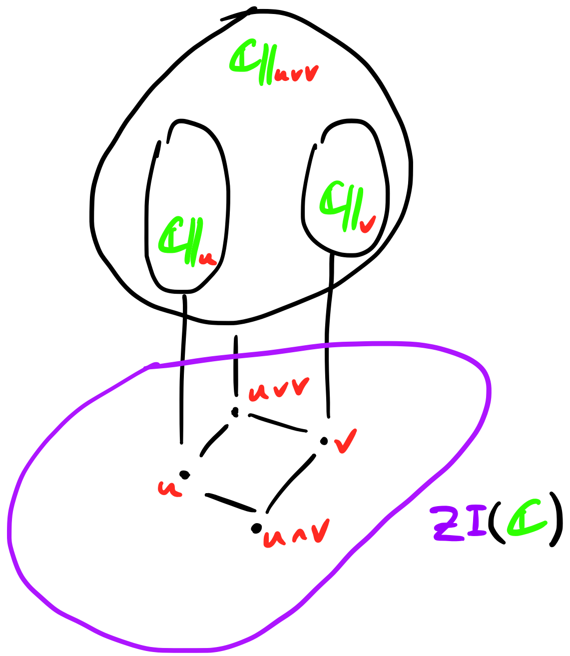
Tensor topology

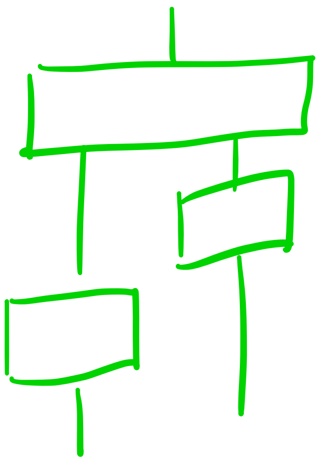
Chris Heunen
<chris.heunen@ed.ac.uk>

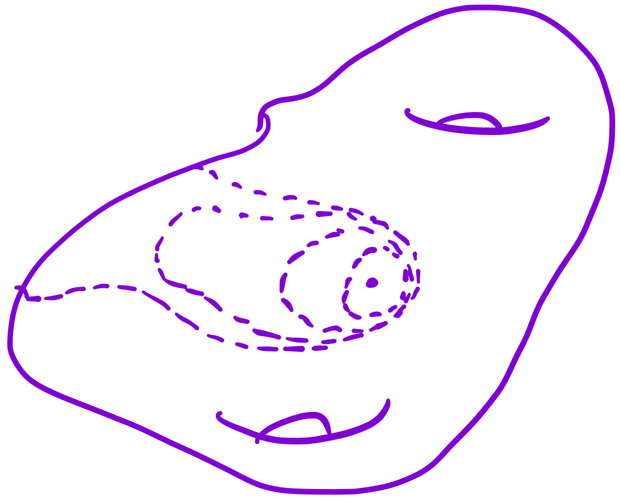


tensor categories ⟷ **topology**

how to combine processes
(in series and in parallel)

where processes take place
(in time and in space)



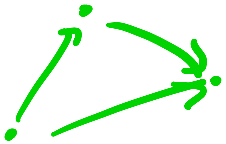


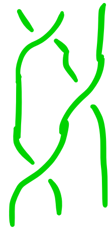
e.g.:
- concurrent computing
- quantum theory
- temporal logic

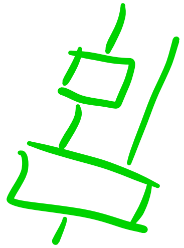
Lecture 1:

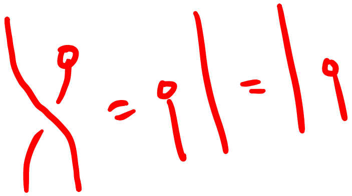
- monoidal categories
- graphical calculus
- the (Drinfeld) centre

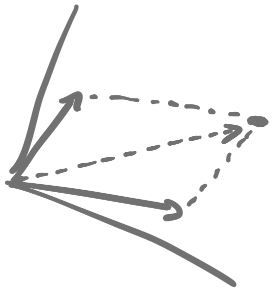
$(\mathcal{C}, \otimes, I)$









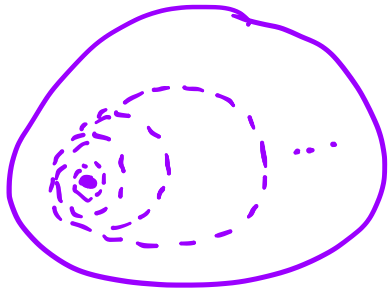


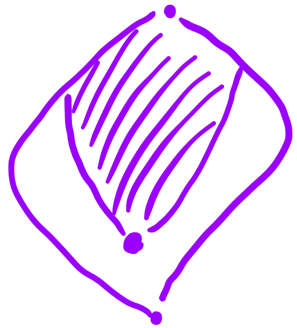
Lecture 2:

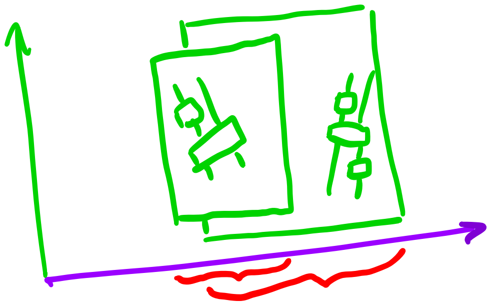
- central idempotents
- examples
- support

Lecture 3:

- pointless topology
- universal joins







Lecture 4:

- sheaves
- decomposition theorem

(**Bonus:**

- localisable monads
- restriction categories)

Literature

"Categories for quantum theory", C. Heunen, J. Vicary, [OUP, 2019]

"Sheaf representation of monoidal categories", R. Barbosa, C. Heunen [arXiv: 2106.08896]

"Tensor topology", P. Enrique Moliner, C. Heunen, S. Tull [arXiv: 1810.01383]

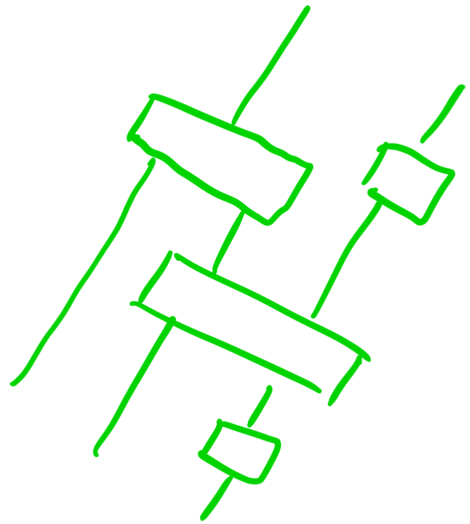
"Space in monoidal categories", P. Enrique Moliner, C. Heunen, S. Tull [arXiv: 1704.08086]

"Tensor-restriction categories", C. Heunen, J.-S. Pacaud Lemay [arXiv: 2009.12432]

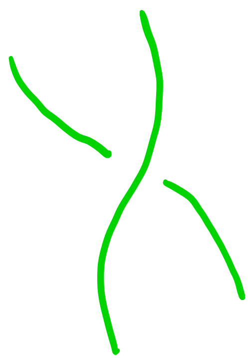
"Localisable monads", C. Constantin, N. Dicaire, C. Heunen [arXiv: 2108.01756]

Lecture 1:

Monoidal categories and centres



$(\mathcal{C}, \otimes, I)$



A **category** $\mathbb{C}$ is:

- a set of **objects** $A, B, C, \dots$
- a set of **morphisms** $\mathbb{C}(A,B) \ni f, g, h, \dots$
  for any objects A, B

$$A \xrightarrow{f} B$$

- an **identity** morphism
  for any object A

$$A \xrightarrow{id_A = A} A$$

- a **composition** $g \circ f \in \mathbb{C}(A,C)$
  for any morphisms $f \in \mathbb{C}(A,B)$ and $g \in \mathbb{C}(B,C)$

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$$

such that:

- $B \circ f = f = f \circ A$
- $(h \circ g) \circ f = h \circ (g \circ f)$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

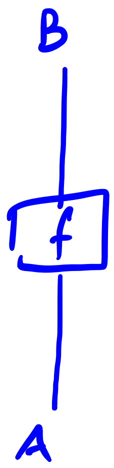
$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, "f"] & B \arrow[loop right, "id_B"] \end{tikzcd}$$

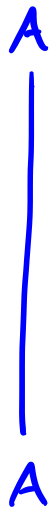
e.g.:

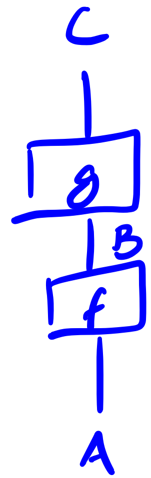
- $\mathbf{Set}$: sets and functions
- $\mathbf{Set}^2$: pairs of sets and functions
- $\mathbf{Vect}$: vector spaces and linear functions
- $\mathbf{Mod}_R$: modules over a commutative ring R and linear functions
- any partially ordered set $(P, \leq)$
  (any locale)

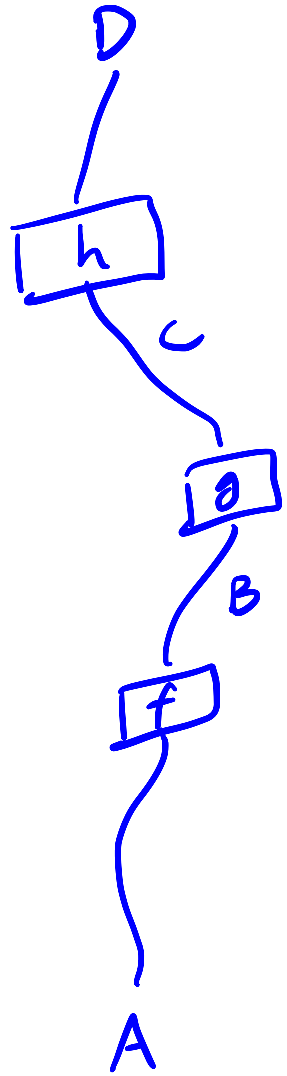
(
- $Sh(X)$: sheaves over a topological space X
- $Hilb_{C_0(X)}$: Hilbert modules over a commutative C^*-algebra
- free categories
)

Graphical calculus in 1 dimension









$A \xrightarrow{f} B$ $\qquad$ $A \xrightarrow{id_A} A$ $\qquad$ $A \xrightarrow{f} B \xrightarrow{g} C$

Monoidal category:

- an object $A \otimes B$ for any objects A, B
- an object I
- a morphism $A \otimes A' \xrightarrow{f \otimes f'} B \otimes B'$ for any morphisms $A \xrightarrow{f} B$ and $A' \xrightarrow{f'} B'$
- isomorphisms $A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C$ ("associator")

  $I \otimes A \xrightarrow{\lambda_A} A$ ("left unitor")

  $A \otimes I \xrightarrow{\rho_A} A$ ("right unitor")

such that:

"bifunctorial"

$$\begin{tikzcd} A \otimes A' \arrow[r, "f \otimes f'"] \arrow[dr, "(g \circ f) \otimes (g' \circ f')"'] & B \otimes B' \arrow[d, "g \otimes g'"] \\ & C \otimes C' \end{tikzcd}$$

$$\begin{tikzcd} A \otimes (B \otimes C) \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "f \otimes (g \otimes h)"] & (A \otimes B) \otimes C \arrow[d, "(f \otimes g) \otimes h"] \\ A' \otimes (B' \otimes C') \arrow[r, "\alpha_{A',B',C'}"'] & (A' \otimes B') \otimes C' \end{tikzcd}$$

"natural"

$$\begin{tikzcd} I \otimes A \arrow[r, "\lambda_A"] \arrow[d, "I \otimes f"] & A \arrow[d, "f"] \\ I \otimes B \arrow[r, "\lambda_B"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \otimes I \arrow[r, "\rho_A"] \arrow[d, "f \otimes I"] & A \arrow[d, "f"] \\ B \otimes I \arrow[r, "\rho_B"'] & B \end{tikzcd}$$

and coherence...

"triangle equations"

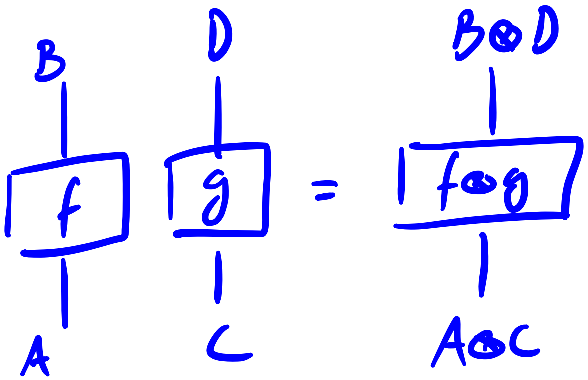
$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes B"] \\
& A \otimes B &
\end{tikzcd}$$

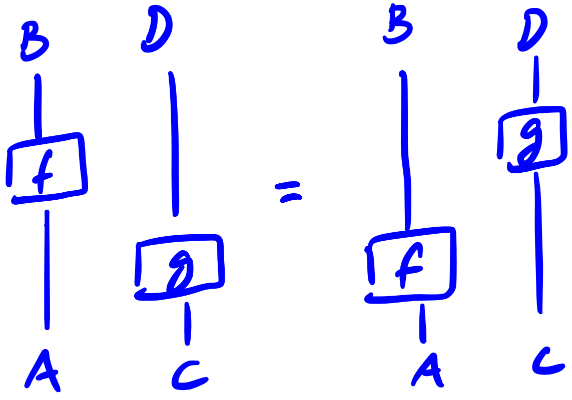
"pentagon equations"

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A \otimes B, C, D}"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "\alpha_{A,B,C \otimes D}"] \arrow[d, "A \otimes \alpha_{BCD}"'] & & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A, B \otimes C, D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{ABC} \otimes D"']
\end{tikzcd}$$

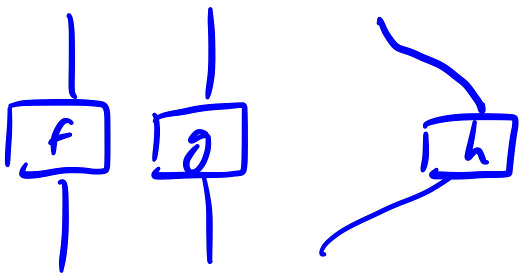
<u>Coherence thm</u>: any well-typed diagram built from α, λ, ρ, id, using $\otimes$, $\circ$, commutes

Graphical calculus in 2 dimensions:



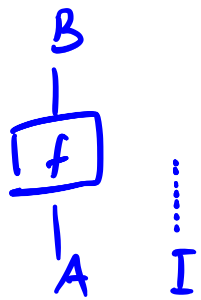


$$(f\otimes D)\circ(A\otimes g) = (B\otimes g)\circ(f\otimes C)$$



$$(f\otimes g)\otimes h =$$

$$\alpha\circ(f\otimes(g\otimes h))$$

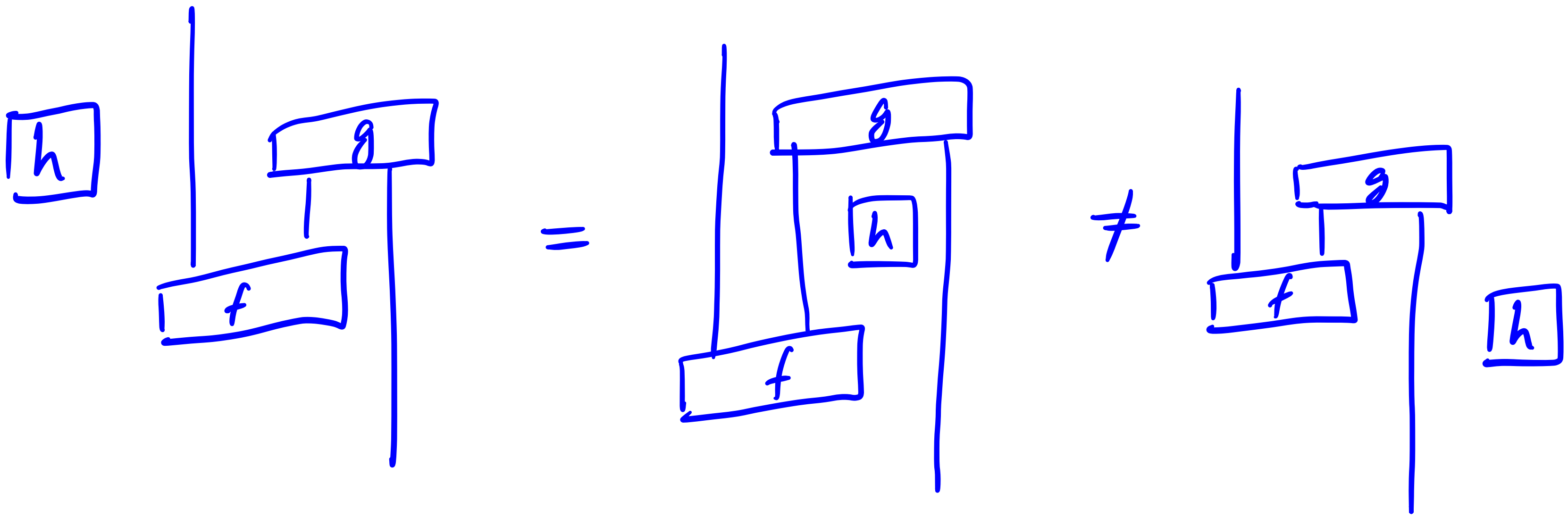


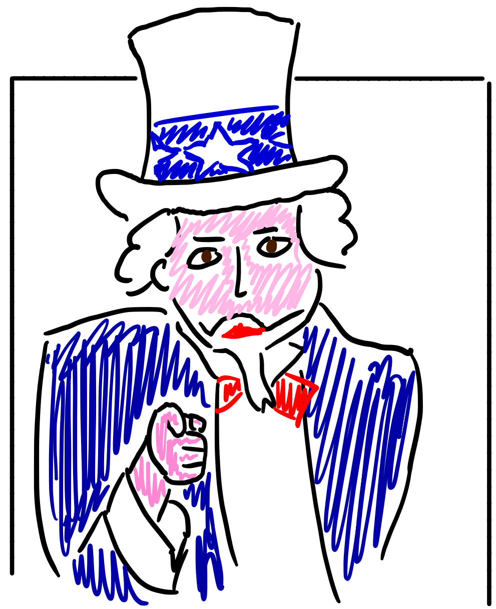
$$\lambda\circ(f\otimes I)$$

Two morphisms are provably equal according to the axioms of monoidal categories

"completeness" $\Uparrow \quad \Downarrow$ "soundness"

Two graphical diagrams can be deformed into each other using planar isotopy


h
g
f
=
g
h
f
≠
g
f
h



e.g.: • $\mathbf{Set}$, $\mathbf{Set}^2$: cartesian product of sets

$$A \otimes B = A \times B$$

$$I = 1$$

$$f \otimes g : (a, b) \longmapsto (f(a), g(b))$$

• $\mathbf{Vect}$, $\mathbf{Mod}_R$: tensor product of modules

$$(a,b) \longmapsto a \otimes b$$

$$\begin{tikzcd}
A \times B \arrow[r] \arrow[dr, "\text{bilinear}"'] & A \otimes B \arrow[d, dashed, "\text{linear}"] \\
& C
\end{tikzcd}$$

$$I = R$$

$$f \otimes g : \sum_i a_i \otimes b_i \longmapsto \sum_i f(a_i) \otimes g(b_i)$$

products:

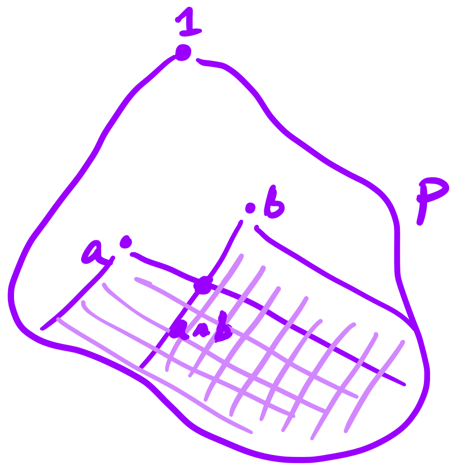
$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dashed, "\langle f, g\rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p"] \arrow[r, "q"'] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, dashed, "!_A"] \\ 1 \end{tikzcd}$$

"terminal object"

e.g. (meet-) semilattice $(P, \leq)$:

$$\frac{c \leq a \qquad c \leq b}{c \leq a \wedge b} \qquad\qquad \frac{}{a \leq 1}$$

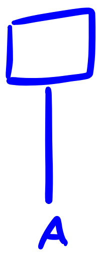


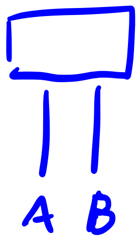
e.g. - security clearance

- memory permission

- regions in space(time)

"effect"



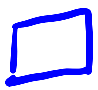


$A \longrightarrow I$

$A \otimes B \longrightarrow I$

- if I terminal unique
- in Mod_R effect = functional

("scalar"



$I \longrightarrow I$

- form commutative monoid
- in Mod_R scalar = homomorphism $R \to R$)

Braiding is isomorphisms $A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$

such that

$$
\begin{tikzcd}
A \otimes B \arrow[r, "\sigma_{A,B}"] \arrow[d, "f \otimes g"] & B \otimes A \arrow[d, "g \otimes f"] \\
A' \otimes B' \arrow[r, "\sigma_{A',B'}"'] & B' \otimes A'
\end{tikzcd}
$$

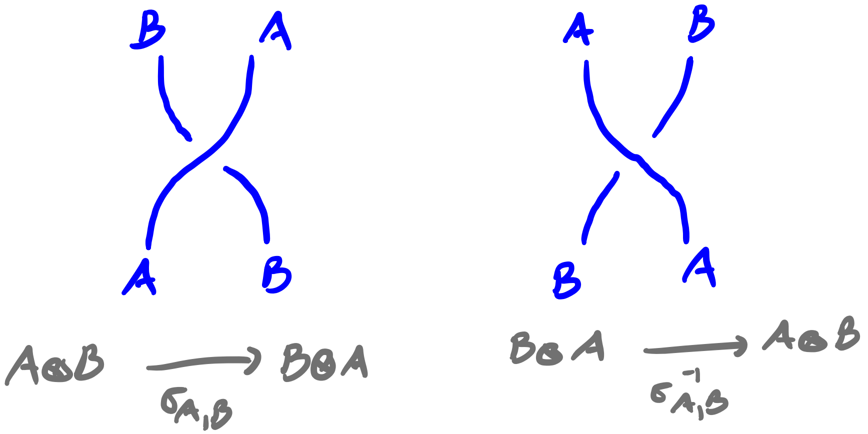
"natural"

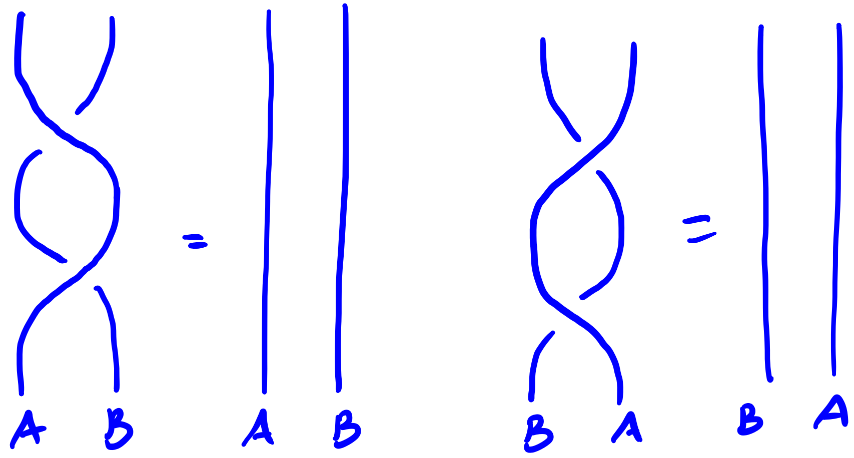
$$
\begin{tikzcd}
 & A \otimes (B \otimes C) \arrow[rr, "\sigma_{A,B \otimes C}"] \arrow[dl, "\alpha_{A,B,C}"'] & & (B \otimes C) \otimes A & \\
(A \otimes B) \otimes C \arrow[dr, "\sigma_{A,B} \otimes C"'] & & & & B \otimes (C \otimes A) \arrow[ul, "\alpha_{B,C,A}"'] \\
 & (B \otimes A) \otimes C \arrow[rr, "\alpha^{-1}_{B,A,C}"'] & & B \otimes (A \otimes C) \arrow[ur, "B \otimes \sigma_{A,C}"'] &
\end{tikzcd}
$$

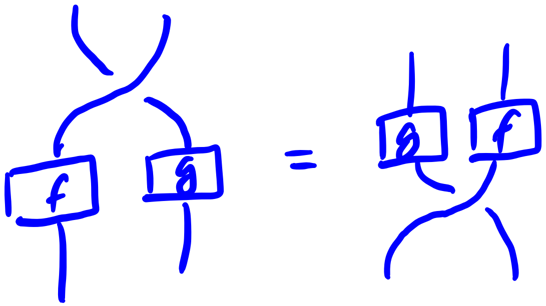
$$
\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[rr, "\sigma_{A \otimes B,C}"] & & C \otimes (A \otimes B) \arrow[dr, "\alpha_{C,A,B}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "A \otimes \sigma_{B,C}"'] & & & & (C \otimes A) \otimes B \\
 & A \otimes (C \otimes B) \arrow[rr, "\alpha_{A,C,B}"'] & & (A \otimes C) \otimes B \arrow[ur, "\sigma_{A,C} \otimes B"'] &
\end{tikzcd}
$$

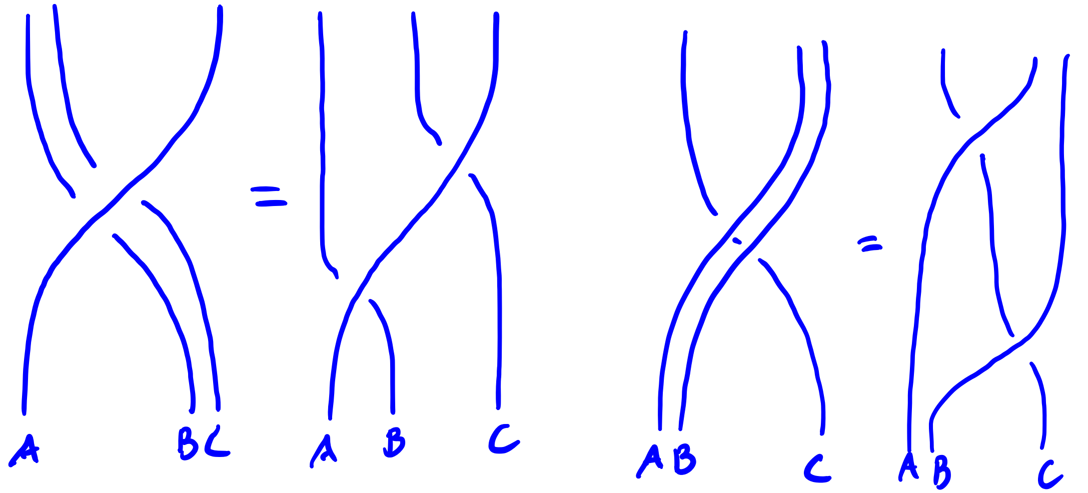
"hexagon equations"

Graphical calculus in 3 dimensions





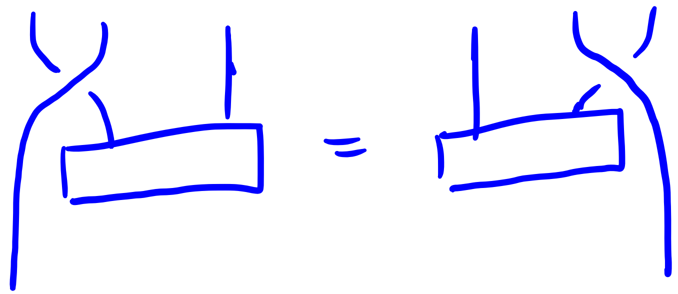


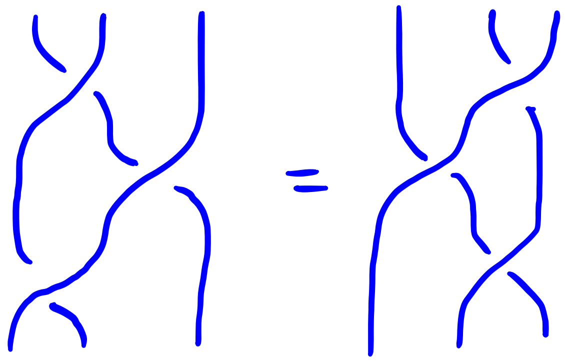


<u>Correctness thm</u>: eqn follows from axioms of braided monoidal cat

$\Longleftrightarrow$

graphical diagrams deformable by spatial isotopy e.g.

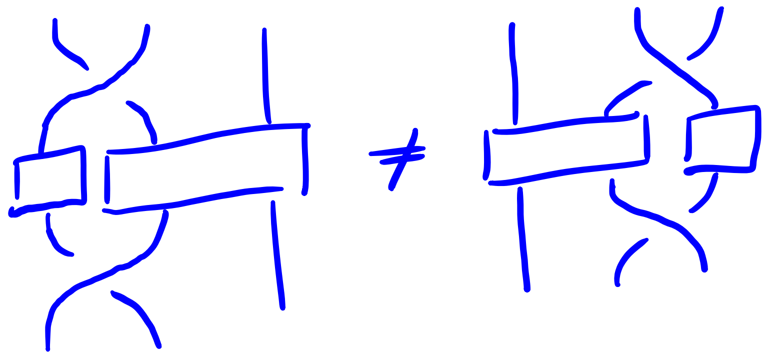




"Yang-Baxter"

in $\mathbf{Set}$: $A \times B \xrightarrow{\sigma_{A,B}} B \times A$

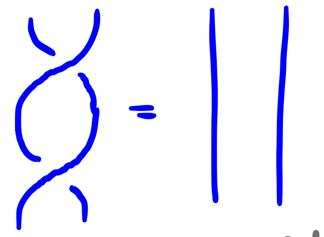
$(a,b) \longmapsto (b,a)$



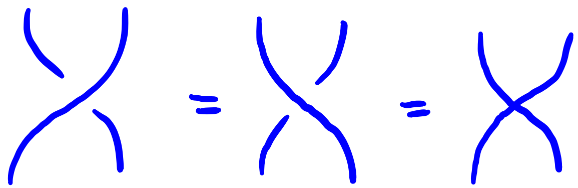
in $\mathbf{Mod}_R$: $A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$

$\sum_i a_i \otimes b_i \longmapsto \sum_i b_i \otimes a_i$

Symmetry is braiding with



$\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}$



Correctness thm: eqn follows from axioms of symmetric mon'l cat $\Longleftrightarrow$ graphical diagrams deformable by 4-dim'l isotopy

e.g.: $\mathbf{Set}$, $\mathbf{Set}^2$, $\mathbf{Vect}$, $\mathbf{Mod}_R$, any cartesian cat

not: $\mathbf{Mod}_A$ modules over a noncommutative Hopf algebra

$([\mathbb{C},\mathbb{C}], \circ, \mathrm{Id}_{\mathbb{C}}$ functor category$)$

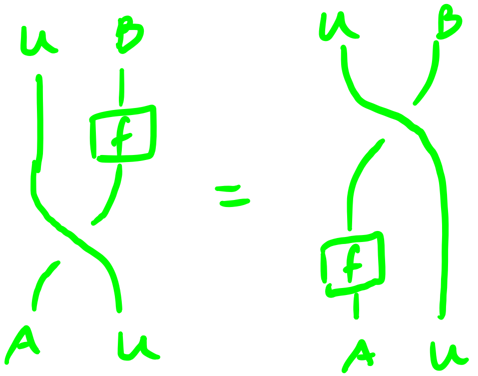
Let U be an object in a monoidal category.

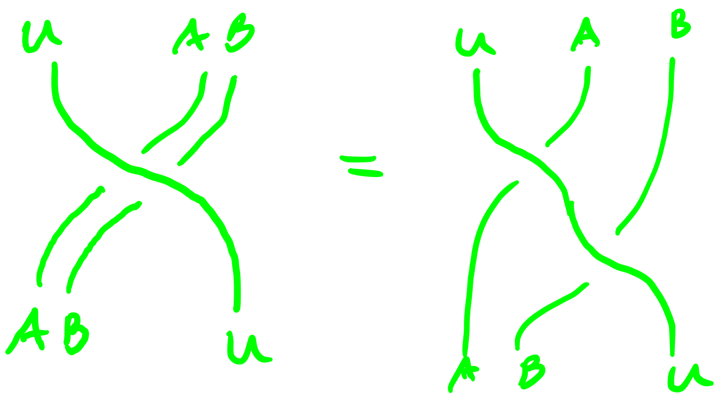
A half-braiding on U consists of

a morphism $A \otimes U \xrightarrow{\sigma_A} U \otimes A$ for each object A

such that:

$$\begin{tikzcd}
A \otimes U \arrow[r, "\sigma_A"] \arrow[d, "f \otimes U"] & U \otimes A \arrow[d, "U \otimes f"] \\
B \otimes U \arrow[r, "\sigma_B"'] & U \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes U \arrow[r, "\sigma_{A \otimes B}"] \arrow[d, "\alpha^{-1}_{A,B,U}"] & U \otimes (A \otimes B) \\
A \otimes (B \otimes U) \arrow[d, "A \otimes \sigma_B"] & (U \otimes A) \otimes B \arrow[u, "\alpha^{-1}_{U,A,B}"'] \\
A \otimes (U \otimes B) \arrow[r, "\alpha^{-1}_{A,U,B}"'] & (A \otimes U) \otimes B \arrow[u, "\sigma_A \otimes B"']
\end{tikzcd}$$

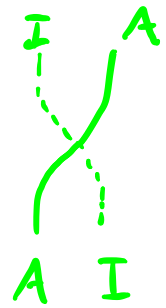




(It follows that $\sigma_I = \rho_U^{-1} \circ \lambda_U$)

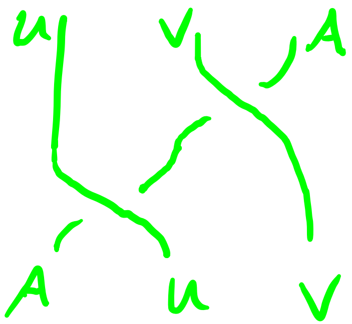
- tensor unit I always carries half-braiding:

$$A\otimes I \xrightarrow{\rho_A} A \xrightarrow{\lambda_A^{-1}} I\otimes A$$



- if (U,σ) and (V,τ) are half-braidings, then $U\otimes V$ has half-braiding:

$$\begin{tikzcd}
(A\otimes U)\otimes V \arrow[r, "\sigma_A\otimes V"'] & (U\otimes A)\otimes V \arrow[d, "\alpha^{-1}_{U,A,V}"] & (U\otimes V)\otimes A \\
A\otimes(U\otimes V) \arrow[u, "\alpha_{A,U,V}"'] & U\otimes(A\otimes V) \arrow[r, "U\otimes\tau_A"'] & U\otimes(V\otimes A) \arrow[u, "\alpha_{U,V,A}"']
\end{tikzcd}$$

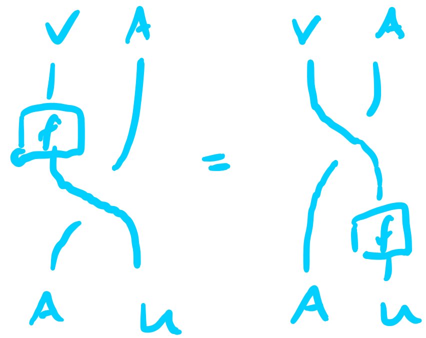


If $\mathbb{C}$ is a monoidal category, define $Z(\mathbb{C})$:

"Drinfeld centre"

- objects in $Z(\mathbb{C})$ are objects U in $\mathbb{C}$ together with half-braiding $A \otimes U \xrightarrow{\sigma_A} U \otimes A$

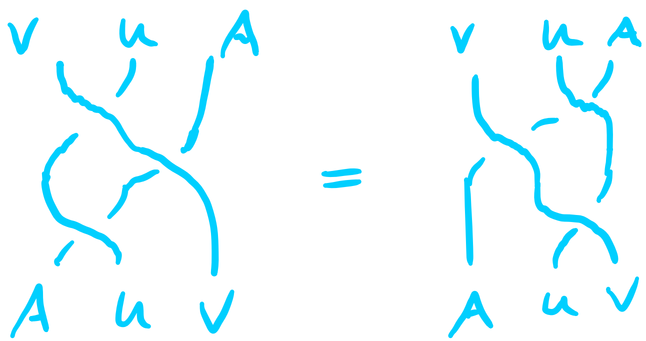
- a morphism in $Z(\mathbb{C})$ from (U, σ) to (V, τ) is a morphism $f : U \longrightarrow V$ in $\mathbb{C}$ satisfying $(f \otimes A) \circ \sigma_A = \tau_A \circ (A \otimes f)$



- composition in $Z(\mathbb{C})$ is as in $\mathbb{C}$
- identity on (U, σ) in $Z(\mathbb{C})$ is as in $\mathbb{C}$

If $\mathcal{C}$ is monoidal, then $Z(\mathcal{C})$ is braided monoidal.

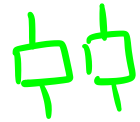
$$\begin{array}{ccccc} \beta_{(U,\sigma),(V,\tau)} & : & (U,\sigma)\otimes(V,\tau) & \longrightarrow & (V,\tau)\otimes(U,\sigma)\\ \| & & \| & & \| \\ \tau_U & : & U\otimes V & \longrightarrow & V\otimes U\end{array}$$

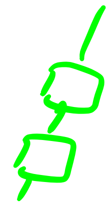


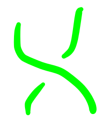
(There is functor $Z(\mathcal{C}) \longrightarrow \mathcal{C}$.
$$(U,\sigma) \longmapsto U$$

If $\mathcal{C}$ was already braided, there is functor $\mathcal{C} \longrightarrow Z(\mathcal{C})$
$$U \longmapsto (U, \sigma_{-,U})\,)$$

Summary:

- in a monoidal category you can compose morphisms in series and in parallel 

- there is a sound and complete 2-dimensional graphical calculus 

- braiding (symmetry) makes graphical calculus 3- (4-) dimensional 

- you can make a monoidal category braided by taking its Drinfeld centre

Next time:

- central idempotents