Lecture 2
Theorems of Fujita and Abhyankar, Eakin, and Heinzer

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Abstract

We give elementary proofs of two well-known theorems. First, if $A[x] \cong \mathbb{C}[x_1, x_2, x_3]$ then $A \cong \mathbb{C}[x_1, x_2]$. Second, if A_1 and A_2 are domains of transcendence degree 1 over $\mathbb{C}$ and $A_1[x_1, \ldots, x_n] \cong A_2[x_1, \ldots, x_n]$ then $A_1 \cong A_2$.

1 $A[x] \cong \mathbb{C}_3$

We abbreviate $R[x_1, \ldots, x_n]$ by R_n.

Theorem 1 (Takao Fujita)**.** $A[x] \cong \mathbb{C}_3$ *implies* $A \cong \mathbb{C}_2$*.*

Definition. For any ring A we can define its *ring of absolute constants* $\mathrm{AK}(A)$. It is the intersection of the rings of constants of all locally nilpotent derivations.

For example though a description of lnds for polynomial rings $\mathbb{C}_n$ are known only when $n = 1$ or $n = 2$, it is easy to compute that $\mathrm{AK}(C_n) = \mathbb{C}$ because all partial derivatives are lnds.

Here is a key Lemma

Lemma 5. If A is a commutative domain, $\mathrm{GK}\dim(A) < \infty$, $(\mathrm{trdeg}(A) < \infty)$, and $\mathrm{AK}(A) = A$ then $\mathrm{AK}(A[x]) = A$.
(We are assuming that the only lnd on A is the zero derivation.)
Proof. The derivative by x is an lnd, just like in the ring $\mathbb{C}[x]$. We will see that all lnds of $A[x]$ are equivalent to this derivation.

If $\partial \in \mathrm{LND}(A[x]$ is a non-zero derivation and $\partial|_A = 0$ consider $\partial(x) = x_0 x^d + x_1 x^{d-1} + \cdots + x_d$ where $x_i \in A$ and $x_0 \neq 0$. The ∂-degree of $\partial(x)$ is

$\deg_\partial(x) - 1$ and $\deg_\partial(x_0x^d + x_1x^{d-1} + \cdots + x_d) = d\deg_\partial(x)$. The equality $\deg_\partial(x) - 1 = d\deg_\partial(x)$ is possible only if $d = 0$ and $\deg_\partial(x) = 1$, i.e. ∂ is equivalent to the derivative by x.

If all lnds of $A[x]$ are zeros on A the lemma is proved. Assume therefore that ∂ is not identically zero on A. Since A has a finite transcendence degree,

$$m = \max(\deg_x(\partial(a))|a \in A) < \infty.$$

To see this take a transcendence basis $\mathcal{T}: \ t_1, \ldots, t_n$ of A, i.e. a maximal set of algebraically independent elements. If $a \in A$ then there is an irreducible dependence of a with this basis given by a polynomial $p(t_1, \ldots, t_n, a) = 0$. Hence $0 = \sum_i p_i\partial(t_i) + p_a\partial(a)$ where all partial derivatives p_i and p_a belong to A. Therefore $\deg_x(\partial(a))$ cannot be larger than $\max(\deg_x(\partial(t_i))|t_i \in \mathcal{T})$.

Let

$$\partial(x) = x_0x^d + x_1x^{d-1} + \cdots + x_d$$

where $x_i \in A$.

To understand what is going on consider the following three possibilities:
(a) $d > m + 1$;
(b) $d < m + 1$;
(c) $d = m + 1$.

If $d > m+1$ then $\deg_x(\partial^2(x)) = 2d-1$. Indeed, $\partial^2(x) = \partial(\sum_{i=0}^d x_ix^{d-i}) = \sum_{i=0}^d[\partial(x_i)x^{d-i}+(d-i)x_ix^{d-i-1}\partial(x)]$ and $\deg_x((\partial(x_i)x^{d-i}+(d-i)x_ix^{d-i-1}\partial(x)) = 2d - i - 1$ since $\deg_x(\partial(x_i)x^{d-i}) \leq m + d - i < d - 1 + d - i$. Similar considerations show that $\deg_x(\partial^j(x)) = jd - j - 1$ and $\partial^j(x) \neq 0$ for any j. This is impossible since ∂ is an lnd.

If $d < m+1$ we can write $\partial(a) = \sum_{i=0}^m \epsilon_i(a)x^{m-i}$ for $a \in A$ where $\epsilon_i(a) \in A$ and ϵ_0 is not identically zero because $\deg_x(\partial(a)) = m$ for at least one $a \in A$.

Operators ϵ_i are derivations of A. Indeed,
$\partial(a_1 + a_2) = \partial(a_1) + \partial(a_2) = \sum_{i=0}^m \epsilon_i(a_1)x^{m-i} + \sum_{i=0}^m \epsilon_i(a_2)x^{m-i}$,
hence $\epsilon_i(a_1 + a_2) = \epsilon_i(a_1) + \epsilon_i(a_2)$ and
$\partial(a_1a_2) = \partial(a_1)a_2 + a_1\partial(a_2) = \sum_{i=0}^m \epsilon_i(a_1)x^{m-i}a_2 + a_1\sum_{i=0}^m \epsilon_i(a_2)x^{m-i}$,
hence $\epsilon_i(a_1a_2) = \epsilon_i(a_1)a_2 + a_1\epsilon_i(a_2)$.

Now, $\partial^2(a) = \partial(\sum_{i=0}^m \epsilon_i(a)x^{m-i}) = \sum_{i=0}^m[\partial(\epsilon_i(a))x^{m-i}+(m-i)\epsilon_i(a)x^{m-i-1}\partial(x)]$ and $\deg_x[\partial(\epsilon_i(a))x^{m-i} + (m-i)\epsilon_i(a)x^{m-i-1}\partial(x)] = 2m - i$ if $\epsilon_0(\epsilon_i(a)) \neq 0$ since $\deg_x[(m-i)\epsilon_i(a)x^{m-i-1}\partial(x)] \leq m - i - 1 + d < 2m - i$. Therefore

$\deg_x(\partial^2(a)) = 2m$ if $\epsilon_0^2(a) \neq 0$ since the coefficient with x^{2m} is $\epsilon_0^2(a)$. Similarly, $\deg_x(\partial^k(a)) = km$ if $\epsilon_0^k(a) \neq 0$. But then ϵ_0 is an lnd of A because ∂ is an lnd of $A[x]$. Therefore $\epsilon_0 = 0$.

The remaining case is $d = m + 1$ (and $m > 0$). In this case $\partial(ax^k) = \partial(a)x^k + kax^{k-1}\partial(x) = \sum_{i=0}^m \epsilon_i(a)x^{m-i}x^k + kax^{k-1}\sum_{i=0}^{m+1} x_i x^{m+1-i} = (\epsilon_0(a) + kax_0)x^{m+k} + \sum_{i=1}^m(\epsilon_i(a) + kax_i)x^{m-i+k} + kax_{m+1}x^{k-1}$.

Hence $\deg_x(\partial(ax^k)) = k+m$ if $\epsilon_0(a) + kax_0 \neq 0$. Consider a derivation D given by $D(a) = \epsilon_0(a)x^m,\ D(x) = x_0x^{m+1}$. We can write $\partial(ax^k) = D(ax^k) + \Delta$ where $\deg_x(\Delta) < k + m$ since $D(ax^k) = \epsilon_0(a)x^{m+k} + kx^{k-1}ax_0x^{m+1}$.

Therefore D is an lnd since if it isn't we will find a monomial ax^k for which $\deg_x(\partial^j(ax^k)) = k + jm$. But this is impossible since if D is an lnd then $\deg_D(x) - 1 = \deg_D(a_0) + (m + 1)\deg_D(x)$.
Lemma is proved. $\square$

Since $A[x] \cong \mathbb{C}_3$ we cannot have $\mathrm{AK}(A) = A$ because, according to Lemma 1 if $\mathrm{AK}(A) = A$ then $\mathbb{C} = \mathrm{AK}(\mathbb{C}_3) = A$. Hence $\mathrm{AK}(A) \neq A$ which means that there is a non-zero lnd ∂ on A.

As we know from the first lecture $\mathrm{trdeg}(A^\partial) = 1$. Therefore $A^\partial = \mathbb{C}[p]$ (Lemma 4).

A derivation ∂ is a non-zero lnd on A. So there are elements of A which do not belong to A^∂, but their images under ∂ belong to $A^\partial = \mathbb{C}[p]$. We can choose among them an $a \in A$ for which $\partial(a) = f(p)$ where degree of a polynomial f is minimal possible. If this degree is zero, i.e. $\partial(a) = c \in \mathbb{C}^*$ then $A = \mathbb{C}[p, a]$. In this case $t \in A$ and the proof that $\mathrm{Nil}_A(\partial) = A^\partial[t]$ is the same as the proof in Lemma 2.

If $A = \mathbb{C}[p, a]$ we are done. Otherwise take a $b \in A \setminus \mathbb{C}[p, a]$. As we know $A \subset \mathrm{Frac}(A^\partial)[\frac{a}{f(p)}] = \mathbb{C}(p)[a]$. Since $b \notin \mathbb{C}[p, a]$ it has a denominator which is a polynomial in p, and we can multiply b by a polynomial $g(p)$ so that the denominator of $g(p)b$ is $p - c$. Thus $g(p)b = \frac{\phi(p,a)}{p-c} = \psi(p, a) + \frac{\xi(a)}{p-c}$ where $\phi,\ \psi \in \mathbb{C}[p, a]$ and $\xi \in \mathbb{C}[a]$.

Therefore $\frac{\xi(a)}{p-c} \in A$. Consider presentation of $\xi(a) = \prod_i(a - \mu_i)$ as a product of irreducible polynomials. One of them should be divisible by an irreducible polynomial $p-c$. This polynomial is a factor in one of the factors $a - \mu_i$. Hence $\frac{a-\mu}{p-c} \in A$ for some $\mu \in \mathbb{C}$. But this leads to a contradiction since $\partial(\frac{a-\mu}{p-c}) = \frac{f(p)}{p-c}$ and $\deg_p(\frac{f}{p-c}) < \deg_p(f)$. $\square$

This is a theorem of Takao Fujita: Fujita, Takao On Zariski problem. Proc. Japan Acad. Ser. A Math. Sci. 55 (1979), no. 3, 106-110.

2 $A_1[x_1, \ldots, x_n] \cong A_2[x_1, \ldots, x_n]$

Now we discuss the following question. Suppose that $R_n = R[x_1, \ldots, x_n]$ is given and R is a domain of transcendence degree one. Can we recover R up to an isomorphism? It turns out that the answer is yes. Though we are not going to discuss it, the answer is no if $\mathrm{trdeg}(R) > 1$.

Theorem 2. *If A_1 and A_2 are domains of transcendence degree 1 over $\mathbb{C}$ and $A_1[x_1, \ldots, x_n] \cong A_2[x_1, \ldots, x_n]$ then $A_1 \cong A_2$.*

Lemma 6. Either $\mathrm{AK}(R_n) = R$ or $\mathrm{AK}(R_n) = \mathbb{C}$.
Proof. Since each of the partial derivatives is in $\mathrm{LND}(R_n)$ it is clear that $\mathrm{AK}(R_n) \subset R$.

So if $\mathrm{AK}(R_n) \neq R$ there exist $\partial \in \mathrm{LND}(R_n)$ which is not identically zero on R. If $\partial(s) \neq 0$ for some $s \in R$ then $\partial(r) \neq 0$ for any $r \in R \setminus \mathbb{C}$. Indeed, if $r_1 \in R \setminus \mathbb{C}$ and $r_2 \in R \setminus \mathbb{C}$ then these two elements are algebraically dependent (since $\mathrm{trdeg}(R) = 1$). If $q(r_1, r_2) = 0$ is an irreducible dependence between these elements then $0 = \partial(q(r_1, r_2)) = q_1(r_1, r_2)\partial(r_1) + q_2(r_1, r_2)\partial(r_2)$ where q_i are the corresponding partial derivatives of $q(r_1, r_2)$. If $\partial(r_1) = 0$ then $q_2(r_1, r_2)\partial(r_2) = 0$ and $\partial(r_2) = 0$ since $q_2(r_1, r_2) \neq 0$. Therefore $\mathrm{AK}(R_n) = \mathbb{C}$. $\square$

Lemma 7. If $\mathrm{AK}(R_n) = \mathbb{C}$ then R is isomorphic to a subring of a polynomial ring with one generator.
Proof. Let $S = \mathrm{Frac}(R)$ and $S_n = \mathrm{Frac}(R_n)$. As we know from the previous Lemma there is a $\partial \in \mathrm{LND}(R_n)$ which is not identically zero on R. Since $\partial \in \mathrm{LND}(R_n)$ there exists $t \in S_n$ for which $\partial(t) = 1$. Of course, $R \subset S_n^{\partial}[t]$ and t-degrees of elements of $R \setminus \mathbb{C}$ are positive.

Since $\mathrm{trdeg}(R) = 1$ it is possible to construct a monomorphism (i.e. one-to-one homomorphism) from R into $\mathbb{C}[x]$.

The ring R is finitely generated. To check it consider all $\deg_{\partial}(r)$ where $r \in R$. This is a semigroup Π relative to addition. Take d, the smallest positive number in Π. Then Π is generated by d and the smallest elements of Π which are congruent to $1, 2, \ldots, d-1$ modulo d. We can chose elements $r_1, ..., r_m \in R$ with the corresponding degrees. Consider a subalgebra A of R generated by these elements. If $r \in R$ there exists an element $a \in A$ with $\deg_{\partial}(a) = \deg_{\partial}(r)$. Suppose $r = \sum_{i=0}^{\nu} r_i t^{\nu-i}$, $a = \sum_{i=0}^{\nu} a_i t^{\nu-i}$; $r_i, a_i \in S_n^{\partial}$. These elements are algebraically dependent: $\sum_{i+j=0}^{i+j=\mu} c_{i,j} a^i r^j = 0$, $c_{i,j} \in \mathbb{C}$. Hence $\sum_{i+j=\mu} c_{i,j} a_0^i r_0^j = 0$, otherwise $\deg_{\partial}(\sum_{i+j=0}^{i+j=\mu} c_{i,j} a^i r^j) = \mu\nu$ and

$\sum_{i+j=0}^{i+j=\mu} c_{i,j} a^i r^j \neq 0$. We can write $\sum_{i+j=\mu} c_{i,j} a_0^i r_0^j = a_0^\mu \sum_{i+j=\mu} c_{i,j} (\frac{r_0}{a_0})^j = a_0^n \prod_{k=0}^{\mu} (\frac{r_0}{a_0} - \lambda_k)$ where $\lambda_1, \dots, \lambda_\mu$ are roots of the polynomial $\sum_{i+j=\mu} c_{i,j} z^j$. Thus $r_0 - \lambda a_0 = 0$ for one of these roots and $\deg_\partial(r - \lambda a) < \deg_\partial(r)$. So, by induction on $\deg_\partial$ we can show that any element $r \in R$ is a linear combination of elements of A, i.e. $R = A = \mathbb{C}[r_1, \dots, r_m]$.

Consider the subfield E of S_n^∂ which is generated by the coefficients of all r_i's as polynomials in t. Since E is finitely generated it has a finite basis of transcendence $t_1, ..., t_k$ over $\mathbb{C}$. Since characteristic of E is zero, $E = \mathbb{C}(t_1, ..., t_k)[\theta]$ where θ is algebraic over $\mathbb{C}(t_1, ..., t_k)$. The element θ is a root of an irreducible polynomial $P(z) \in \mathbb{C}(t_1, \dots, t_k)[z]$.

The coefficients of P are rational functions in $t_1, ..., t_k$ and the coefficients of r_i are rational functions in $t_1, ..., t_k$ and polynomials in θ. Say, $f(t_1, \dots, t_k; \theta)$ is the coefficient with the highest degree of t in r_1. Since P is irreducible, polynomials $P(z)$ and $F(z) = f(t_1, \dots, t_k; z)$ are relatively prime and we can find a linear combination $\phi P + \psi F = 1$ where $\phi,\ \psi \in \mathbb{C}(t_1, \dots, t_k)$, of these polynomials.

We can find complex numbers $c_1, \dots, c_k$ which satisfy the following conditions:
(a) denominators of ϕ and ψ are not zeros;
(b) denominators of coefficients of P are not zeros;
(c) denominators of coefficients of all r_i are not zeros.

Now, substitute $c_1, \dots, c_k$ in $P(z)$ and find a root $\theta' \in \mathbb{C}$ of this polynomial. After that substitute $c_1, \dots, c_k$ and θ' in the coefficients of r_i. We will obtain m polynomials $r'_1, \dots, r'_m \in \mathbb{C}[t]$ and $\deg(r'_1) > 0$. The mapping $\alpha : R \to \mathbb{C}[t]$ sending $r_i \to r'_i,\ i = 1, \dots, m$ is a homomorphism of R into $\mathbb{C}[t]$. The image $\alpha(R)$ has the transcendence degree one since r'_1 is not a constant. Hence this in a monomorphism. (If the kernel of α contains a non-zero element the image will be just $\mathbb{C}$.) $\square$

From now on we assume that $R \subseteq \mathbb{C}[y]$. I also assume that $S = \mathrm{Frac}(R) = \mathbb{C}(y)$. This follows from the Lüroth theorem.

Lemma 8. There exists an $r \in R$ such that $r\mathbb{C}[y] \subset R$.
Proof. By assumptions of the lemma there exist two elements $f, g \in R$ such that $g = yf$. Let $\deg_y(f) = k + 1$ and let us assume that f is monic. Since elements $g^i f^{k-i} = y^i f^k \in R$ it is clear that $f^k \mathbb{C}[y] \subset R$. (It is enough to notice that if $i > k$ then $y^i = y^{i-k-1} f + r_i(y)$ with $\deg(r_i) < i$. So we can use induction on i to observe that $y^i f^k \in R$ for all i.) $\square$

Lemma 9. Denote the extension of $\partial \in \mathrm{Der}(R)$ on $\mathbb{C}(y)$ also by ∂. Then $\partial(\mathbb{C}[y]) \subset \mathbb{C}[y]$.
Proof. Let us assume that $\partial(y) \in \mathbb{C}(y) \setminus \mathbb{C}[y]$. Then making a change of variable if necessary we may assume that $\partial(y)$ has a pole of order $m-1$ at $y=0$. If $r \in R$ and $r = r(0) + y^i s$ where $s(0) \neq 0$ then i is divisible by m. Otherwise several applications of ∂ will take r out of $\mathbb{C}[y]$ and hence out of R. Since in Lemma 8 $g = yf$ where $f,\ g \in R$ this is impossible. $\square$

Lemma 10. There exists an $h \in R$ such that $h\mathbb{C}[y] \subset R$ and $\partial(h) \in h\mathbb{C}[y]$ for any $\partial \in \mathrm{Der}(R)$.
Proof. By the previous lemma there are non-zero ideals of $\mathbb{C}[y]$ which belong to R. Let us take the maximal ideal with this property and denote by h its generator.

If $\partial \in \mathrm{Der}(R)$ then $R \supset \partial(h\mathbb{C}[y]) = \partial(h)\mathbb{C}[y] + h\partial(\mathbb{C}[y])$. Since $\partial(\mathbb{C}[y]) \subset \mathbb{C}[y]$ it implies that $\partial(h)\mathbb{C}[y] \subset R$ and thus $\partial(h) \in h\mathbb{C}[y]$. $\square$

Lemma 11. $R = \mathbb{C}[y]$.
Proof. $R_n \subseteq \mathbb{C}[y, x_1, \ldots, x_n] = A$ by Lemma 7, and, using Lüroth theorem $\mathrm{Frac}(R_n) = \mathbb{C}(y, x_1, \ldots, x_n)$. By Lemma 10 there exists an $h \in R$ such that $h\mathbb{C}[y] \subset R$ and $\partial(h) \in h\mathbb{C}[y]$ for any $\partial \in \mathrm{Der}(R)$. If $h \in \mathbb{C}$ the Lemma is proved. So let us assume that $h \notin \mathbb{C}$.

By the assumption of the Lemma there is a $\partial \in \mathrm{LND}(R_n)$ which is not zero on R. As we know if $h \notin \mathbb{C}$ then $\partial(h) \neq 0$.

We can write $\partial(r) = \sum_{\mathbf{i}} \epsilon_{\mathbf{i}}(a)\mathbf{x}^{\mathbf{i}}$ where $\mathbf{i} = (i_1, \ldots, i_n)$ is a multi-index and $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \ldots x_n^{i_n}$. As above, $\epsilon_{\mathbf{i}} \in \mathrm{Der}(R)$. Hence by Lemma 10 $\partial(h) = ha$ where $a \in A$. By Lemma 9 $\partial(A) \subset A$. Hence $\partial^i(h) = ha_i,\ a_i \in A$. Consider $g = \partial^k(h)$ where $k = \deg_\partial(h)$. Then $g = ha_k$ and $\partial(g) = 0$.

We know from Lemma 10 that $hA \subseteq R_n$. Therefore $g^2 = h^2a_k^2 = h(ha_k^2) \in R_n$ and $\partial(g^2) = 0$. Since $ha_k^2 \in R_n$ both $\partial(h) = 0$ and $\partial(ha_k^2) = 0$ contrary to our assumption that $\partial(h) \neq 0$. Therefore $h \in \mathbb{C}$ and $R_n = A$. $\square$

Our claim is proved. If $\mathrm{AK}(R_n) = R$ we know what R is, if $\mathrm{AK}(R_n) = \mathbb{C}$ then $R = \mathbb{C}$.

This is a theorem of Abhyankar, Eakin, and Heinzer: Abhyankar, Shreeram S.; Heinzer, William; Eakin, Paul On the uniqueness of the coefficient ring in a polynomial ring. J. Algebra 23 (1972), 310-342.