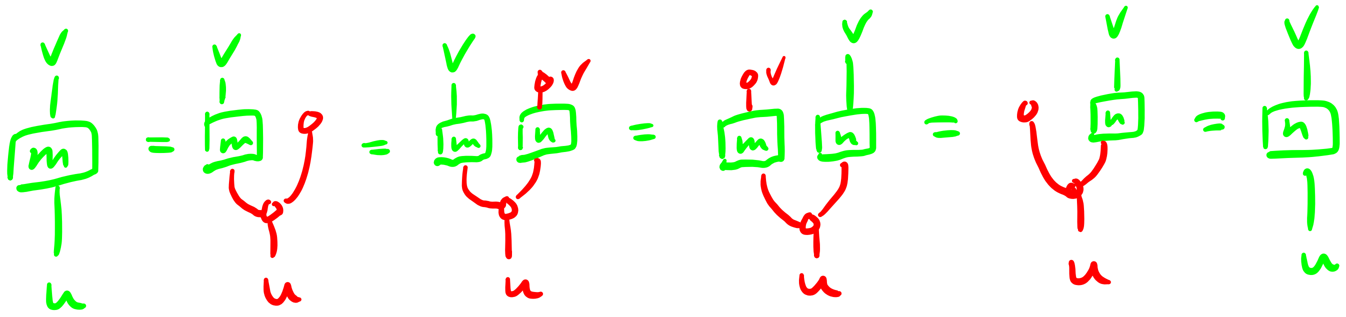
If u, v central idempotents, the following are equivalent:

(1) $u = v \circ m$ for unique m respecting half-braidings

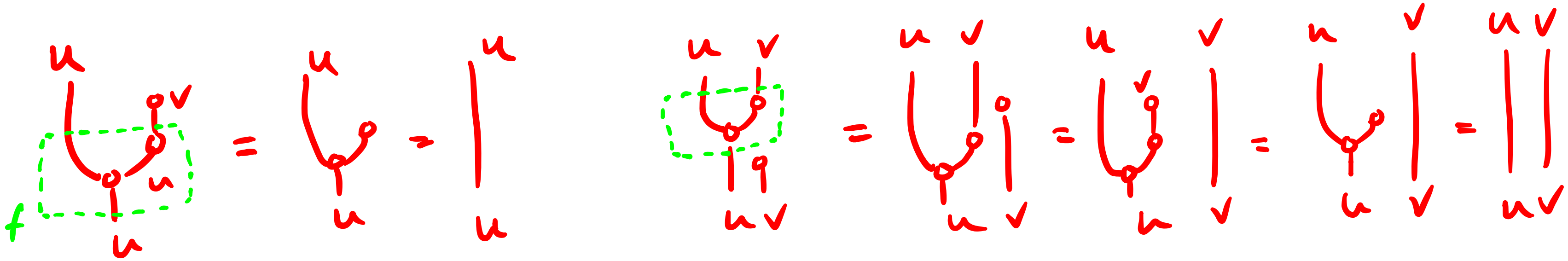
(2) $U \otimes v : U \otimes V \longrightarrow U \otimes I$ invertible

(3) $v \otimes U : V \otimes U \longrightarrow I \otimes U$ invertible

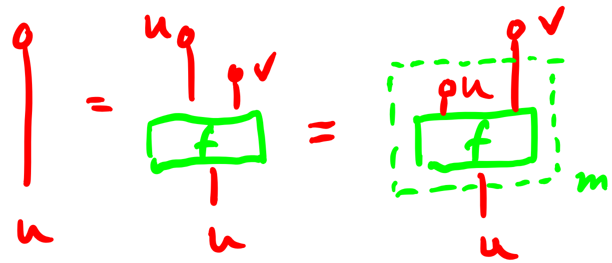
• m unique:



• (1) $\Rightarrow$ (2):



• (2) $\Rightarrow$ (1): if $f = (U \otimes v)^{-1}$ then



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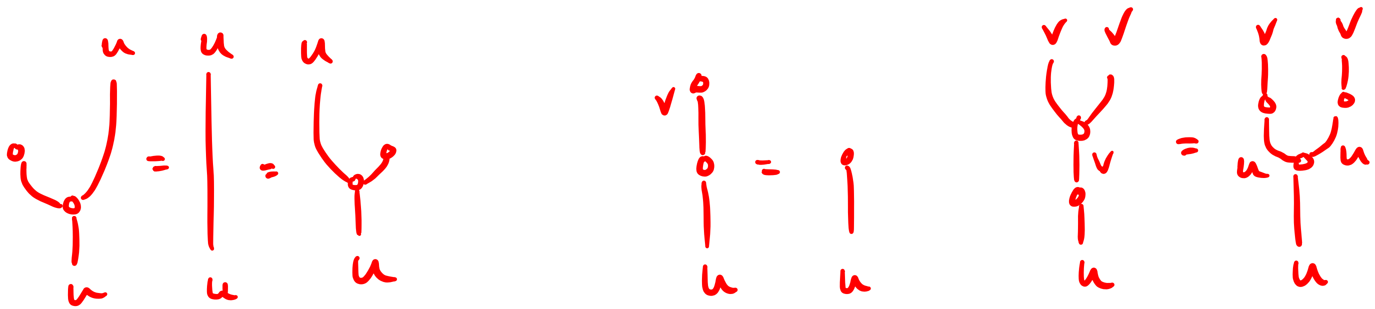
Say $u \leq v$; identify u, v when $u \leq v \leq u$

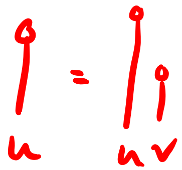
$$\boxed{ZI(\mathbb{C}) = \{\text{central idempotents}\} \text{ partially ordered set}}$$

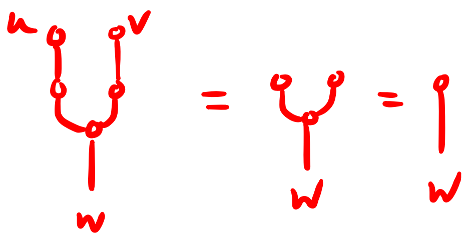


central idempotent $U \xrightarrow{u} I$ is completely determined by its domain U

$ZI(\mathbb{C})$ is a (meet-)semilattice with

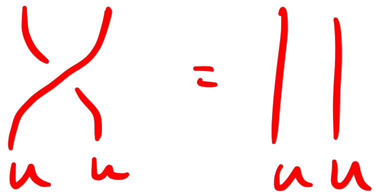
$u \wedge v =$  $\qquad 1 = \mathrm{id}_I$

Pf:  $\Longrightarrow u \wedge v \leq u$

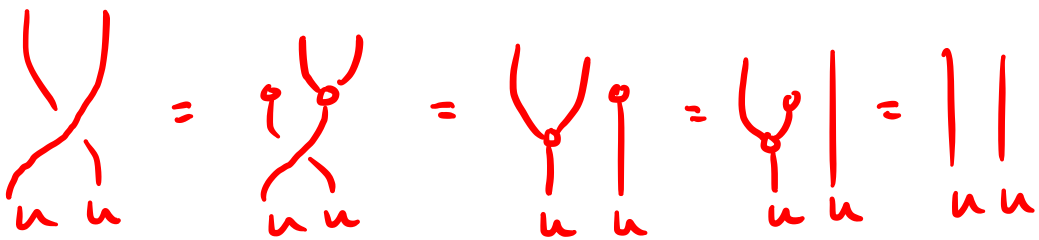
$w \leq u, v \Longrightarrow$  $\Longrightarrow w \leq u \wedge v$

Lemma: a half-braiding σ makes a left idempotent into a central idempotent

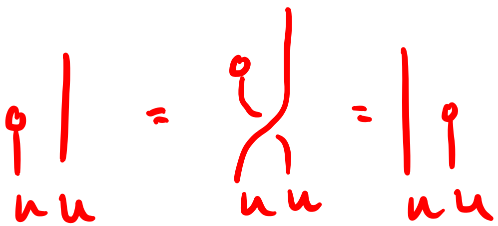
$\Longleftrightarrow$ $\sigma_{u,u} = u \otimes u$



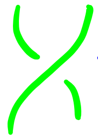
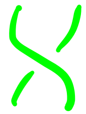
Pf: $\Longrightarrow$:

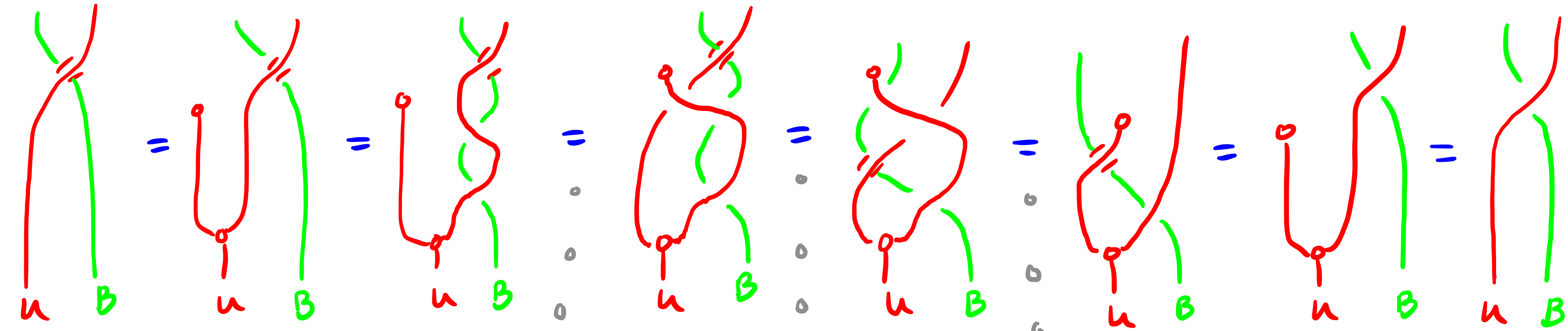


$\Longleftarrow$:



Lemma: If $\mathcal{C}$ is braided, then half-braiding = braiding

Pf:  = half-braiding  = braiding  = inverse braiding

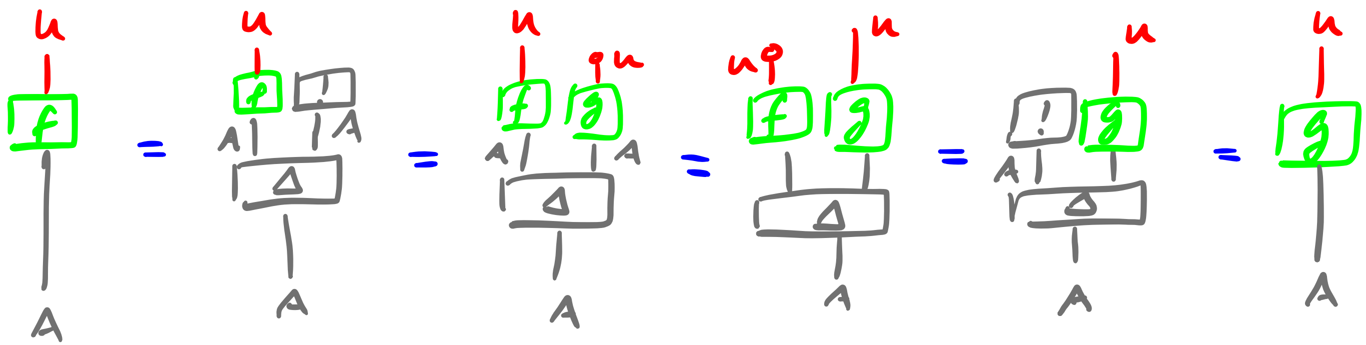


Ex: if $\otimes$ is product: $U \xrightarrow{u} I$ central idempotent

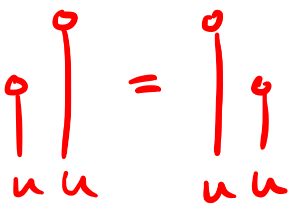
$\Longleftrightarrow$

$U \rightarrowtail 1$ subterminal $\quad \left(A \overset{f}{\underset{g}{\rightrightarrows}} U \Rightarrow f = g \right)$

Pf: $\Rightarrow$:

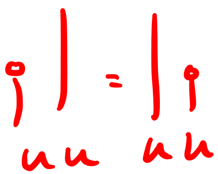


$\Leftarrow$:



and

mono, so



Ex: • $ZI(\mathbf{Set}) = \{\emptyset, 1\}$

(• $ZI(Sh(X)) = \{U \subseteq X \text{ open}\}$ for topological space X)

Ex: any (meet-)semilattice S is a monoidal category

- <u>objects</u>: $s \in S$
- <u>arrow</u> $s \to t$ iff $s \leq t$
- <u>tensor</u>: $s \otimes t = s \wedge t$
- <u>unit</u>: $I = 1$

$$ZI(S) = S$$

$$\begin{tikzcd} \mathbf{SLat} \arrow[loop left, "\mathrm{Id}"] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{MonCat} \arrow[l, shift left, "ZI"] \end{tikzcd}$$

monotone maps that preserve 1

lax monoidal functors with $I \to F(I)$ iso

__Ex__: a __quantale__ is a complete lattice Q with multiplication and unit e

s.t. $u\cdot\left(\bigvee v_i\right) = \bigvee u\cdot v_i \qquad \left(\bigvee u_i\right)\cdot v = \bigvee u_i\cdot v \qquad e\cdot u = u = u\cdot e$

__Ex__:

- $Q = \{R \subseteq X\times X\}$ for set X
- $Q = \{U \subseteq M\}$ for monoid M
- $Q = \{J \subseteq M \text{ ideal}\}$ for monoid M
- $Q = \{U \subseteq X \text{ open}\}$ for topological space X

(• $Q = \{\text{arrows of } G\}$ for étale groupoid G)

$$ZI(Q) = \{ q\in Q \mid q^2 = q \leq e \}$$

$$\left(\begin{tikzcd} \mathrm{Frame} \arrow[loop left, "\mathrm{id}"] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{cQuantale} \arrow[l, shift left, "ZI"] \end{tikzcd}\right)$$

Ex: $\mathbb{M}od_R$ modules over commutative ring R

$A \xrightarrow{a} R$ central: $a(x)\cdot y = a(y)\cdot x$

idempotent: $\forall x\ \exists x_i, y_i:\ x = \sum_{i=1}^{n} a(x_i)\cdot y_i$

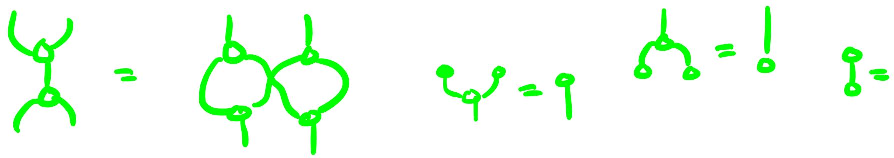
$a(x)\cdot y = 0 \iff x \otimes y = 0$

Ex: • if $e \in ZI(R)$, then $A = eR \xrightarrow{a} R = I$ in $ZI(\mathbb{M}od_R)$

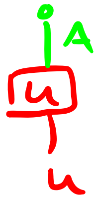
• $a(A) \subseteq R$ always idempotent ideal

• if all A faithfully flat, then $ZI(\mathbb{M}od_R) = ZI(R)$
(e.g. A free module)

Ex: if A bialgebra over field, then $\mathbb{M}\mathrm{od}_A$ is monoidal (not braided)



if $U \xrightarrow{u} A$ central idempotent in $\mathbb{M}\mathrm{od}_A$,

then  central idempotent in $\mathbb{V}\mathrm{ect}$,

so

$$ZI(\mathbb{M}\mathrm{od}_A) = ZI(A) = \{ u \in Z(A) \mid u = u^2 \}$$

- commutative quantale A is bialgebra in category of complete lattices.

e.g. $A = ([0,\infty], +, \geq) \Rightarrow ZI(A) = \left(\begin{array}{c} [0,\infty] \\ | \\ (0,\infty] \\ | \\ \{0,\infty\} \\ | \\ \{\infty\} \\ | \\ \emptyset \end{array} \right)$

Ex: if X is locally compact Hausdorff space,

$C_0(X) = \{ f: X \to \mathbb{C} \text{ continuous} \mid \forall \varepsilon > 0 \ \exists K \subseteq X \text{ compact } \forall x \notin K: |f(x)| < \varepsilon \}$

is C^*-algebra. Hilbert module over $C_0(X)$ is

$C_0(X)$-module A with $\langle - | - \rangle : A \times A \to C_0(X)$ complete.

($\mathrm{Hilb}_{C_0(X)} \simeq$ continuous field of Hilbert spaces

$\simeq \mathrm{Hilb}(\mathrm{Sh}(X))$

$$\begin{array}{l} E \\ \downarrow p \\ X \end{array} \quad \text{s.t. } p^{-1}(x) \in \mathrm{Hilb}, \quad \langle - | - \rangle : E \times_X E \to \mathbb{C} \text{ cts.}$$)

$$\textcolor{purple}{ZI}(\mathrm{Hilb}_{C_0(X)}) \simeq \{ \underbrace{C_0(U)}_{\{ f \in C_0(X) \mid f(X \setminus U) = 0 \}} \rightarrowtail C_0(X) \mid U \subseteq X \text{ open} \} \simeq \{ U \subseteq X \text{ open} \}$$

(but $\mathrm{Hilb}_{C_0(X)}$ not abelian or topos)

Ex: if $\mathcal{C}$ is category, $[\mathcal{C}, \mathcal{C}]$ is monoidal (not braided)

idempotent = idempotent comonad T on $\mathcal{C}$

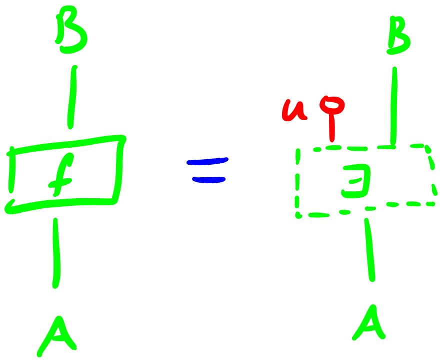
$(\delta : T \Rightarrow T^2 \text{ iso})$

central $\Leftrightarrow$ $T = \mathrm{Id}$

category	central idempotents
cartesian cat	subterminal objs
$Sh(X)$	opens of X
semilattice	everything
quantale	largest subframe $\{q^2=q\leq 1\}$
Mod_R — comm. ring (nonunital)	idempotent ideals $I^2=I\subseteq R$ (+...?)
Mod_A — bialgebra over field	central idemp. elts. of A
Hilbert modules over $C_0(X)$	opens of X
endofunctors $C \to C$	trivial

A morphism $A \xrightarrow{f} B$ <u>restricts</u> to central idempotent $U \xrightarrow{u} I$ if

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, dashed] & & B \\
U \otimes B \arrow[r, "u \otimes B"'] & I \otimes B \arrow[ur, "\lambda_B"'] &
\end{tikzcd}$$

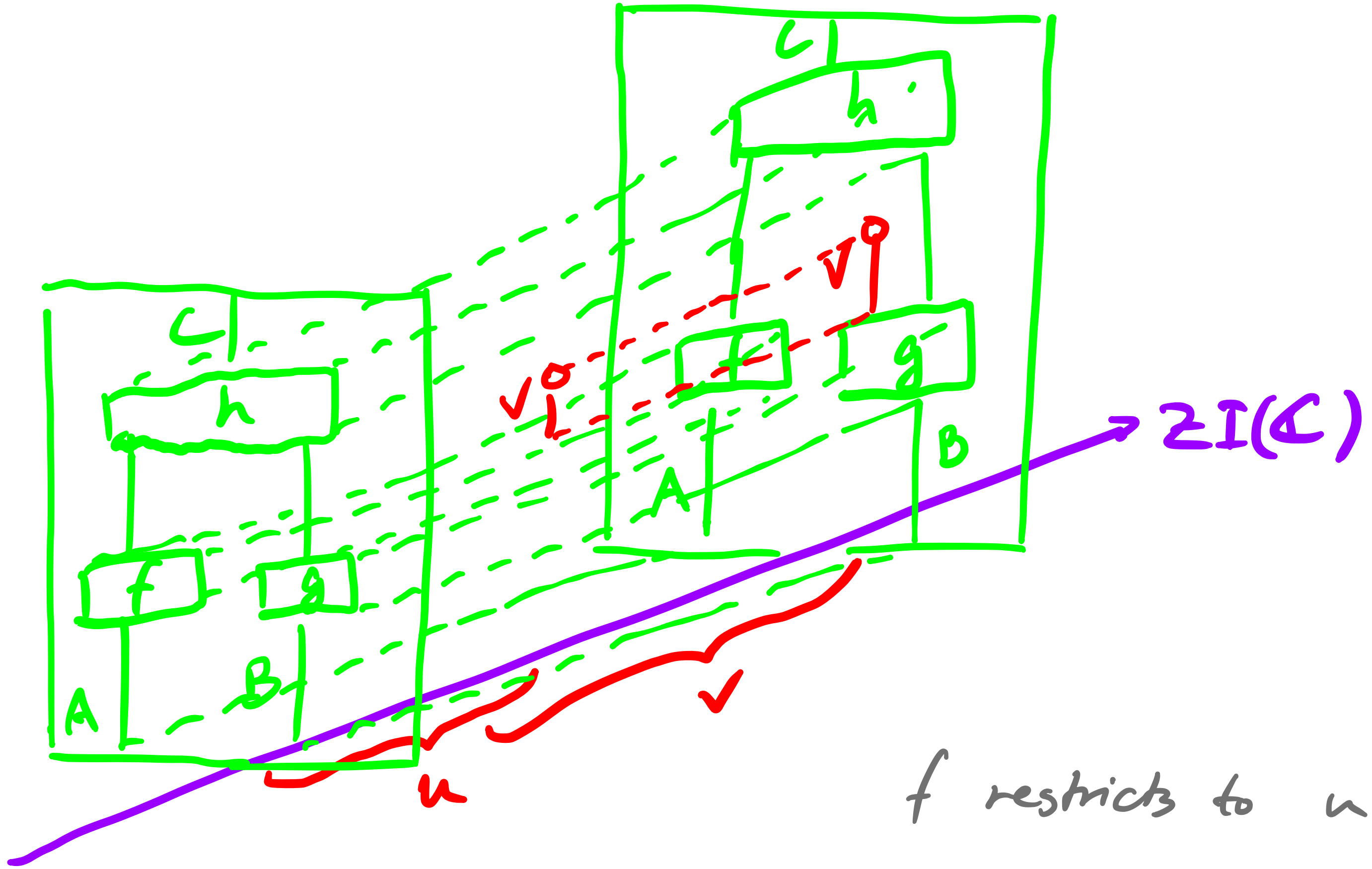


e.g.:

- in semilattice: $a \leq b$ restricts to $u \leq 1$ if $a \leq u \wedge b$
- in Mod_R: f restricts to eR if $e \cdot f = f$
- in $\mathrm{Hilb}_{C_0(X)}$: $A \xrightarrow{f} B$ restricts to $C_0(U)$ if $\forall a \in A\ \forall x \notin U$: $\langle f(a) | f(a) \rangle (x) = 0$

(• in $\mathrm{Sh}(X)$: $F \overset{\alpha}{\Rightarrow} G$ restricts to $U \subseteq X$ if $\alpha_V = \emptyset$ for $U \cap V \neq \emptyset$)

Extra dimension in graphical calculus?



f restricts to u

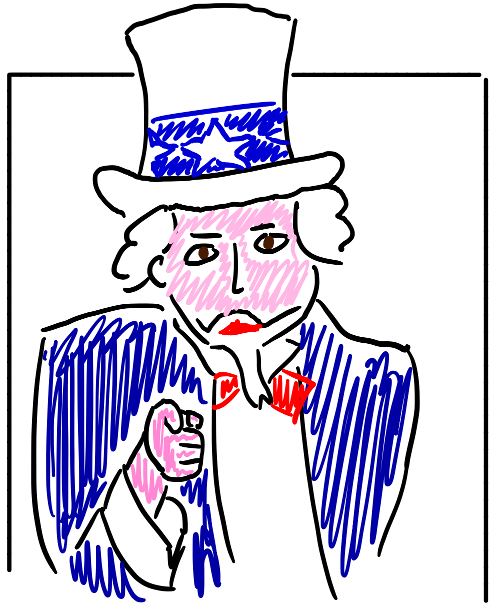
$\Longleftrightarrow$

"f only lives above u"

Lemma:

$\left[\begin{array}{ll} f \text{ restricts to } u \\ g \text{ restricts to } v \end{array}\right] \Longrightarrow f \circ g \text{ and } f \otimes g \text{ restrict to } u \wedge v$

Pf:



Lemma: $\left[\begin{array}{ll} f & \text{restricts to } u \\ g & \text{restricts to } v \end{array}\right] \Longrightarrow$ $f \circ g$ and $f \otimes g$ restrict to $u \wedge v$

Pf:

$$\begin{tikzcd}
A \otimes A' \arrow[rr, "f \otimes g"] \arrow[d, dashed] & & B \otimes B' \\
U \otimes B \otimes V \otimes B' \arrow[rr, "u \otimes B \otimes v \otimes B'"'] \arrow[d, "\simeq"] & & I \otimes B \otimes I \otimes B' \arrow[u, "\simeq"] \\
(U \otimes V) \otimes (B \otimes B') \arrow[rr, "(u \wedge v) \otimes B \otimes B'"'] & & I \otimes B \otimes B' \arrow[u, "="]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, dashed, "m"] & & B \arrow[rr, "g"] \arrow[d, dashed, "n"] & & C \\
U \otimes B \arrow[r, "u \otimes B"] \arrow[dr, "U \otimes n"'] & I \otimes B \arrow[ur, "\lambda_B"] & V \otimes C \arrow[rr, "v \otimes C"] & & I \otimes C \arrow[u, "\lambda_C"'] \\
& U \otimes V \otimes C \arrow[urrr, "(u \wedge v) \otimes C"'] & & &
\end{tikzcd}$$

Lemma: the following are equivalent:

(1) $U\otimes A \xrightarrow{u\otimes A} I\otimes A$ iso

(2) $U\otimes A \simeq A$

(3) $U\otimes B \simeq A$ for some B

(4) $A \xrightarrow{id} A$ restricts to U

"A only lives above u"

Pf: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) ✓

(3) $\Rightarrow$ (4): $A \simeq U\otimes B \simeq U\otimes U\otimes B = U\otimes A$ $\Longrightarrow$

$$\begin{tikzcd} A \arrow[loop left, "id"] \arrow[r, shift left, "m"] & U\otimes A \arrow[l, shift left, "e"] \end{tikzcd}$$

$$\begin{tikzcd} U\otimes A \arrow[r, "U\otimes m"] \arrow[d, "u\otimes A"] & U\otimes U\otimes A \arrow[r, "U\otimes e"] \arrow[d, "u\otimes U\otimes A", "\simeq"'] & U\otimes A \arrow[d, "u\otimes A"] \\ I\otimes A \arrow[r, "I\otimes m"'] \arrow[rr, bend right, "id"'] \arrow[urr, dashed] & I\otimes U\otimes A \arrow[r, "I\otimes e"'] & I\otimes A \end{tikzcd}$$

((4) $\Rightarrow$ (1): similar)

Support datum on $\mathbb{C}$ is $F : \mathbb{C}^{\leq} \longrightarrow L$

obj: f in $\mathbb{C}$

arr: $f \to g$ iff

g restricts to $u \Rightarrow f$ restricts to u

complete lattice

such that $\quad F(f) = \bigwedge \{ F(u) : u \in ZI(\mathbb{C}) \mid f \text{ restricts to } u \}$

Lemma:

- $F(f) = \bigwedge \{ F(A) \mid f \text{ factors through } A \}$ — support of maps determined by support of objects
- $F(f \otimes g) \leq F(f) \wedge F(g)$

Any $\mathbb{C}$ has canonical support datum

$$\begin{array}{ccc} \mathbb{C}^{\leq} & \xrightarrow{\text{supp}} & D(\mathcal{Z}\mathcal{I}(\mathbb{C})) \\ f & \longmapsto & \{u \mid f \text{ restricts to } v \implies u \leq v\} \end{array}$$

down-closed subsets S of $\mathcal{Z}\mathcal{I}(\mathbb{C})$
$(u \leq v \in S \implies u \in S)$
= free complete lattice on $\mathcal{Z}\mathcal{I}(\mathbb{C})$

Initial among support data:

$$\begin{tikzcd} \mathbb{C}^{\leq} \arrow[r, "\text{supp}"] \arrow[dr, "F"'] & D(\mathcal{Z}\mathcal{I}(\mathbb{C})) \arrow[d, dashed] & \{u_i\} \arrow[d, mapsto] \\ & L & \bigvee F(u_i) \end{tikzcd}$$

(Next time: the support as a single central idempotent)

Summary:

- any monoidal category has a (meet-)semilattice of central idempotents
- they behave like open sets of a hidden base space
- in many examples they are open sets of a base space
- support says "where" a morphism/object lives

Next time:

- unions of central idempotents