COMPOSITION ALGEBRAS AND THEIR GRADINGS

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AAC Mini Course

ABSTRACT. The goal of this course is the introduction of the basic properties of the classical composition algebras (that is, those algebras which are analogous to the real, complex, quaternion or octonion numbers), and how these basic properties are enough to get all the possible gradings on them. Then a new class of (non unital) composition algebras will be defined and studied, the so called *symmetric composition algebras*. Finally, the gradings on these two families of composition algebras will be shown to induce some interesting gradings on the exceptional simple Lie algebras.

1. UNITAL COMPOSITION ALGEBRAS. THE CAYLEY-DICKSON PROCESS.

Composition algebras constitute a generalization of the classical algebras of the real $\mathbb{R}$, complex $\mathbb{C}$, quaternion $\mathbb{H}$ (1843), and octonion numbers $\mathbb{O}$ (1845).

Definition 1.1. A *composition algebra* (over a field $\mathbb{F}$) is a not necessarily associative algebra C, endowed with a nondegenerate quadratic form (the *norm*) $q : C \to \mathbb{F}$ (i.e., the bilinear form $q(x, y) = q(x + y) - q(x) - q(y)$ is nondegenerate) which is multiplicative: $q(xy) = q(x)q(y) \ \forall x, y \in C$.

The unital composition algebras will be called *Hurwitz algebras*

Easy consequences:

- $q(xy, xz) = q(x)q(y, z) = q(yx, zx) \ \forall x, y, z.$ (l_x and r_x are similarities of norm $q(x)$.)
- $q(xy, tz) + q(ty, xz) = q(x, t)q(y, z) \ \forall x, y, z, t.$
  Assume now that C is unital:
- $t = 1 \ \Rightarrow q(xy, z) = q\big(y, (q(x, 1)1 - x)z\big) = q(y, \bar{x}z)$ ($\bar{x} = q(x, 1)1 - x$ is an order 2 orthogonal map). That is:

$$l_x^* = l_{\bar{x}}, \qquad r_x^* = r_{\bar{x}}.$$

  Then $l_x l_{\bar{x}} = r_x r_{\bar{x}} = q(x)id$, and applied to 1 this gives:

$$\boxed{x^2 - q(x, 1)x + q(x)1 = 0, \quad \forall x} \quad \text{(quadratic algebras)}$$

- $q(\overline{xy}, z) = q(xy, \bar{z}) = q(x, \bar{z}\bar{y}) = q(zx, \bar{y}) = q(z, \bar{y}\bar{x})$, so that $\overline{xy} = \bar{y}\bar{x}$. That is, $x \mapsto \bar{x}$ is an involution (the *standard involution*), which satisfies $x\bar{x} = q(x)1 = \bar{x}x$, and $x + \bar{x} = q(x, 1)1 \ \forall x.$
- $l_x l_{\bar{x}} = q(x)id \Rightarrow l_x^2 - q(x, 1)l_x + q(x)id = 0 \Rightarrow l_x^2 = l_{x^2} \ (x(xy) = x^2 y)$, and in the same vein $(yx)x = yx^2$. That is, $\boxed{\text{Hurwitz algebras are } alternative}$.

Cayley-Dickson doubling process:

Let Q be a subalgebra of a Hurwitz algebra C such that $q|_Q$ is nondegenerate, and let $u \in C$ such that $q(u) \neq 0 = q(u, Q)$. Then $1 \in Q$, so that $q(u, 1) = 0$ and hence $u^2 = -q(u)1$. Then for any $x \in Q$, $q(xu, 1) = q(x, \bar{u}) = -q(x, u) = 0$, so that $\overline{xu} = -xu$. Then:

$$x(yu) = -x(\overline{yu}) = -x(\bar{u}\bar{y}) = u(\bar{x}\bar{y}) = u(\overline{yx}) = -(yx)\bar{u} = (yx)u,$$

$$(yu)x = -\bar{x}(\overline{yu}) = \bar{x}(yu) = (y\bar{x})u,$$

$$(xu)(yu) = -\bar{y}((\overline{xu})u) = \bar{y}((xu)u) = \bar{y}(xu^2) = \alpha\bar{y}x,$$

(for $\alpha = -q(u) \neq 0$).

Thus $Q \oplus Qu$ is a subalgebra of C and $q|_{Q \oplus Qu}$ is nondegenerate.

Conversely, assume that Q is a Hurwitz algebra with norm q and $0 \neq \alpha \in \mathbb{F}$. Consider the vector space $C := Q \oplus Qu$ (this is formal: just the direct sum of two copies of Q), with multiplication:

$$(a + bu)(c + du) = (ac + \alpha\bar{d}b) + (da + b\bar{c})u,$$

and quadratic form

$$q(x + yu) = q(x) - \alpha q(y).$$

Notation: $C = CD(Q, \alpha)$.

Then:

$$q\big((a + bu)(c + du)\big) = q(ac + \alpha\bar{d}b) - \alpha q(da + b\bar{c}),$$

$$q(a + bu)q(c + du) = \big(q(a) - \alpha q(b)\big)\big(q(c) - \alpha q(d)\big)$$
$$= q(ac) + \alpha^2 q(cd) - \alpha\big(q(da) + q(ba)\big).$$

and these expressions are equal for any $a, b, c, d \in Q$ if and only if:

$$q(ac, \bar{d}b) = q(da, b\bar{c}) \quad \forall a, b, c, d \in Q$$

$$\Leftrightarrow \quad q\big(d(ac), b\big) = q\big((da)c, b\big) \quad \forall a, b, c, d \in Q$$

$$\Leftrightarrow \quad d(ac) = (da)c \quad \forall a, c, d \in Q$$

$$\Leftrightarrow \quad Q \text{ is associative.}$$

Theorem 1.2. *Let Q be a Hurwitz algebra with norm q and let $0 \neq \alpha \in \mathbb{F}$. Let $C = CD(Q, \alpha)$ as above. Then:*

(i) *C is a Hurwitz algebra if and only if Q is associative.*

(ii) *C is associative if and only if Q is commutative. (As $x(yu) = (yx)u$.)*

(iii) *C is commutative if and only if $Q = \mathbb{F}1$. (As $xu = u\bar{x}$, so we must have $x = \bar{x}$ for any x.)*

Remark 1.3. $\mathbb{F}$ *is a Hurwitz algebra if and only if* $\operatorname{char} \mathbb{F} \neq 2$.

Notation: $CD(A, \alpha, \beta) = CD\big(CD(A, \alpha), \beta\big)$.

Generalized Hurwitz Theorem 1.4. *Every Hurwitz algebra over a field $\mathbb{F}$ is isomorphic to one of the following types:*

(i) *The ground field $\mathbb{F}$ if its characteristic is $\neq 2$.*

(ii) *A quadratic commutative and associative separable algebra $K(\mu) = \mathbb{F}1 + \mathbb{F}v$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. Its norm is given by the generic norm.*

(iii) *A quaternion algebra $Q(\mu, \beta) = CD\big(K(\mu), \beta\big)$. (These are associative but not commutative.)*

(iv) *A Cayley algebra* $C(\mu, \beta, \gamma) = CD\big(Q(\mu, \beta), \gamma\big)$. *(These are alternative but not associative.)*

In particular, the dimension of any Hurwitz algebras is finite and restricted to 1, 2, 4 *or* 8.

Corollary 1.5. *Two Hurwitz algebras are isomorphic if and only if its norms are isometric.*

Isotropic Hurwitz algebras: Let C be a Cayley algebra such that its norm q represents 0 (*split Cayley algebra*). (This is always the situation if $\mathbb{F}$ is algebraically closed.)

Take $0 \neq x \in C$ with $q(x) = 0$ and take $y \in C$ with $q(x, \bar{y}) = 1$ ($q(.,.)$ is nondegenerate), then

$$q(xy, 1) = q(x, \bar{y}) = 1.$$

Let $e_1 = xy$, so $q(e_1) = 0$, $q(e_1, 1) = 1$, and hence $e_1^2 = e_1$. Let $e_2 = \bar{e}_1 = 1 - e_1$, so $q(e_2) = 0$, $e_2^2 = e_2$, $e_1 e_2 = 0 = e_2 e_1$ and $q(e_1, e_2) = q(e_1, 1) = 1$.

Then $\boxed{K = \mathbb{F}e_1 + \mathbb{F}e_2 \text{ is a composition subalgebra of } C}$.

For any $x \in K^{\perp}$, $xe_1 + \overline{xe_1} = q(xe_1, 1)1 = q(x, \bar{e}_1)1 = q(x, e_2)1 = 0$. Hence $xe_1 = -\bar{e}_1 \bar{x} = e_2 x$. We get:

$$xe_1 = e_2 x, \qquad xe_2 = e_1 x.$$

Also, $x = 1x = e_1 x + e_2 x$, and $e_2(e_1 x) = (1 - e_1)(e_1 x) = ((1 - e_1)e_1)x = 0 = e_1(e_2 x)$. Therefore,

$$K^{\perp} = U \oplus V$$

with

$$U = \{x \in C : e_1 x = x = xe_2, \ e_2 x = 0 = xe_1\},$$
$$V = \{x \in C : e_2 x = x = xe_1, \ e_1 x = 0 = xe_2\}.$$

For any $u \in U$, $q(u) = q(e_1 u) = q(e_1)q(u) = 0$, and hence U and V are isotropic subspaces of C. And for any $u_1, u_2 \in U$ and $v \in V$:

$$q(u_1 u_2, K) \subseteq q(u_1, K u_2) \subseteq q(U, U) = 0,$$
$$q(u_1 u_2, v) = q(u_1 u_2, e_2 v) = -q(e_2 u_2, u_1 v) + q(u_1, e_2)q(u_2, v) = 0.$$

Hence U^2 is orthogonal to K and V, so it must be contained in V. Also $V^2 \subseteq U$. Besides,

$$q(U, UV) \subseteq q(U^2, V) \subseteq q(V, V) = 0,$$
$$q(UV, V) \subseteq q(U, V^2) \subseteq q(U, U) = 0,$$

so $UV + VU \subseteq K$. Moreover, $q(UV, e_1) \subseteq q(U, e_1 V) = 0$, so that $UV \subseteq \mathbb{F}e_1$ and $VU \subseteq \mathbb{F}e_2$.

Therefore the decomposition $C = K \oplus U \oplus V$ is $\mathbb{Z}/3\mathbb{Z}$-grading of C.

Moreover, the trilinear map

$$U \times U \times U \longrightarrow \mathbb{F}$$
$$(x, y, z) \mapsto q(xy, z),$$

is alternating (for any $x \in U$, $q(x) = 0 = q(x, 1)$, so $x^2 = 0$ and hence $q(x^2, z) = 0$; but $q(xy, y) = -q(x, y^2) = 0$ too).

Take a basis $\{u_1, u_2, u_3\}$ of U with $q(u_1 u_2, u_3) = 1$ (this is always possible because $q(U^2, U) \neq 0$ since q is nondegenerate). Then $\{v_1 = u_2 u_3, v_2 = u_3 u_1, v_3 = u_1 u_2\}$ is the dual basis in V (relative to q) and the multiplication table is:

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

(For instance, $q(u_1v_1, e_2) = -q(u_1, e_2v_1) = -q(u_1, v_1) = -q(u_1, u_2u_3) = -1$, so that $u_1v_1 = -e_1$; $v_1v_2 = v_1(u_3u_1) = -v_1(u_1u_3) = u_1(v_1u_3) - (u_1v_1 + v_1u_1)u_3 = 1u_3 = u_3$, ...)

Notation: The split Cayley algebra above is denoted by $C(\mathbb{F})$ and the basis considered is called a canonical basis of $C(\mathbb{F})$.

Theorem 1.6. *Let $n = 2, 4$ or 8. Then there is, up to isomorphism, a unique Hurwitz algebra with isotropic norm of dimension n:*

(i) *$\mathbb{F}e_1 + \mathbb{F}e_2$ in dimension 2, which is just the cartesian product of two copies of $\mathbb{F}$.*

(ii) *$\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}u_1 + \mathbb{F}v_1$ in dimension 4, which is isomorphic to $\mathrm{Mat}_2(\mathbb{F})$, with the norm given by the determinant.*

(iii) *$C(\mathbb{F})$ in dimension 8.*

What about real Hurwitz algebras?

If Q is a real Hurwitz algebra which is not split (q does not represent 0) then q is positive definite, the norm of $CD(Q, \alpha)$ is positive definite if and only if $\alpha < 0$, and in this case (change u to $\sqrt{-\alpha}^{-1}u$) $CD(Q, \alpha) = CD(Q, -1)$. Thus the list of real Hurwitz algebras is:

- the split ones: $\mathbb{R} \oplus \mathbb{R}$, $\mathrm{Mat}_2(\mathbb{R})$, $C(\mathbb{R})$,
- the "division" ones: $\mathbb{R}$, $\mathbb{C} = CD(\mathbb{R}, -1)$, $\mathbb{H} = CD(\mathbb{C}, -1)$, and $\mathbb{O} = CD(\mathbb{H}, -1)$.

There are many good references that cover the material in this section. Let us mention, for instance, [KMRT98, Chapter VIII] or [ZSSS82, Chapter 2].

2. Symmetric composition algebras. Triality.

Definition 2.1. A composition algebra $(S, *, q)$ is said to be a *symmetric composition algebra* if $l_x^* = r_x$ for any $x \in S$ (that is, $q(x * y, z) = q(x, y * z)$ for any $x, y, z \in S$).

Theorem 2.2. *Let $(S, *, q)$ be a composition algebra. The following conditions are equivalent:*

(a) $(S, *, q)$ *is symmetric.*
(b) *For any $x, y \in S$, $(x * y) * x = x * (y * x) = q(x)y$.*

Proof. If $(S, *, q)$ is symmetric, then for any $x, y, z \in S$,

$$q\big((x * y) * x, z\big) = q(x * y, x * z) = q(x)q(y, z) = q\big(q(x)y, z\big)$$

whence (b), since q is nondegenerate. Conversely, take $x, y, z \in S$ with $q(y) \neq 0$, so that l_y and r_y are bijective, and hence there is an element $z' \in S$ with $z = z' * y$. Then:

$$q(x * y, z) = q(x * y, z' * y) = q(x, z')q(y) = q\big(x, y * (z' * y)\big) = q(x, y * z).$$

This proves (a) assuming $q(y) \neq 0$, but any isotropic element is the sum of two non isotropic elements, so (a) follows. □

Remark 2.3.

- Condition (b) above implies that $((x * y) * x) * (x * y) = q(x * y)x$, but also $((x * y) * x) * (x * y) = q(x)y * (x * y) = q(x)q(y)x$, so that condition (b) already forces the quadratic form q to be multiplicative.
- Let $(S, *, q)$ be a symmetric composition algebra. Take an element $a \in S$ with $q(a) \neq 0$ and define a new multiplication and nondegenerate quadratic form on S by means of

$$x \bullet y = (a * x) * (y * a), \qquad \tilde{q}(x) = q(x)q(a).$$

Then $(S, \bullet, \tilde{q})$ is again a composition algebra. Consider the element $e = \frac{1}{q(a)^2} a * a$. Then

$$e \bullet x = (a * e) * (x * a) = \frac{1}{q(a)^2}(a * (a * a)) * (x * a) = \frac{1}{q(a)} a * (x * a) = x,$$

and $x \bullet e = x$ too for any $x \in S$. Hence $(S, \bullet, \tilde{q})$ is a Hurwitz algebra. Therefore the dimension of any symmetric composition algebra is restricted to 1, 2, 4 or 8.

Examples 2.4. (Okubo 1978 [Oku78])

- **Para-Hurwitz algebras:** Let C be a Hurwitz algebra with norm q and consider the composition algebra $(C, \bullet, q)$ with the new product given by

$$x \bullet y = \bar{x}\bar{y}.$$

Then $q(x \bullet y, z) = q(\bar{x}\bar{y}, z) = q(\bar{x}, zy) = q(x, \overline{zy}) = q(x, y \bullet z)$, for any x, y, z, so that $(C, \bullet, q)$ is a symmetric composition algebra. (Note that $1 \bullet x = x \bullet 1 = \bar{x} = q(x, 1)1 - x \; \forall x$: 1 is the *para-unit* of $(C, \bullet, q)$.)

- **Okubo algebras:** Assume $\operatorname{char} \mathbb{F} \neq 3$ (the case of $\operatorname{char} \mathbb{F} = 3$ requires a different definition), and let $\omega \in \mathbb{F}$ be a primitive cubic root of 1. Let A be a central simple associative algebra of degree 3 with trace tr, and let $S = A_0 = \{x \in A : \operatorname{tr}(x) = 0\}$ with multiplication and norm given by:

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$

$$q(x) = -\frac{1}{2} \operatorname{tr}(x^2), \quad \text{(it is valid in characteristic 2!)}$$

Then, for any $x, y \in S$:

$$(x * y) * x = \omega(x * y)x - \omega^2 x(x * y) - \frac{\omega - \omega^2}{3} \operatorname{tr}\big((x * y)x\big)1$$

$$= \omega^2 xyx - yx^2 - \frac{\omega^2 - 1}{3} \operatorname{tr}(xy)x - x^2 y + \omega xyx + \frac{1 - \omega}{3} \operatorname{tr}(xy)x$$

$$- \frac{\omega - \omega^2}{3} \operatorname{tr}\big((\omega - \omega^2)x^2 y\big)1 \quad (\operatorname{tr}(x) = 0)$$

$$= -\big(x^2 y + yx^2 + xyx\big) + \operatorname{tr}(xy)x + \operatorname{tr}(x^2 y)1 \quad ((\omega - \omega^2)^2 = -3).$$

But if $\operatorname{tr}(x) = 0$, then $x^3 - \frac{1}{2} \operatorname{tr}(x^2)x + \det(x)1 = 0$, so

$$x^2 y + yx^2 + xyx - \big(\operatorname{tr}(xy)x + \frac{1}{2} \operatorname{tr}(x^2)y\big) \in \mathbb{F}1.$$

Since $(x * y) * x \in A_0$, we have $(x * y) * x = -\frac{1}{2} \operatorname{tr}(x^2)y = x * (y * x)$.

Therefore $(S, *, q)$ is a symmetric composition algebra.

In case $\omega \notin \mathbb{F}$, take $\mathbb{K} = \mathbb{F}[\omega]$ and a central simple associative algebra A of degree 3 over $\mathbb{K}$ endowed with a $\mathbb{K}/\mathbb{F}$-involution of second kind J. Then take $S = K(A, J)_0 = \{x \in A_0 : J(x) = -x\}$ (this is a $\mathbb{F}$-subspace) and use the same formulae above to define the multiplication and the norm.

For instance, for $\mathbb{F} = \mathbb{R}$, take $A = \operatorname{Mat}_3(\mathbb{C})$, $S = \mathfrak{su}_3 = \{x \in \operatorname{Mat}_3(\mathbb{C}) : \operatorname{tr}(x) = 0, \ \overline{x}^T = -x\}$

Remark 2.5. Given an Okubo algebra, note that for any $x, y \in S$,

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$

$$y * x = \omega yx - \omega^2 xy - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$

so that

$$\omega x * y + \omega^2 y * x = (\omega^2 - \omega)xy - (\omega + \omega^2)\frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$

and

$$xy = \frac{\omega}{\omega^2 - \omega} x * y + \frac{\omega^2}{\omega^2 - \omega} y * x + \frac{1}{3} q(x, y)1,$$

and the product in A is determined by the product in the Okubo algebra.

Classification (char $\mathbb{F} \neq 3$)**:**

We can go in the reverse direction of Okubo's construction. Given a symmetric composition algebra $(S, *, q)$ over a field containing ω, define the algebra $A = \mathbb{F}1 \oplus S$ with multiplication determined by the formula

$$xy = \frac{\omega}{\omega^2 - \omega} x * y + \frac{\omega^2}{\omega^2 - \omega} y * x + \frac{1}{3} q(x, y)1,$$

for any $x, y \in S$. Then A is a separable alternative algebra of degree 3.

In case $\omega \notin \mathbb{F}$, then we must consider $A = \mathbb{F}[\omega]1 \oplus \big(\mathbb{F}[\omega] \otimes S\big)$, with the same formula for the product. In $\mathbb{F}[\omega]$ we have the Galois automorphism $\omega^\tau = \omega^2$. Then the conditions $J(1) = 1$ and $J(s) = -s$ for any $s \in S$ induce a $\mathbb{F}[\omega]/\mathbb{F}$-involution of the second kind in A.

Theorem 2.6. *(char $\mathbb{F} \neq 3$)*

$\boxed{\omega \in \mathbb{F}}$**:** *The symmetric composition algebras are, up to isomorphism, the algebras $(A_0, *, q)$ for A a separable alternative algebra of degree 3.*

Two symmetric composition algebras are isomorphic if and only if so are the corresponding alternative algebras.

$\boxed{\omega \notin \mathbb{F}}$: *The symmetric composition algebras are, up to isomorphism, the algebras $\big(K(A; J)_0, *, q\big)$ for A a separable alternative algebra of degree 3 over $\mathbb{K} = \mathbb{F}[\omega]$, and J a $\mathbb{K}/\mathbb{F}$-involution of the second kind.*

Two symmetric composition algebras are isomorphic if and only if so are the corresponding alternative algebras, as algebras with involution.

Possibilities for such algebras A: Let $\mathbb{K} = \mathbb{F}[\omega]$, so that $\mathbb{K} = \mathbb{F}$ if $\omega \in \mathbb{F}$.

- $A = \mathbb{K} \times C$, with $\deg C = 2$ ($\Rightarrow C$ is a Hurwitz algebra!), then $(A_0, *, q)$ is isomorphic to the para-Hurwitz algebra attached to C if $\mathbb{K} = \mathbb{F}$, and $(KA, J)_0, *, q)$ to the one attached to $\hat{C} = \{x \in C : J(x) = \bar{x}\}$ if $\mathbb{K} \neq \mathbb{F}$.
- A is a central simple associative algebra of degree 3, and hence $(A_0, *, q)$ or $\big(K(A, J)_0, *, q\big)$ is an Okubo algebra.
- $A = \mathbb{K} \otimes_{\mathbb{F}} L$, for a cubic field extension L of $\mathbb{F}$ (if $\omega \notin \mathbb{F}$ $L = \{x \in A : J(x) = x\}$) and $\dim_{\mathbb{F}} S = 2$.

Remark 2.7. The classification in characteristic 3 follows a different path to arrive at a similar result: any symmetric composition algebra is either para-Hurwitz or "Okubo", with a few exceptions in dimension 2.

Remark 2.8. Assume that $(S, *, q)$ is a two-dimensional symmetric composition algebra.

If there is an element $a \in S$ such that $q(a) \neq 0$ and $a * a \in \mathbb{F}a$, then we may scale a and get an element $e \in S$ such that $e * e = e$ (so that $q(e) = 1$). Then S is the para-Hurwitz algebra attached to the Hurwitz algebra defined over S with the multiplication

$$x \cdot y = (e * x) * (y * e),$$

with unity $1 = e$.

Otherwise, take $a \in S$ with $q(a) = 1$ (this is always possible). Then $a * a \notin \mathbb{F}$, so that $S = \mathbb{F}a \oplus \mathbb{F}(a * a)$, and the multiplication is completely determined by the scalar $\alpha = q(a, a * a)$:

$$a * (a * a) = (a * a) * a = q(a)a = a,$$
$$(a * a) * (a * a) = -((a * a) * a) * a + q(a, a * a)a = a * a - \alpha a.$$

Triality:

Assume char $\mathbb{F} \neq 2$, and let $(S, *, q)$ be a symmetric composition algebra. Consider the associated orthogonal Lie algebra

$$\mathfrak{so}(S, q) = \{d \in \operatorname{End}_{\mathbb{F}}(S) : q\big(d(x), y\big) + q\big(x, d(y)\big) = 0 \ \forall x, y \in S\}.$$

The *triality Lie algebra* of $(S, *, q)$ is defined as the following Lie subalgebra of $\mathfrak{so}(S, q)^3$ (with componentwise bracket):

$$\mathfrak{tri}(S, *, q) = \{(d_0, d_1, d_2) \in \mathfrak{so}(S, q)^3 : d_0(x * y)d_1(x) * y + x * d_2(y) \ \forall x, y, z \in S\}.$$

Note that the condition $d_0(x * y) = d_1(x) * y + x * d_2(y)$ for any $x, y \in S$ is equivalent to the condition

$$q\big(x * y, d_0(z)\big) + q\big(d_1(x) * y, z\big) + q\big(x * d_2(y), z\big) = 0,$$

for any $x, y, z \in S$. But $q(x * y, z) = q(y * z, x) = q(z * x, y)$. Therefore, the linear map:

$$\theta : \mathfrak{tri}(S, *, q) \longrightarrow \mathfrak{tri}(S, *, q)$$
$$(d_0, d_1, d_2) \mapsto (d_2, d_0, d_1),$$

is an automorphism of the Lie algebra $\mathfrak{tri}(S, *, q)$.

Theorem 2.9. *Let $(S, *, q)$ be an eight-dimensional symmetric composition algebra over a field of characteristic $\neq 2$. Then:*

(i) **Principle of Local Triality:** *The projection map:*
$$\pi_0 : \mathfrak{tri}(S, *, q) \longrightarrow \mathfrak{so}(S, q)$$
$$(d_0, d_1, d_2) \mapsto d_0$$

is an isomorphism of Lie algebras.

(ii) *For any $x, y \in S$, the triple*
$$t_{x,y} = \left(\sigma_{x,y} = q(x, .)y - q(y, .)x, \frac{1}{2}q(x,y)id - r_x l_y, \frac{1}{2}q(x,y)id - l_x r_y \right)$$

belongs to $\mathfrak{tri}(S, *, q)$, and $\mathfrak{tri}(S, *, q)$ is spanned by these elements. Moreover, for any $a, b, x, y \in S$:
$$[t_{a,b}, t_{x,y}] = t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}.$$

Proof. Let us first check that $t_{x,y} \in \mathfrak{tri}(S, *, q)$:
$$\sigma_{x,y}(u * v) = q(x, u * v)y - q(y, u * v)x$$
$$r_x l_y(u) * v = ((y * u) * x) * v = -(v * x) * (y * u) + q(y * u, v)x,$$
$$u * l_x r_y(v) = u * (x * (v * y)) = -u * (y * (v * x)) + q(x, y)u * v$$
$$= (v * x) * (y * u) + q(u, v * x)y + q(x, y)u * v,$$

and hence
$$\sigma_{x,y}(u * v) - \left(\frac{1}{2}q(x,y)id - r_x l_y\right)(u) * v - u * \left(\frac{1}{2}q(x,y)id - l_x r_y\right)(v) = 0.$$

Also $\sigma_{x,y} \in \mathfrak{so}(S, q)$ and $\left(\frac{1}{2}q(x,y)id - r_x l_y\right)^* = \frac{1}{2}q(x,y)id - r_y l_x$ (adjoint relative to the norm q), but $r_x l_x = q(x)id$, so $r_x l_y + r_y l_x = q(x,y)id$ and hence $\left(\frac{1}{2}q(x,y)id - r_x l_y\right)^* = -\left(\frac{1}{2}q(x,y)id - r_x l_y\right)$, so that $\frac{1}{2}q(x,y)id - r_x l_y \in \mathfrak{so}(S, q)$, and $\frac{1}{2}q(x,y)id - l_x r_y \in \mathfrak{so}(S, q)$ too. Therefore, $t_{x,y} \in \mathfrak{tri}(S, *, q)$.

Since the Lie algebra $\mathfrak{so}(S, q)$ is spanned by the $\sigma_{x,y}$'s, it is clear that the projection π_0 is surjective (and hence so are π_1 and π_2). Consider an element (d_0, d_1, d_2) in $\ker \pi_0$. Hence $d_0 = 0$ and $d_1(x) * y + x * d_2(y) = 0$ for any $x, y \in S$. But since π_1 is onto, the subspace $\{d_1 \in \mathfrak{so}(S, q) : \exists d_2 \in \mathfrak{so}(S, q) \ (0, d_1, d_2) \in \mathfrak{tri}(S, *, q)\}$ is an ideal of the simple Lie algebra $\mathfrak{so}(S, q)$. Hence either $\ker \pi_0 = 0$ or for any $d \in \mathfrak{so}(S, q)$ there is another element $d' \in \mathfrak{so}(S, q)$ such that $d(x) * y + x * d'(y) = 0$ for any $x, y \in S$. This is impossible: take $d = \sigma_{a,b}$ for linearly independent elements $a, b \in S$ and take x orthogonal to a, b and not isotropic. Then $d(x) = 0$, so we would get $x * d'(y) = 0$ for any $y \in S$. This forces $d' = 0$ since l_x is a bijection, and we get a contradiction. Therefore, π_0 is an isomorphism.

Finally the formula $[t_{a,b}, t_{x,y}] = t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}$ follows from the "same" formula for the σ's and the fact that π_0 is an isomorphism. $\square$

For the results in this section one may consult [EM93] or [KMRT98, Chapter VIII].

3. GRADINGS ON COMPOSITION ALGEBRAS.

All the gradings considered here will be group gradings: $C = \oplus_{g \in G} C_g$, G a group generated by $\{g \in G : C_g \neq 0\}$ and $C_g C_{g'} \subseteq C_{gg'}$ $\forall g, g' \in G$.

3.1. Gradings on Hurwitz algebras.

Let $C = \oplus_{g \in G} C_g$ be a graded Hurwitz algebra. For any $x \in C$, $x^2 - q(x, 1)x + q(x)1 = 0$. Always $1 \in C_e$, and hence if $x \in C_g$, with $g \neq e$:

- $q(x, 1) = 0$ so that $\bar{C}_h = C_h$ for any $h \in G$,
- $q(x) = 0$ unless $g^2 = e$.

Take now $x \in C_g$, $y \in C_h$, then $q(x, y) = q(x\bar{y}, 1) = 0$ unless $gh = e$. But then for $g \neq h^{-1}$, $0 = q(x\bar{y}, 1)1 = x\bar{y} + y\bar{x}$, so that either $C_g C_h = 0 = C_h C_g$, or $gh = hg$.

Thus, if $g, h \in G$, with $g \neq h$ and $C_g \neq 0 \neq C_h$, $q(C_g + C_{g^{-1}}) \neq 0$ (q is nondegenerate), so that $(C_g + C_{g^{-1}})C_h \neq 0$, and hence either

- $C_g C_h \neq 0$, and then $gh = hg$, or
- $C_{g^{-1}} C_h \neq 0$, and then $g^{-1}h = hg^{-1}$, so $gh = hg$ too.

We conclude that G is abelian. In what follows, additive notation for G will be used.

Examples 3.1.

(1) **Gradings induced by the Cayley-Dickson doubling process:**
- If $C = CD(Q, \alpha) = Q \oplus Qu$, this is a $\mathbb{Z}_2$-grading: $C_{\bar{0}} = Q$, $C_{\bar{1}} = Qu$.
- If, moreover, $Q = CD(K, \beta) = K \oplus Kv$, then $C = K \oplus Kv \oplus Ku \oplus (Kv)u$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading.
- Finally, if $K = CD(\mathbb{F}, \gamma) = \mathbb{F}1 \oplus \mathbb{F}w$, then C is $\mathbb{Z}_2^3$-graded.

(2) **Cartan grading:** Take a canonical basis of the split Cayley algebra $B = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$. Then C is $\mathbb{Z}^2$-graded with

$$C_{(0,0)} = \mathbb{F}e_1 \oplus \mathbb{F}e_2,$$
$$C_{(1,0)} = \mathbb{F}u_1, \quad C_{(-1,0)} = \mathbb{F}v_1,$$
$$C_{(0,1)} = \mathbb{F}u_2, \quad C_{(0,-1)} = \mathbb{F}v_2,$$
$$C_{(1,1)} = \mathbb{F}v_3, \quad C_{(-1,-1)} = \mathbb{F}u_3.$$

Theorem 3.2. *Any proper grading of a Cayley algebra is either a grading induced by the Cayley-Dickson doubling process or it is a coarsening of the Cartan grading of the split Cayley algebra.*

Proof. Let $C = \oplus_{g \in G} C_g$ be a grading of the Cayley algebra C. Then C_0 is a composition subalgebra of C.

First case: Assume that G is 2-elementary. Then take $0 \neq g_1 \in G$ with $C_{g_1} \neq 0$. The restriction $q|_{C_{g_1}}$ is nondegenerate so we may take an element $u \in C_{g_1}$ with $q(u) \neq 0$, so that $C_{g_1} = C_0 u$ and $C_0 \oplus C_{g_1} = C_0 \oplus C_0 u = CD(C_0, \alpha)$ with $\alpha = -q(u)$. This is a composition subalgebra of C, and hence either $C = C_0 \oplus C_{g_1}$ and $G = \mathbb{Z}_2$, or there is another element $g_2 \in G \setminus \{0, g_1\}$ with $C_{g_2} \neq 0$. Again take $v \in C_{g_2}$ with $q(v) \neq 0$ and we get $C_0 \oplus C_{g_1} \oplus C_{g_2} \oplus C_{g_1+g_2} = (C_0 \oplus C_{g_1}) \oplus (C_0 \oplus C_{g_1})v = CD(C_0 \oplus C_{g_1}, \beta) = CD(C_0, \alpha, \beta)$, which is a $\mathbb{Z}_2^2$-graded composition subalgebra of C. Again, either this is the whole C or we can repeat once more the process to get $C = CD(C_0, \alpha, \beta, \gamma)$ $\mathbb{Z}_2^3$-graded (and $\dim C_0 = 1$).

Second case: Assume that G is not 2-elementary, so there exists $g \in G$ with $C_g \neq 0$ and the order of g is > 2. Then $q(C_g) = 0$, so q is isotropic and hence C is the split Cayley algebra. Take elements $x \in C_g$, $y \in C_{-g}$ with $q(x, y) = -1$ (q is nondegenerate). That is, $q(xy, 1) = q(x, \bar{y}) = -q(x, y) = 1$.

Our considerations on isotropic Hurwitz algebras show that $e_1 = xy$ satisfies $e_1^2 = e_1$, $q(e_1) = 0$, $\bar{e}_1 = 1 - e_1 =: e_2$. Therefore $\mathbb{F}e_1 \oplus \mathbb{F}e_2$ is a composition subalgebra of C_0 and hence the subspaces $U = \{x \in C : e_1 x = x = x e_2\}$ and $V = \{x \in C : e_2 x = x = x e_1\}$ are graded subspaces of C and we may choose a basis $\{u_1, u_2, u_3\}$ of U consisting of homogeneous elements and such that $q(u_1 u_2, u_3) = 1$. With $v_1 = u_2 u_3$, $v_2 = u_3 u_1$ and $v_3 = u_1 u_2$ we get a canonical basis $B = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of C formed by homogeneous elements and such that $\deg(e_1) = \deg(e_2) = 0$. Let $g_i = \deg(u_i)$, $i = 1, 2, 3$. From $u_i v_i = -e_1$ we conclude that $\deg(v_i) = -g_i$, and from $v_1 = u_2 u_3$ we conclude that $g_1 + g_2 + g_3 = 0$. The grading is a coarsening of the Cartan grading. $\qquad\square$

Up to symmetry, any coarsening of the Cartan grading is obtained as follows (here $g_1 = (1, 0)$ and $g_2 = (0, 1)$):

$\boxed{g_1 = 0}$: Then we obtain a 3-grading over $\mathbb{Z}$: $C = C_{-1} \oplus C_0 \oplus C_1$, with $C_0 = \langle e_1, e_2, u_1, v_1 \rangle$, $C_1 = \langle u_2, v_3 \rangle$, $C_{-1} = \langle u_3, v_2 \rangle$. Its proper coarsenings are all "2-elementary".

$\boxed{g_1 = g_2}$: Here we obtain a 5-grading over $\mathbb{Z}$, with $C_{-2} = \mathbb{F}u_3$, $C_{-1} = \langle v_1, v_2 \rangle$, $C_0 = \langle e_1, e_2 \rangle$, $C_1 = \langle u_1, u_2 \rangle$ and $C_2 = \mathbb{F}v_3$, which has two proper coarsenings which are not 2-elementary:

$\boxed{g_1 = g_2 = g_3}$: This gives a $\mathbb{Z}_3$-grading: $C_{\bar{0}} = \langle e_1, e_2 \rangle$, $C_{\bar{1}} = U$, $C_{\bar{2}} = V$.

$\boxed{g_3 = -g_3}$: This gives a $\mathbb{Z}_4$-grading.

$\boxed{g_1 = -g_1}$: Here we get a $\mathbb{Z} \times \mathbb{Z}_2$-grading

$$C = C_{(0,\bar{0})} \oplus C_{(1,\bar{0})} \oplus C_{(-1,\bar{0})} \oplus C_{(0,\bar{1})} \oplus C_{(1,\bar{1})} \oplus C_{(1,\bar{1})}$$
$$\| \qquad\quad \| \qquad\quad \| \qquad\quad \| \qquad\quad \| \qquad\quad \|$$
$$\langle e_1, e_2 \rangle \quad \mathbb{F}u_2 \quad \mathbb{F}v_2 \quad \langle u_1, v_1 \rangle \quad \mathbb{F}u_3 \quad \mathbb{F}v_3$$

Any of its coarsenings is a coarsening of the previous gradings.

$\boxed{g_1 = -g_2}$: In this case $g_3 = 0$, and this is equivalent to the grading obtained with $g_1 = 0$.

Theorem 3.3. *Up to equivalence, the gradings of the split Cayley algebra are:*

(i) *The $\mathbb{Z}_2^r$-gradings induced by the Cayley-Dickson doubling process.*

(ii) *The Cartan grading over $\mathbb{Z}^2$.*

(iii) *The 3-grading: $C_0 = span\{e_1, e_2, u_3, v_3\}$, $C_1 = span\{u_1, v_2\}$, $C_{-1} = span\{u_2, v_1\}$.*

(iv) *The 5-grading: $C_0 = span\{e_1, e_2\}$, $C_1 = span\{u_1, u_2\}$, $C_2 = span\{v_3\}$, $C_{-1} = span\{v_1, v_2\}$, $C_{-2} = span\{u_3\}$.*

(v) *The $\mathbb{Z}_3$-grading: $C_{\bar{0}} = span\{e_1, e_2\}$, $C_{\bar{1}} = U$, $C_{\bar{2}} = V$.*

(vi) *The $\mathbb{Z}_4$-grading: $C_{\bar{0}} = span\{e_1, e_2\}$, $C_{\bar{1}} = span\{u_1, u_2\}$, $C_{\bar{2}} = span\{u_3, v_3\}$, $C_{\bar{3}} = span\{v_1, v_2\}$.*

(vii) *The $\mathbb{Z} \times \mathbb{Z}_2$-grading.*

Remark 3.4. The gradings on quaternion algebras are obtained in a similar but simpler way. Any grading is either induced by the Cayley-Dickson doubling process ($\mathbb{Z}_2^r$-grading for $0 \leq r \leq 2$) or it is the Cartan grading of $\mathrm{Mat}_2(\mathbb{F})$.

3.2. Gradings on symmetric composition algebras.

Let $S = \oplus_{g \in G} S_g$ be a grading of the symmetric composition algebra $(S, *, q)$. Take nonzero homogeneous elements $x \in S_a$, $y \in S_b$ and $z \in S_c$. Then

$$(x * y) * z + (z * y) * x = q(x, z)y,$$

so $q(S_a, S_c) = 0$ unless $abc = b$ or $cba = b$. With $b = a$ we get $q(S_a, S_c) = 0$ unless $c = a^{-1}$. With $c = a^{-1}$, since q is nondegenerate we may take x and z with $q(x, z) = 1$, and hence either $aba^{-1} = b$ or $a^{-1}ba = b$. In any case $ab = ba$. Hence again the grading group must be abelian and additive notation will be used.

Proposition 3.5. *Let $(S, *, q)$ be a para-Hurwitz algebra of dimension 4 or 8, so that $x * y = \bar{x} \cdot \bar{y}$ for a Hurwitz product. Then the gradings on $(S, *, q)$ and on the Hurwitz algebra $(S, \cdot, q)$ coincide.*

Proof. We know that given any grading $S = \oplus_{g \in G} S_g$ of the Hurwitz algebra $(S, \cdot, q)$, $\bar{S}_g = S_g$ for any g, and hence this is a grading too of $(S, *, q)$. Conversely, let $S = \oplus_{g \in G} S_g$ be a grading of $(S, *, q)$. Then

$$K = \{x \in S : x * y = y * x \ \forall y \in S\}$$
$$= \{x \in S : \bar{x} \cdot y = y \cdot \bar{x} \ \forall y \in S\} = \mathbb{F}1,$$

because the dimension is at least 4. Thus $\mathbb{F}1$ is a graded subspace of $(S, *, q)$ and as $1 * 1 = 1$, it follows that $1 \in S_0$. But then it is clear that $\bar{S}_g = S_g$ for any $g \in G$ (because $q(S_g, 1) = 0$ unless $g = 0$) and the grading is a grading of the Hurwitz algebra. $\square$

Therefore it is enough to study the gradings of the Okubo algebras. (And of the two-dimensional symmetric composition algebras, but this is quite easy: one gets either the trivial grading or a $\mathbb{Z}_2$-grading of a para-Hurwitz algebra or some $\mathbb{Z}_3$-gradings.)

Theorem 3.6. *Let $\mathbb{F}$ be a field of characteristic $\neq 3$ containing the cubic roots of 1. Then any grading of an Okubo algebra over $\mathbb{F}$ is a coarsening of either a $\mathbb{Z}^2$-grading or of a $\mathbb{Z}_3^2$-grading.*

Proof. Let $(S, *, q)$ be an Okubo algebra over $\mathbb{F}$ and $S = \oplus_{g \in G} S_g$ be a grading over the abelian group G. Let $A = \mathbb{F}1 \oplus S$ be the central simple associative algebra of degree 3 with multiplication determined by

$$xy = \frac{\omega}{\omega^2 - \omega} x * y + \frac{\omega^2}{\omega^2 - \omega} y * x + \frac{1}{3} q(x, y)1,$$

for any $x, y \in S$. Then since $q(S_g, S_h) = 0$ unless $g + h = 0$, the grading on S induces a grading on A. By well-known results on gradings on associative algebras, this is a coarsening of either the Cartan grading over $\mathbb{Z}^2$ of $\mathrm{Mat}_3(\mathbb{F})$, or a $\mathbb{Z}_3^2$-grading on either $\mathrm{Mat}_3(\mathbb{F})$ or a central division algebra of degree 3. $\square$

Remark 3.7. The gradings on Okubo algebras have been completely determined over arbitrary fields, but the methods needed are different.

What do these $\mathbb{Z}^2$ and $\mathbb{Z}_3^2$-gradings look like?

$\mathbb{Z}^2$**-grading:** The type of this grading on $\mathrm{Mat}_3(\mathbb{F})$ ($\omega \in \mathbb{F}$) is $(6, 0, 1)$, so its type on S is $(6, 1)$, with $\dim S_0 = 2$ and $\dim S_g \leq 1$ for $g \neq 0$. Take $0 \neq g \in \mathbb{Z}^2$ with $S_g \neq 0 = S_{2g}$. Then $S_0 \oplus S_g \oplus S_{-g}$ is a para-quaternion subalgebra S with "para-unit" $e \in S_0$. Consider the Hurwitz algebra $(S, \cdot, q)$ with multiplication

$$x \cdot y = (e * x) * (y * e),$$

and unity e.

Lemma 3.8. *The map $\tau : S \to S$, such that $\tau(x) = q(x,e)e - x * e$ is an order 3 automorphism of both $(S,*)$ and $(S,\cdot)$.*

Proof. Define $\bar{x} = q(x,e)e - x$, then $\tau(x) = \bar{x} * e = \overline{x * e}$ ($q(e,x) = q(e*e,x) = q(e,e*x)$), so that $\tau(x) = r_e(\bar{x}) = \overline{r_e(x)}$, and hence $\tau^3(x) = r_e^3(\bar{x})$. But

$$((x*e)*e)*e = -(e*e)*(x*e) + q(x*e,e)e = -e*(x*e) + q(x,e)e = -x + q(e,x)e = \bar{x}.$$

Therefore, $\tau^3 = id$, and $\tau \neq id$, because otherwise e would be a "para-unit" of $(S,*,q)$ and this would force this algebra to be para-Hurwitz. Also $\tau^2(x) = (x * e) * e = q(e,x)e - x * e = l_e(\bar{x}) = \overline{l_e(x)}$. Now,

$$
\begin{aligned}
\tau(x) * \tau(y) &= (q(e,x)e - x * e) * (q(e,y)e - y * e) \\
&= q(e,x)q(e,y)e - q(e,x)y - q(e,y)(x*e)*e + (x*e)*(y*e) \\
&= q(e,x)q(e,y)e - q(e,x)y - q(e,y)\Big(q(e,x)e - e*x\Big) \\
&\quad + \Big(q(x*e,e)y - e*(y*(x*e))\Big) \\
&= q(e,y)e*x - e*(y*(x*e)) \\
&= e*(e*(x*y)) = q(e,x*y)e - (x*y)*e = \tau(x*y),
\end{aligned}
$$

and hence τ is an automorphism of $(S,*,q)$. Since $\tau(e) = e$, it follows that τ is an automorphism too of $(S,\cdot,q)$. $\square$

Note that the restriction of τ to the subalgebra $S_0 \oplus S_g \oplus S_{-g}$ is the identity, that all the homogeneous subspaces are invariant under τ and that for any $x,y \in S$ $x*y = (e*(x*e))*((e*y)*e) = (x*e)\cdot(e*y) = \tau(\bar{x})\cdot\tau^2(\bar{y})$. That is,

$$x*y = \tau(\bar{x})\cdot\tau^2(\bar{y}), \tag{3.9}$$

for any $x,y \in S$.

The automorphism τ being of order 3, it induces a $\mathbb{Z}_3$-grading of the Cayley algebra $(S,\cdot,q)$ with $\dim S_{\bar{0}} = 4$. There is just one possibility for such a grading (which is a $\mathbb{Z}$-grading too). It follows that there exists a canonical basis $B = \{e_1,e_2,u_1,u_2,u_3,v_1,v_2,v_3\}$ with $S_{\bar{0}} = \text{span}\{e_1,e_2,u_1,v_1\}$, $S_{\bar{1}} = \text{span}\{u_2,v_3\}$ and $S_{\bar{2}} = \text{span}\{u_3,v_2\}$. That is, $\tau|_{S_{\bar{0}}} = id$, $\tau|_{S_{\bar{1}}} = \omega id$ and $\tau|_{S_{\bar{2}}} = \omega^2 id$. The $\mathbb{Z}^2$-grading is given by the canonical $\mathbb{Z}^2$-grading on the Hurwitz algebra $(S,\cdot,q)$ relative to this basis, with the product given by (3.9). The grading is thus expressed in terms of the Cartan grading of the split Cayley algebra.

$\mathbb{Z}_3^2$**-grading:** Here the type of this grading on the central simple associative algebra A is (9), and hence the type in S is (8) with $S_0 = 0$ and $\dim S_g = 1$ for any $g \neq 0$. The associative algebra A appears as a crossed product

$$A = \text{alg}\left\langle x,y : x^3 = \alpha,\ y^3 = \beta,\ yx = \omega xy\right\rangle.$$

Think for example in $A = \text{Mat}_3(\mathbb{F})$, and x and y given by:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this situation $S = A_0 = \text{span}\left\{x^i y^j : 0 \le i,j \le 2,\ (i,j) \neq (0,0)\right\}$ and $S_{(\bar{i},\bar{j})} = \mathbb{F}x^i y^j$.

The material in this section is taken from [Eld98] and [Eldpr1].

4. THE MAGIC SQUARE OF EXCEPTIONAL LIE ALGEBRAS. INDUCED GRADINGS.

Throughout this lecture the characteristic of the ground field $\mathbb{F}$ will always be assumed to be $\neq 2, 3$.

Given two symmetric composition algebras $(S, *, q)$ and $(S', \star, q')$, consider the vector space:

$$\mathfrak{g} = \mathfrak{g}(S, S') = \left(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\right) \oplus \left(\oplus_{i=0}^{2} \iota_i(S \otimes S')\right),$$

where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$ $(i = 0, 1, 2)$ and we write $\mathfrak{tri}(S)$, $\mathfrak{tri}(S')$ instead of $\mathfrak{tri}(S, *, q)$ and $\mathfrak{tri}(S', \star, q')$ for short. Define now an anticommutative bracket on $\mathfrak{g}$ by means of:

- the Lie bracket in $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\left(d_i(x) \otimes x'\right)$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = \iota_i\left(x \otimes d'_i(x')\right)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\left((x * y) \otimes (x' \star y')\right)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x, y)\theta'^i(t'_{x',y'}) \in \mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$.

Theorem 4.1. *With this bracket, $\mathfrak{g}(S, S')$ is a Lie algebra and, if S_r and S'_s denote symmetric composition algebras of dimension r and s, then the Lie algebra $\mathfrak{g}(S_r, S'_s)$ is a (semi)simple Lie algebra whose type is given by Freudenthal's Magic Square:*

	S_1	S_2	S_4	S_8
S'_1	A_1	A_2	C_3	F_4
S'_2	A_2	$A_2 \oplus A_2$	A_5	E_6
S'_4	C_3	A_5	D_6	E_7
S'_8	F_4	E_6	E_7	E_8

Proof. "Straightforward" (but lengthy). □

The Lie algebra $\mathfrak{g} = \mathfrak{g}(S, S')$ is naturally $\mathbb{Z}_2^2$-graded with

$$\mathfrak{g}_{(\bar{0},\bar{0})} = \mathfrak{tri}(S) \oplus \mathfrak{tri}(S'),$$

$$\mathfrak{g}_{(\bar{1},\bar{0})} = \iota_0(S \otimes S'), \quad \mathfrak{g}_{(\bar{0},\bar{1})} = \iota_1(S \otimes S'), \quad \mathfrak{g}_{(\bar{1},\bar{1})} = \iota_2(S \otimes S').$$

Now, this $\mathbb{Z}_2^2$-grading can be combined with gradings on S and S' to obtain some nice gradings of the exceptional simple Lie algebras.

Also, the triality automorphisms θ and θ' induce an order 3 automorphism $\Theta \in \text{Aut } \mathfrak{g}$ such that

$$\begin{cases} \Theta|_{\mathfrak{tri}(S)} = \theta, \quad \Theta|_{\mathfrak{tri}(S')} = \theta', \\ \Theta\left(\iota_i(x \otimes x')\right) = \iota_{i+1}(x \otimes x') \quad \text{(indices modulo 3)} \end{cases}$$

If $\omega \in \mathbb{F}$ this gives a $\mathbb{Z}_3$-grading which can be combined too with the gradings on S and S'.

Examples 4.2.
- The $\mathbb{Z}_2^3$-grading on a Cayley algebra C give a fine grading of the simple Lie algebra $\mathfrak{g}_2 = \mathfrak{Der}\, C$, where

$$\mathfrak{g}_2 = \oplus_{0 \neq \alpha \in \mathbb{Z}_2^3}(\mathfrak{g}_2)_\alpha,$$

and

$$\boxed{(\mathfrak{g}_2)_\alpha \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_2^3!}$$

(The only fine gradings on $\mathfrak{g}_2$ (char $\mathbb{F} = 0$) are this $\mathbb{Z}_2^3$-grading and the Cartan grading. This has been proved independently by Draper and Martín [DM06] and by Bahturin and Tvalavadze [BT09].)

It induces too a $\mathbb{Z}_2^3$-grading on $\mathfrak{d}_4 = \mathfrak{so}(C, q)$ with

$$\mathfrak{d}_4 = \oplus_{0 \neq \alpha \in \mathbb{Z}_2^3} (\mathfrak{d}_4)_\alpha,$$

where again

$$\boxed{(\mathfrak{d}_4)_\alpha \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_2^3!}$$

But this grading is not fine. It can be refined (if $\omega \in \mathbb{F}$) by means of the triality automorphism θ of $\mathfrak{tri}(\bar{C}) \simeq \mathfrak{so}(C, q)$ to get a fine $\mathbb{Z}_2^3 \times \mathbb{Z}_3$-grading of type $(14, 7)$.

- Let $(\mathcal{O}, *, q)$ be an Okubo algebra and assume that $\omega \in \mathbb{F}$. The $\mathbb{Z}_3^2$- grading on $\mathcal{O}$, combined with the automorphism Θ, induces a $\mathbb{Z}_3^3$-grading of $\mathfrak{f}_4 = \mathfrak{g}(\mathbb{F}, \mathcal{O})$. Again,

$$\mathfrak{f}_4 = \oplus_{0 \neq \alpha \in \mathbb{Z}_3^3} (\mathfrak{f}_4)_\alpha,$$

with $\dim(\mathfrak{f}_4)_\alpha = 2$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$, and

$$\boxed{(\mathfrak{f}_4)_\alpha \oplus (\mathfrak{f}_4)_{-\alpha} \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_3^3!}$$

This can be extended to a $\mathbb{Z}_3^3$-grading on $\mathfrak{e}_6 = \mathfrak{g}(S_2, \mathcal{O})$ with similar properties: $\dim(\mathfrak{e}_6)_\alpha = 3$ and

$$\boxed{(\mathfrak{e}_6)_\alpha \oplus (\mathfrak{e}_6)_{-\alpha} \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_3^3!}$$

- Consider now two $\mathbb{Z}_2^3$-graded para-Cayley algebras $\bar{C}$ and $\bar{C}'$. The natural $\mathbb{Z}_2^2$-grading of $\mathfrak{g}(\bar{C}, \bar{C}')$ combined with the $\mathbb{Z}_2^3$-grading on $\bar{C} \otimes \bar{C}'$ induces a $\mathbb{Z}_2^5$-grading:

$$\mathfrak{e}_8 = \oplus_{0 \neq \alpha \in \mathbb{Z}_2^5} (\mathfrak{e}_8)_\alpha,$$

such that

$$\boxed{(\mathfrak{e}_8)_\alpha \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_3^3!}$$

This is a famous *Dempwolff* decomposition considered by Thompson [Tho76].

Jordan gradings: Alekseevskii [Al74] considered *Jordan subgroups* A of Aut $\mathfrak{g}$ for the simple complex Lie algebras. Any such group is abelian and:

(i) its normalizer is finite,

(ii) A is a minimal normal subgroup of its normalizer,

(iii) its normalizer is maximal among the normalizers of abelian subgroups satisfying (i) and (ii).

He classified (1974) these groups and gave detailed models of all the possibilities for classical simple Lie algebras. The exceptional cases are:

$\mathfrak{g}$	A	$\dim \mathfrak{g}_\alpha$ $(\alpha \neq 0)$
G_2	$\mathbb{Z}_2^3$	2
F_4	$\mathbb{Z}_3^3$	2
E_8	$\mathbb{Z}_5^3$	2
D_4	$\mathbb{Z}_2^3$	4
E_8	$\mathbb{Z}_2^5$	8
E_6	$\mathbb{Z}_3^3$	3

With the exception of the $\mathbb{Z}_5^3$-grading of E_8, these are precisely the gradings considered in the previous examples.

Some other related results: Assume $\mathbb{F}$ algebraically closed of characteristic 0.
- **Fine gradings of F_4 [DMpr]:**
  – Cartan grading (over $\mathbb{Z}^4$),
  – The $\mathbb{Z}_2^5$-grading on $\mathfrak{f}_4 = \mathfrak{g}(k, \bar{C})$ (C a Cayley algebra) obtained by combining the natural $\mathbb{Z}_2^2$-grading on $\mathfrak{g}(k, \bar{C})$ and the $\mathbb{Z}_2^3$-grading on $\bar{C}$.
  – The $\mathbb{Z}_3^3$-grading on $\mathfrak{f}_4 = \mathfrak{g}(k, \mathcal{O})$ obtained by combining the $\mathbb{Z}_3^2$-grading on $\mathcal{O}$ with the $\mathbb{Z}_3$-grading induced by the automorphism Θ.
  – A $\mathbb{Z}_2^3 \times \mathbb{Z}$-grading: the $\mathbb{Z}_2^2$-grading on $\mathfrak{g}(k, \bar{C})$ can be "unfolded" to a $\mathbb{Z}$-grading compatible with the $\mathbb{Z}_2^3$-grading on $\bar{C}$. (This is related to the fact that C is a *structurable algebra*.)
- **Fine gradings of D_4 ([DMVpr], [Eldpr2]):**
  Among the **17** fine gradings of D_4, there are 3 of them which have no counterparts for D_n, $n > 4$:
  – A $\mathbb{Z}^2 \times \mathbb{Z}_3$-grading obtained by combining the $\mathbb{Z}^2$-grading on C and the $\mathbb{Z}_3$-grading given by the triality automorphism.
  – A $\mathbb{Z}_2^3 \times \mathbb{Z}_3$-grading obtained by combining the $\mathbb{Z}_2^3$-grading on C and the $\mathbb{Z}_3$-grading given by the triality automorphism.
  – A $\mathbb{Z}_3^3$-grading obtained by combining the $\mathbb{Z}_3^2$- grading on $\mathcal{O}$ and the $\mathbb{Z}_3$-grading given by the triality automorphism.

It is hoped that the construction $\mathfrak{g}(S, S')$ will allow nice descriptions of a large portion of the fine gradings on the exceptional Lie algebras E_6, E_7, E_8.

The results in this section are taken from [Eld04], [Eldpr1] and [Eld09].

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