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$k = \bar{k}$, H Hopf algebra $/k$, A assoc. unital $/k$.

<u>Def</u> A right coaction of H on A is an H-comodule structure $\rho: A \to A \otimes H$ which is an algebra homom.

<u>Dual notion:</u> A left action of H on A is an H module structure $H \otimes A \to A$ $h \otimes a \mapsto h(a)$ s.t. $h(ab) = h_1(a)\, h_2(b)$
$h(1) = \varepsilon(\varepsilon)(1)$ $\Delta h = h_1 \otimes h_2$

H f.d. $\Rightarrow$ H-coaction $= H^*$-action $=$ quantum finite symmetry

<u>Def.</u> An action is <u>inner faithful</u> if it does not factor through a nontrivial quotient Hopf algebra (on a module V)
~~a sub Hopf alg.~~

A coaction - - - factor through a sub Hopf algebra

Any (co)module can be cannonically replaced by inner faithful one.

<u>Ex.</u> Group action (classical symmetry) $G \curvearrowright A \Rightarrow kG$ acts on A
finite

<u>Main question</u> When is finite symmetry possible?

<u>Ex 1</u> V repr. of $H \Rightarrow H$ acts on $\mathrm{End}_k V$ by adjoint action $(V \otimes V^*)$

So we'll restrict to A integral domain.

<u>Ex. 2</u> H acts on TV V inner faith $\Rightarrow$ TV is inner faithf.

We will consider domains with additional properties.

1. A commutative domain Skryabin $H \curvearrowright A \Rightarrow H \curvearrowright Q_A$ $\left(\begin{array}{c}\text{enough to}\\ \text{consider fields}\end{array}\right)$

THM (EW) H semisimple + cosemisimple $\Rightarrow$ action factors through a group alg.

Pf. A f.g., $X = \operatorname{spec} A$ $\forall \chi \in X$ $\chi : A \to k$ $\rho_\chi : A \to H^*$
$\rho_\chi = (\chi \otimes 1)(\rho)$, $\operatorname{Im} \rho_\chi = A_\chi \subseteq H^*$ L. A_χ is a one-sided coideal subalgebra

Schneider: H s.s., cos.s. $\Rightarrow$ H has finitely many coideal subalgebras.
$\exists$ dense open $U \subseteq X$ s.t. $\forall \chi \in U$, A_χ has same dimension
$\Rightarrow A_\chi = B$ $\chi \in U$ is indep. of χ.

ΣA_χ - subcoalgebra $\Rightarrow$ $B \subset H^*$ Hopf subalg.
commutative
$\rho : A \to A \otimes B$ $\Rightarrow$ action factors through B^* cocommutative cosemisimple
$\Rightarrow B^* = kG$ $\square$

Skryabin H s.s <u>or</u> cos.s. $\Rightarrow$ finitely many coideal subalg.

Cor. H cos.s. $\Rightarrow$ action factors through a group
s.s $\Rightarrow$ ——— s.s. finite group scheme

Non-semis. case $H = \text{Sweedler} = \langle g, X \rangle$, $g^2 = 1$, $X^2 = 0$, $gX = -Xg$.

$A = \mathbb{C}[z]$, $gz = -z$, $Xz = 1$ (char $k \neq 2$) $\Delta g = g \otimes g$, $\Delta X = X \otimes g + 1 \otimes X$

Def. Action of H on A is Hopf-Galois if $A \otimes_{A^H} A \to A \otimes H^*$ is an isom.

Def. H is Galois-theoretical if it can ~~be~~ act inner faithfully on a commut. domain ($\Leftrightarrow$ a field)

Ex. H_{Sweedler} is GT.

Ex. $H_q(sl_2)$ is GT but $\operatorname{gr}(u_2(sl_2))$ is <u>not</u>.

2. A noncommut. (A division alg.) 13-6-16

Thm. (Skryabin, v. Oyst.) A ~~...~~ Dre domain $H \hookrightarrow A \Rightarrow H \hookrightarrow Q_A$

Q. Can any ~~...~~ f.d. Hopf alg. act inner faithfully on a div. alg.?

Conjecture (~~Artamonov~~) Any f.d. Hopf alg. can act inner faithf. on a quantum torus $k < x_i^{\pm 1}, x_i x_j = q_{ij} x_j x_i >$

Q. Is this already true if q_{ij} are roots of unity? Can any H act. i.f. on a central division algebra?

Special case: $H = \mathrm{Fun}(G, k)$ H-action $= G$-grading $A = \bigoplus_{g \in G} A_g$!
inn. faithf. $=$ faithf. grading, $A_g \neq 0$ $\forall g$

Thm (GE) Let G be a finite group $\exists A$, of deg. d with a faithful G-grading $\Leftrightarrow$ ~~$A = \bigoplus_{g \in G} A_g$~~ G has a normal abelian subgroup of index dividing d.

Q. If a semisimple H acts on a div. alg. A inner faithfully $\boxed{\overset{?}{\Rightarrow}}$ Hopf-Galois
char $k=0$

Thm (CE, J Zhang) $H = kG^J$. If V a faith repres. then $H \hookrightarrow (SV)_J$ and $(SV)_J$ is a domain $\Rightarrow H \hookrightarrow Q_{(SV)_J}$

Thm If $H \hookrightarrow A$ central div. alg. of degree d and action Hopf-Galois $\Rightarrow$ PI deg $(H^*) \le d^2$. ($\Leftrightarrow$ All irrep. of H^* have dim $\le d^2$)
Sharp: attained for $H = \mathbb{C} G^J$

3. A = Weyl alg., char $k=0$ X_i, Y_i $[X_i, X_j]=0$, $[Y_i, Y_j]=0$, $[Y_i, X_j]=\delta_{ij}$.

<u>Thm</u> (CEW) $H \curvearrowright A$ Weyl alg. $\Rightarrow$ action factors through a group.

<u>Proof</u> (H semisimple)

<u>Key lemma</u>: If H acts on a division alg. D of degree d and $(\dim H!, d) = 1$ then $H \curvearrowright Z = Z(D)$ and $D = Z \cdot D^H$

<u>Proof of the lemma</u>

CFM: $[D : D^H] \le \dim H \Rightarrow \deg(D^H) = d. \underset{\uparrow}{\Rightarrow} D = ZD^H \Rightarrow Z = C(D^H) \Rightarrow$

$\Rightarrow H \curvearrowright Z$ $\square$ Classical theory of div. alg.

<u>Pf. of thm.</u> <u>Reduce mod. p.</u> $H_p \curvearrowright A_p$ over $\overline{\mathbb{F}_p}$ inner faithful for $p \gg 0$.

H_p s.s. or cos.s.

$\Rightarrow H_p$ acts on $Q(A_p)$ – central div. alg of degree $d = p^{2n}$. $p > \dim H \Rightarrow$

$(p, \dim H!) = 1 \underset{\text{key lemma}}{\Longrightarrow} H \underset{\text{inner faith.}}{\curvearrowright} Z(Q(A_p))$ $Q(A_p) = Z(Q(A_p))Q(A_p)^H$

$\Rightarrow$ EW H_p-action factors through gp. alg. $\Rightarrow H_p$ is cocomm.

$\Rightarrow H$ cocomm $= H = kG$. $\square$

Thm (CEW) $H \curvearrowright A$ - Weyl algebra $\quad H/k,\ k=\bar{k}$, char $k=0$ $\quad$ 14-6-16

Then action factors through a group algebra

First assumed H s.s.

Key lemma If $H \curvearrowright D$ div. alg. $/k$ of deg d, $(\dim H!, d)=1$ then $D = Z\cdot D^H$ and $H \curvearrowright Z$.

$H \curvearrowright A$ inn. faithf. $\quad H_p \curvearrowright A_p$ over $\overline{\mathbb{F}_p}$ $\quad H_p$ s.s., cos.s. $\quad H_p \curvearrowright Z_p = \text{center}(A_p)$

inner faithf. $\overset{EW}{\Longrightarrow} H_p$ is cocomm. $\Rightarrow H$ cocomm. $\Rightarrow H = kG$.

What if H is not semisimple?

Z field $/\overline{\mathbb{F}_p}$, $H \curvearrowright Z$ $\quad k = \overline{\mathbb{F}_p}$

Def Action is Frobenius if $H \curvearrowright Z^{p^i}$ $\quad \forall i \geq 0$

Ex. H = Sweedler $= \langle g, x\rangle$ $\quad Z = k(z)$, $gz=-z$, $xz=1$, $xz^2=0$, $xz^p = xz(z^2)^{\frac{p-1}{2}} = z^{p-1} \notin Z^p$

not Frobenius p odd

Lemma If $H \curvearrowright Z$ Frobenius, $\bigcap_{i\geq 0} Z^{p^i} = k$ then action factors through a group.

(Pf. (Idea) $Z_{\mathcal{X}} \subset Z\otimes H^*$ is defined over k).

Pf. of thm: To show that $H_p \curvearrowright Z_p^{p^i}$, use reduction $H_{p^i} \curvearrowright A_{p^i}$, algebras over $W_i(\overline{\mathbb{F}_p})$, an alg. $/\mathbb{Z}/p^i\mathbb{Z}$

Generalizations char $k=0$. $\quad A$ filtered quantization of A_0 - a commutative f.g. domain $/k$, $A = \bigcup_{i\geq 0} F_i A$, $F_i A \subseteq F_{i+1} A$ $\quad \operatorname{gr} A = A_0$.

Then can reduce mod $p \gg 0$, get A_p, filtered quant of A_{0p}, a comm. f.g. domain $/\overline{\mathbb{F}_p}$

Bad definition A is residually PI if A_p are PI for $p \gg 0$. (bad since this terminology usually means smth. else)

Question Are all A residually PI? E.g. $A_0 = k[X_1,\dots,X_n]$, $\deg(X_i)=d_i$.

Prop. If A is residually PI then PI deg $(A_p) = p^m$.

Thm. A residually PI, $H \curvearrowright A$, H semisimple $\Rightarrow$ action factors through a group

14-6-16

Pf. Repeats Weyl algebra case using Prop.

Examples 1. (Twisted) diff operators on X.
2. Finite W-alg., quotients by central char
3. Enveloping alg. of f.d. Lie alg.
4. Spherical sympl. refl. alg.
5. Quantized quiver varieties.
6. Tensor products thereof.

When can we generalize to nonsemisimple H?

Thm. This holds even if H is not semisimple if $X = \operatorname{Spec} A_0$ is generally symplectic (has an open symplectic leaf).

Non-filtered quantiz. $k\langle x,y\rangle / xy = q\,yx$ quantum poly-alg.

Problem: PI deg $A(q) = \operatorname{ord}(q)$
q transcendental $\Rightarrow$ no problem, we can map it to a root of 1 of any degree
q algebraic $\operatorname{ord}(q \bmod p)$ is not p^m.

Thm. If $q \in \overline{\mathbb{Q}}^\times$ is not a root of 1 then $\forall N \geq 1$ $\exists \infty$ many primes p s.t. $\operatorname{ord}(q \bmod p)$ is coprime to N (positive Dirichlet density).

Using this thm. from ANT for $N = \dim H!$, prove:
Thm $H \curvearrowright A(q)$, q is not a root of 1, then action factors trough group actions.

Generalization: $k\langle x_1^{\pm 1}, \dots, x_n^{\pm 1}\rangle / x_i x_j = q_{ij} x_j x_i$, $q_{ij} \in k^\times$, $q_{ij} q_{ji} = 1$, $q_{ii} = 1$
$q = (q_{ij})_{i<j} \in (k^\times)^{n(n-1)/2}$

14-6-16

ANT theorem (Perucca, 2009) Let G be a torus or abelian variety $/k$ (number field), $\sigma \in G(k)$

G_σ = Zariski closure of $\langle\sigma\rangle$ in G. Let $(N, |G_\sigma/G_\sigma^0|) = 1$. Then $\exists$ a positive Dirichlet density of primes p s.t. $\sigma \bmod p$ has order coprime to N. $G = (k^\times)^{n(n-1)/2}$

Thm. If $(\dim H!, |G_q/G_q^0|) = 1$ then any action of H on A factors through a group.

How to compute $|G_q/G_q^0|$: $|G_q/G_q^0|$ = maximal finite order of $\chi(q)$, $\chi: G \to k^\times$

Twisted homog. coord. rings: ATV X abelian ~~group~~ var, $\mathcal{L}$ ample line bdle$/X$, $\sigma \in X$, $B(X, \mathcal{L}, \sigma) = \bigoplus_{m \geq 0} H^0(X, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \ldots \otimes \mathcal{L}^{\sigma^{m-1}})$

Thm $H \curvearrowright B(X, \mathcal{L}, \sigma)$, σ of inf. order, H ss $\Rightarrow$ action factors through a group. X ell curve $\Rightarrow$ even true in nonsem. case.